Generalized stick fragmentation and Benford's Law

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2. Our Problem: Stick Breaking
3. Results
4. Further Directions

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Benford's Law

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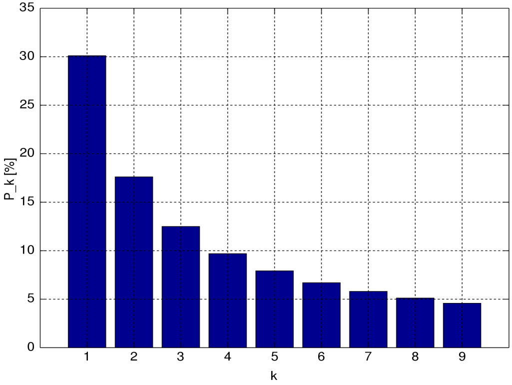
For example, when $B = 10$ (figure from Wikipedia):



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Examples:

- special sequences and functions (e.g., $n!$ and the Fibonaccis)
- iterations of the $3x + 1$ map [[1]]
- financial data (fraud detection)
- **products of random variables**

Strong Benford

Definition (Significand, Mantissa)

For $x > 0$, the **significand** of x base B is $S_B(x) \in [1, B)$ such that

$$x \ = \ S_B(x) \cdot B^k$$

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A data set $\{x_n\}$ is **strong Benford base** B if $\{M_B(x_n)\}$ is distributed uniformly in $[0, 1]$. In other words, if

$$\mathbb{P}(M_B(x_n) \in [a, b]) = b - a$$

for all $[a, b] \subseteq [0, 1]$.

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Basic Stick Breaking Model

Start with a stick of length L. Choose a random point on the stick to break it in two, and repeat the process on each new stick obtained.

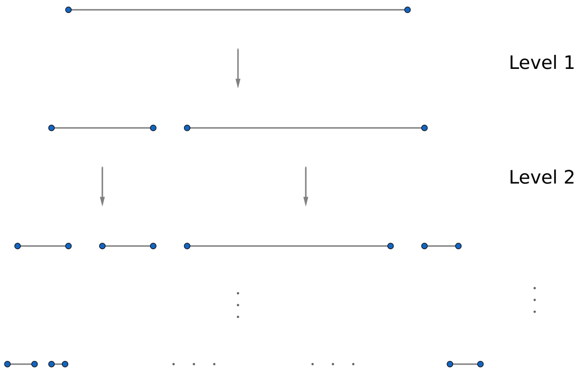

Figure 1: Illustration of stick breaking

Motivation from Physics

This process and its variations may be of interest to nuclear physicists for modelling particle decay ([2]).

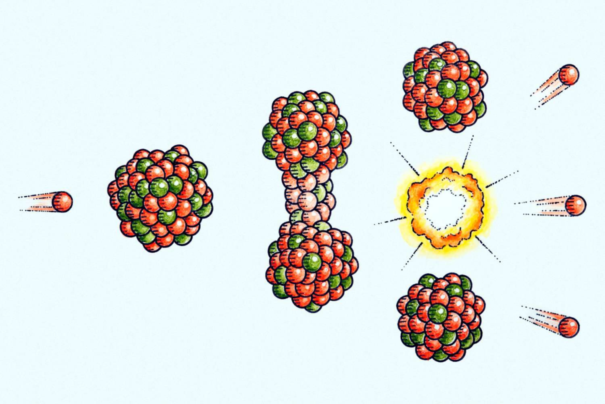

Figure 2: Random Stick Breaking Shares Similarities with Nuclear Fission

Figure source: https://www.thoughtco.com/nuclear-fission-definition-and-examples-4065372

Previous Results: Unrestricted Continuous Breaking

Theorem ([3])

Fix some distribution $\mathcal{D}$ on $(0, 1)$ satisfying the Mellin transform condition[a].

Start with a stick of length L, and break it in two with ratio sampled from $\mathcal{D}$. If we repeat this on both fragments for N levels, then the final collection of stick lengths converges to strong Benford as $N \to \infty$.

[a] Precisely, this means that

$$\lim_{n\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} \mathcal{M}f_{\mathcal{D}}\left(1 - \frac{2\pi i\ell}{\log B}\right) = 0.$$

Previous Results: Discrete One-Side Breaking

Theorem ([3])

Start with a stick of integer length L. Choose an integer $X \in \{1, \cdots, L\}$ uniformly, and break off a fragment of length X. Repeat this process on the remaining stick $L - X$, until no more such breaking can be done. The final collection converges to strong Benford as $L \to \infty$.

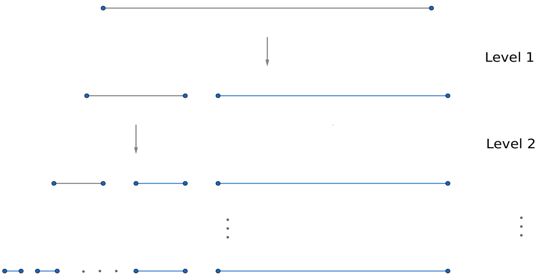

Figure 3: Illustration of discrete one-side breaking

Our Generalization: Discrete Breaking with Stopping Set

What if we break on both sides with extra *stopping conditions*?

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Question

Which sets $\mathfrak{S}$ would lead to strong Benford behavior as $L \to \infty$?

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Result 1: Stop at Evens

Theorem (F.-Miller-S.-Verga, 2023)

Start with a stick of odd integer length L. Let the stopping set be $\mathfrak{S} = \{1\} \cup \{2m : m \in \mathbb{Z}_+\}$. Then the distribution of lengths of all dead sticks at the end approaches strong Benfordness as $L \to \infty$.

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Question

Can we generalize this to other sets defined by residue classes mod *n*?

Simulation Results: Stop At Odds, Many Trials

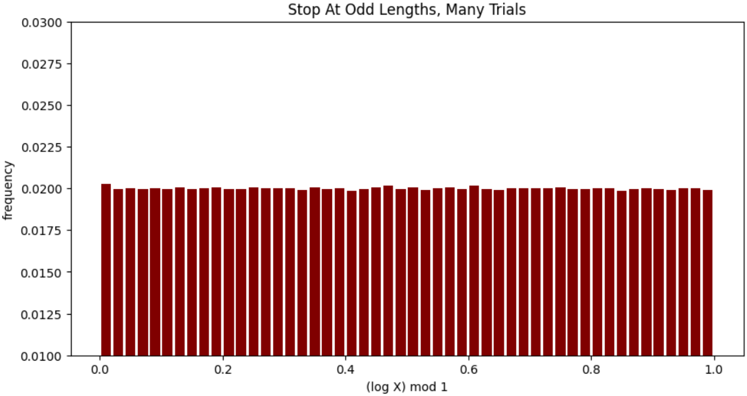

Figure 4: Histogram for $M_{10}(X)$, $L \approx 10^{1000}$, $R = 1000$ [1]

[1] R is the number of trials run with the same starting length L. The figure depicts the aggregated distribution of ending sticks from these trials.

Simulation Results: $n = 3$, stop at 1 residue class

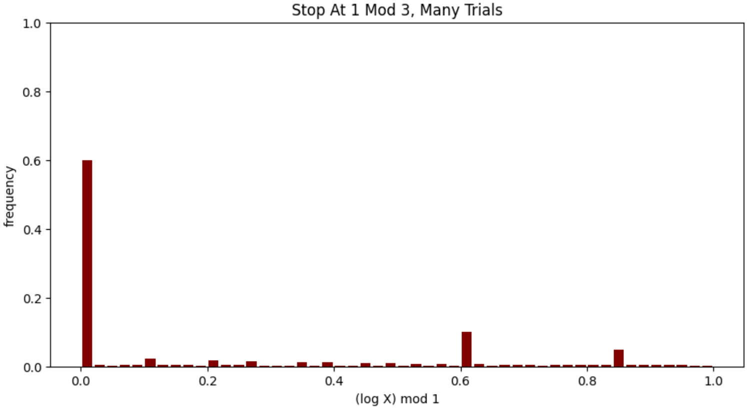

Figure 5: Histogram for $M_{10}(X)$, $L \approx 8 \cdot 10^{11}$, $R = 1000$

Simulation Results: $n = 3$, stop at 2 residue classes

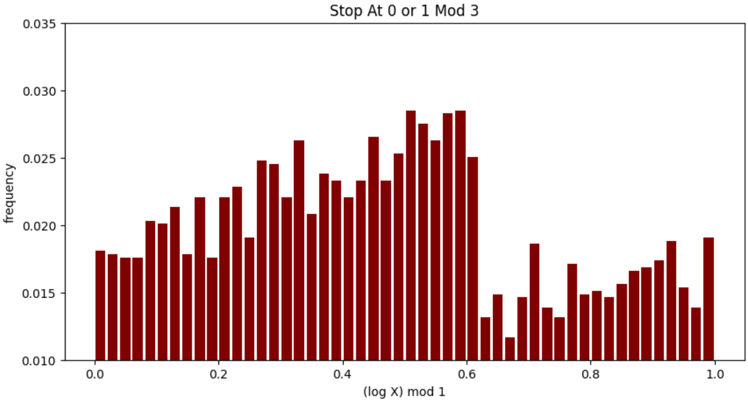

Figure 6: Histogram for $M_{10}(X)$, $L \approx 4 \cdot 10^{502}$, $R = 1000$

Simulation Results: $n = 4$, stop at 2 residue classes

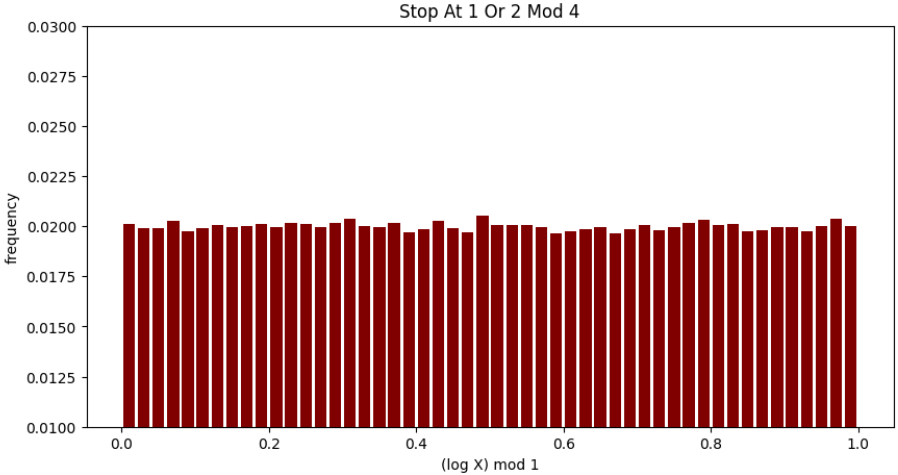

Figure 7: Histogram for $M_{10}(X)$, $L \approx 4 \cdot 10^{502}$, $R = 1000$

Result 2: Stop at Half of the Residues

After lots of trials up to $n = 10$...

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Theorem (F.-Miller-S.-Verga, 2023)

Fix an even modulus $n \geq 2$ *and a subset* $S \subset \{0, \ldots, n-1\}$ *of size* $n/2$*. Let the stopping set be*

$$\mathfrak{S} \;:=\; \{1\} \cup \{m \in \mathbb{Z}_+ : m = qn + r,\ \ r \in S, q \in \mathbb{Z}\}.$$

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$$\mathfrak{S} \ := \ \{1\} \cup \{m \in \mathbb{Z}_+ : m = qn + r,\ \ r \in S, q \in \mathbb{Z}\}.$$

If we start with R identical sticks of positive integer length $L \notin \mathfrak{S}$, then the collection of ending stick lengths converges to strong Benford behavior given that $R > (\log L)^3$ as $L \to \infty$.

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2. Show that the continuous analogue results in strong Benford behavior. (Easier!) [Key Input]
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Our contribution:

1. Generalize and adapt the "continuous approximation" strategy
2. Prove the Key Input
3. Give a new proof of the Key Lemma

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- Repeat until all sticks die.

Theorem (F.-Miller-S.-Verga, 2023)

The above process ends in finitely many levels with probability 1, and the collection of ending stick lengths almost surely converges to strong Benford behavior as $R \to \infty$.

When $|S| < n/2$: Non-Benford!

Theorem (F.-Miller-S.-Verga, 2023)

If $|S| < n/2$, then as $R \to \infty$ and $L \to \infty$, the collection of mantissas of ending stick lengths does not converge to any continuous distribution on $[0, 1]$. *In particular, it does not converge to strong Benford behavior.*

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Future Work: $|S| > n/2$?

Conjecture

When $|S| > n/2$, the result does not converge to strong Benford.

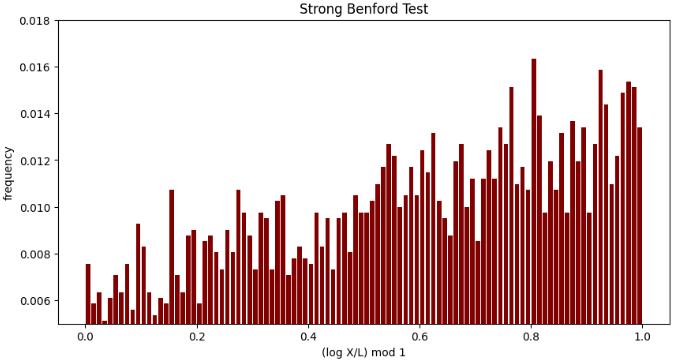

Figure 8: Stop at 8 Residue Classes Mod 12, $L = 82 \cdot 10^{12000}$, $R = 1000$

Future Work: General Number of Parts?

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In fact, we have a more general version of the Key Input.

Theorem (F.-Miller-S.-Verga, 2023)

Fix some $k \geq 2$. Consider the continuous breaking process in which we start from R sticks of length L, break each stick into k pieces, and let a new stick die with probability $1 - 1/k$. The process ends in finitely many levels with probability 1, and the collection of ending stick lengths almost surely converges to strong Benford behavior as $R \to \infty$.

Future Work: General Number of Parts?

Conjecture

Fix $k \geq 2$. If we break each active stick into k pieces and stop at $(k-1)n/k$ residue classes modulo n, where n is a multiple of k, then the result converges to strong Benford.

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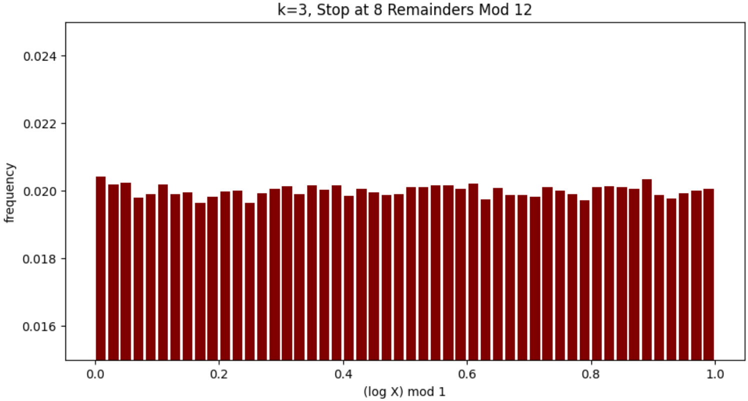

Thank you!

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(1) A. V. Kontorovich and S. J. Miller, *Acta Arithmetica*, 2005, **120**, 269–297.

(2) J.-C. Pain, *Physical Review E*, 2008, **77**, DOI: 10.1103/physreve.77.012102.

(3) T. Becker, D. Burt, T. C. Corcoran, A. Greaves-Tunnell, J. R. Iafrate, J. Jing, S. J. Miller, J. D. Porfilio, R. Ronan, J. Samranvedhya, F. W. Strauch and B. Talbut, *Ann. Physics*, 2018, **388**, 350–381.