BENFORD’S LAW, VALUES OF $L$-FUNCTIONS AND THE $3x+1$ PROBLEM

STEVEN J. MILLER

Abstract. Below are slides for talks at Boston College (10/19/04), the University of Michigan (11/15/04) and the University of Arizona (1/11/06). Many systems exhibit a digit bias. For example, the first digit (base 10) of the Fibonacci numbers or $2^n$ equals 1 about 30% of the time. This phenomena was first noticed by observing which pages of log tables were most worn with age – it’s a good thing there were no calculators 100 years ago! We show that the first digit of values of $L$-functions near the critical line also exhibit this bias. A similar bias exists (in a certain sense) for the first digit of terms in the $3x+1$ problem, provided the base is not a power of two. For $L$-functions the main tool is the Log-Normal law; for $3x+1$ it is the rate of equidistribution of $n \log_B 2 \mod 1$ and understanding the irrationality measure of $\log_B 2$. This work is joint with Alex Kontorovich.
Benford's Law Values of $L$-Frac $3x + 1$

(1) History

- Newcomb: 1881 \( \text{Prob}(d) = \log_b \left( \frac{d+1}{d} \right) \)
- Benford: 1938
- Lonyi Street, invariant under rescaling: define $y_n$

(2) Lucas + Benford

\[ x_n, \quad y_n = \log_b x_n \]
\[ y_n \equiv 0 \mod 1 \iff x_n \text{ Benford} \]

Example: \( x_n = \varphi^n \), \( \log_b \varphi \in \mathbb{Q} \)

- D. + \( \mathbb{Q} \) \( \varphi \) (Fibonacci)

Let each: \[ a_n = 2a_{n-1} - a_{n-2} \]
\[ a_0 = a_1 = 1 \]
\[ a_0 = 0 \quad a_1 = 1 \]
(3) Poisson Sum + Benford

P. Kibria (1961), Feller: errors

Motivation: \( x > 0 \): \( M_k(x) \), extend to \( x \in \mathbb{C} \)

Sets: \( \bar{P}(A) = \lim_{t \to 0} \frac{\# \{ a \in A \cap \mathbb{N} t > \}}{t} = \lim_{t \to 0} \frac{\# \{ a \in \mathbb{N} t > \leq \text{a} \}}{t} \)

Setup: \( X_T \circ \hat{Y}_T, B = \log_B \hat{X}_T \rightarrow \hat{Y}_B \)

Say \( \mathcal{F} \) nice density \( f \) st \( \hat{Y}_{T,B} \) is sparse \( f \) plus error.

CDF: \( \hat{P}_{T,B}(x) = \int_{-\infty}^{x} \frac{1}{t} f \left( \frac{t}{T} \right) dt + E_T(x) \)

Assume \( \mathcal{F} \) normal inc \( b(T) \to \infty \) st \( \mathcal{P} = \mathcal{E} \)

Corollary: \( F_T(-\infty) = F_T(\hat{h}(T)) = 0(1) \)

\( F_T(-\hat{h}(T)) = F_T(-\infty) = o(1) \)

\( \frac{1}{T} \sum_{k \geq T(h)} f \left( \frac{x + k}{T} \right) dx = o(1) \)

\( \sum_{k \geq 0} \left| \frac{f(Tk)}{k} \right| = o(1) \)

\( \sum_{k \geq 0} \left[ E_T(b+k) - E_T(b+k) \right] = o(1) \)
**THM (M-1, ?)**

Conds 1-4 imply Benford (if \( f \) is \( \epsilon \)-intermediate prob dst)

\[
P_T(a, b) = \mathbb{P}(\bar{T}_b \text{ mod } 1 \in [a, b])
\]

\[
= \sum_{r} \mathbb{P}(a + r \leq \bar{T}_b \leq b + r)
\]

\[
\sim \text{Poisson Sum}
\]

\[
= f(0)(b-a) + \sum_{k=0}^{\infty} f(\pi k) \frac{e^{2\pi i \beta} - e^{-2\pi i \beta}}{2\pi i \beta} + o(1)
\]

\( b + f(0) \sim 1 \) as prob dst

**Corr:** Geo. Brownian Motions are Benford

**General Idea**

- Structure Thm of sorts

  In many cases spread out so something nice

  apply Poisson Sum

- Control of errors
VALUES OF $L$-FUNCTIONS

Unconditionally, $\zeta(s)$, Dirichlet, holomorphic, class $L^2$, orthogonality (Density Conjecture replaces GRH) \( N(T, \delta) = O \left( T^{-1/2} \log^{1+\delta} T \right) \).

**Structure**

Selberg's $LG$-Normal Law

\[
M(t) \leq \frac{1}{\sqrt{2\pi t^3}} \int_{-\infty}^{\infty} \frac{e^{-\frac{s^2}{2}}}{{s}^2} ds + O \left( \frac{\sqrt{\pi}}{T} \right)
\]

Error term due to lack of pointwise summation

Neal Hejhal's refinement

**Ingredients of proof**

1. Approx $\log L(s+it)$ with $\sum_{n=x}^{\infty} \frac{\chi(n) \Lambda(n)}{n} e^{-it} n^{-s}$

2. Look at moments $\int_{-T}^{T} |L(\frac{1}{2}+it)|^{2k} dt$, $k \leq \log \frac{1}{\log T}$

3. Mat.-Weiner-Haar:

\[
\int_{-T}^{T} (\sum a_n e^{-it}) (\sum b_n e^{-it}) dt = H \sum a_n b_n + O(1) \sum \Lambda^2 n \sqrt{n} \sum \Lambda n \Lambda^2
\]

4. Work @ test frs: Check for $\chi_{a,b}$

Do for $\sigma = \frac{1}{2} + \frac{1}{\log^3 T}$

$\cdots$
3X+1 and Benford

- Kahutani: Conspiracy
- Erdos: not ready

\[ X \text{ odd: } T(X) = \frac{3X+1}{2^k} \quad \text{for } k \geq 1 \]

Conj: eventually 1

\[ 7 \rightarrow 11 \rightarrow 17 \rightarrow 13 \rightarrow 7 \rightarrow 1 \]

Structure Thm \((S, K-S)\)

Given \(p\) ints \((k_1, \ldots, k_m)\): two arithmetic progressions of form \(X, X+6, \ldots, X+k_m\)

full (shift initially)

\[ \Rightarrow \text{ get natural density } \]

\[ P(A) = \lim_{N \to \infty} \frac{\# \{ n \in N, n \leq N \mid A \}}{N} \]

\[ \Rightarrow \text{ Geo Brownian Motion in a sense } \quad P(n) = \left( \frac{1}{2} \right)^n, \alpha = 1.33 \ldots \]

- \(k_j\) are iid rv \(\exp\) dist \(\exp\) param \(\frac{1}{2}\)

\[ P \left( \frac{X_m}{\log_2(3) X_0} \leq \alpha \right) = \prod_{j=1}^{m} P \left( S_m - 2m \leq n \right) \]

where \(S_m\) is sum \(\exp\) dist \(\exp\) dist \(\frac{1}{2}\)
THEM (K-M)

As \( m \to \infty \), \( \frac{x_m}{(\frac{3}{4})^m x_0} \) is Benford.

\textbf{Proof:} Lattice bad errors (Abelian approx)

\textbf{Proof:} CLT: \( S_m - 2m \to N(0, \ln m) \)

1. \( \text{Prob} \left( \frac{S_m}{\sqrt{m}} = \frac{k}{\sqrt{m}} \right) = \sqrt{\frac{\pi}{2m}} + O \left( \frac{1}{\sqrt{m}} \right) \), \( k \) is std normal

\( \Rightarrow \) \( O \left( \frac{1}{\sqrt{g(m)\sqrt{m}}} \right) \)

2. \( I_1 = \{ 1, 1+1, \ldots, (l+1)M-1 \} \)

\( M = m^c, \ c < \frac{1}{2} \)

- \( k, k_2 \in I_1 \Rightarrow \left\lfloor \frac{1}{\sqrt{m}} \ln \left( \frac{k_2}{k_1} \right) - \frac{1}{2} \ln \left( \frac{k_2}{k_1} \right) \right\rfloor \) is measurable

\( \Rightarrow \) allows us to use just left endpoints

- assume \( C: \log_2^2 \) is irrational of type \( k < 1 \)

\( \nu(\mathbb{Z}) + \{ k \in I_1 : kC \text{ mod } 1 \in [0, c] \} = M(l-1) + O(M + \frac{c}{2}) \)

(quotient euclidean + irrationality measure)

3. \( \text{Prob}S_m: \ \frac{1}{6} \sum \frac{1}{n} e^{-\pi n^2 \sqrt{m}} = \sum \frac{1}{n} e^{-\pi n^2 / m} \)

\( Y_m = \log_2 \frac{x_m}{(\frac{3}{4})^m x_0} \) : mult by \( \frac{1}{\log_2 \sqrt{2}} = \frac{1}{\log_2 2} \)

Study \( S_m \cdot \log_2 \mod 1 \in [0,1] \)
\[ P_m(a, b) = \sum_{k \in \mathbb{Z}} \text{Prob} \left( \overline{s_m} = k \in I_k : k \in \mathbb{Z} \text{ and } 1 \in G(s) \right) \\
+ \sum \text{case } l \]

**Rate of Error**

Given seq \( x_1, x_2, \ldots \)

\[ D_N = \frac{1}{N} \sup_{|N| = k} \left| N(k - m) - \sum_{n \in k \in \mathbb{R}} x_n \right| \]

Erdos-Turan: \( \exists \ C > 0 \) m

\[ D_N \leq C \cdot \left( \frac{1}{m} + \frac{\sum_{k=1}^m}{h} \right) \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h x_n} \right| \]

Say \( X_n = 1 \) not mod 1

Exp sum is \( \leq \frac{1}{|\sin \theta|} \leq \frac{1}{2|\sin \theta|} \) distance to nearest 1/2

Must control

\[ \sum_{h=1}^m \frac{1}{h \langle h \theta \rangle} \]

Now see why can’t be too close to \( m^{1/2} \)

Say of type \( k \) if \( \sum \langle k \theta \rangle \sum 2 < k \theta \rangle = 0 \)

\( \Rightarrow \) Rel: alg \#s of type \( 1 \): \( |k - \frac{n}{2}| > \frac{\pi}{\theta} \)

\( \Rightarrow \) Gives \( \sum_{h=1}^m \frac{1}{h \langle h \theta \rangle} = O(m^{k-1/2}) \), take \( m = \lfloor N^{1/k} \rfloor \)
\[ \left| \log_{10} 2 - \frac{p}{2} \right| = \left| \frac{\log 2}{\log 10} - \frac{p}{2} \right| = \left| \frac{\log 2 - p \log 10}{2 \cdot \log 10} \right| \]

Enough to show \( | \log 2 - p \log 10 | > \frac{1}{2^n} \)

(Literature always works with log not \( \log \) -
- choose get \( \log 2 \) \& integer powers)

THM (Baker)

\[ \lambda_1, \ldots, \lambda_n \text{ alg roots of } -1 \text{ at most } A^2(x, y) \]

\[ \beta_1, \ldots, \beta_n \text{ rational ints } \text{ at most } B(x, y) \]

\[ \Lambda = \beta_1 \log x + \cdots + \beta_n \log x \]

If \( \Lambda \neq 0 \), \( | \Lambda | > \frac{1}{B \cdot C \cdot S \cdot \log S} \)

where \( C = (16 \pi d)^{2n-1} \)

\[ K = \Omega(x, \beta) \text{ of deg } d \]

\[ S = \log A_1 \cdots \log A_n \]

\[ S' = S / \log A_n \]

(Consider special for few poles center \( S, \ldots \))

For \( u, d = 1, n = 2, C = 2^{2000} \)

\[ \Omega = 6 \pi \cdot \log 10 \quad S' = \log y \]

\[ K = 1 + C \Omega \log 2' = 2^{2007}(\log 4 \cdot \log 10)(\log 10 \cdot \log 4) + 1 = 1,197052,10602 \]
E-mail address: sjmiller@math.brown.edu

Department of Mathematics, Brown University, Providence, RI 02912