Benford's Law, Values of L-Functions and the $3x+1$ Problem, or: Why the IRS should care about Number Theory

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Five Colleges Number Theory Seminar
January 31, 2012

Introduction

Interesting Question

For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of Five Colleges Number Theory Seminar participants, ..., what percent of the leading digits are 1?

Plausible answers:

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Plausible answers: 10%, 11%, about 30%.

Summary

- State Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.

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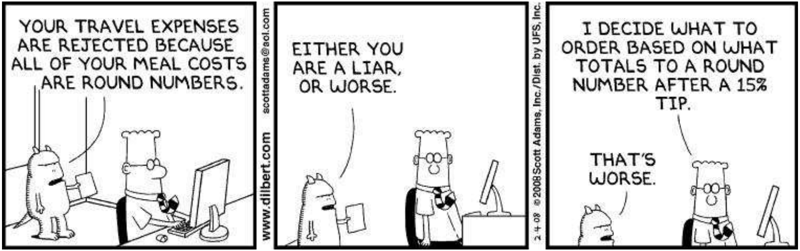

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.

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  - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
  - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ◇ Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x+1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

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Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

Mantissas (or Significands)

Mantissa: $x = M_{10}(x) \cdot 10^k$, k integer.

$M_{10}(x) = M_{10}(\widetilde{x})$ if and only if x and $\widetilde{x}$ have the same leading digits.

Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x})$ mod 1 if and only if x and $\widetilde{x}$ have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \to b - a.$$

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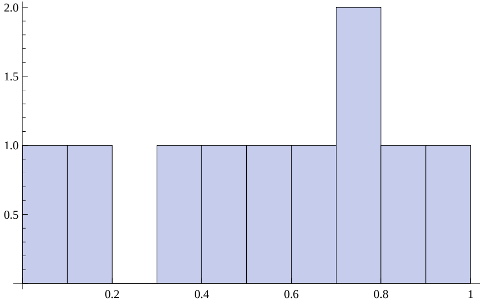
- Thm: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
- Examples: $\log_{10} 2, \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 10$

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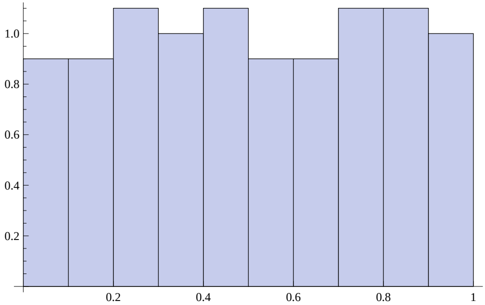

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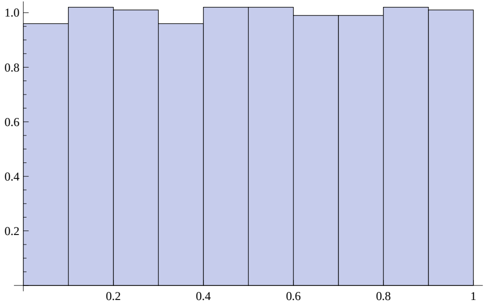

$n\sqrt{\pi}$ mod 1 for $n \leq 1000$

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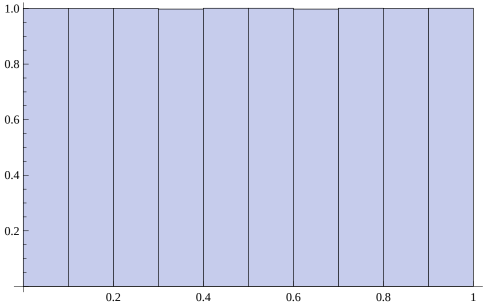

$n\sqrt{\pi}$ mod 1 for $n \leq 10,000$

Logarithms and Benford's Law

Fundamental Equivalence

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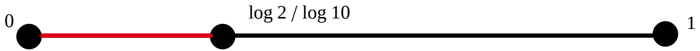

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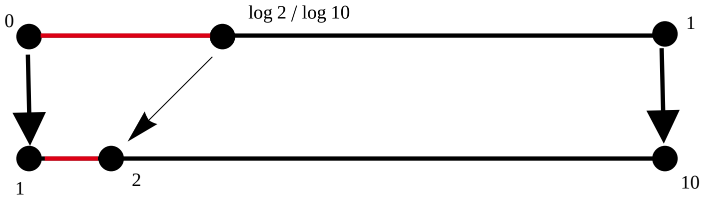

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  $a_{n+1} = a_n + a_{n-1}$.

  Binet: $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

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- Most linear recurrence relations Benford.
  $a_{n+1} = 2a_n$
  $a_{n+1} = 2a_n - a_{n-1}$
  Take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
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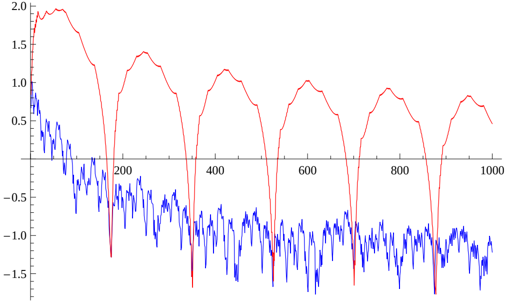
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

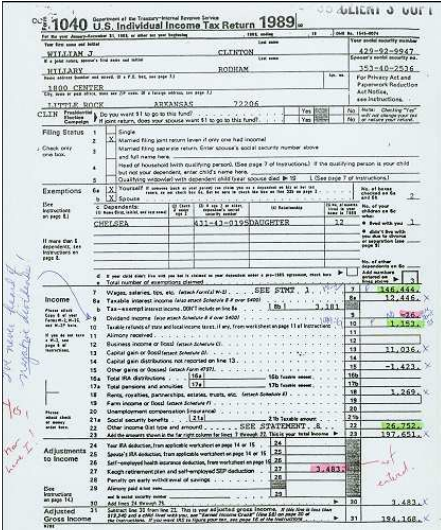
Logarithms and Benford's Law: Base 10

$\log_{10}(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5% and 8 d.f., $\log_{10}(\chi^2) \approx .44$).



Applications

Applications for the IRS: Detecting Fraud


CLIENT'S COPY

1040 Department of the Treasury—Internal Revenue Service
U.S. Individual Income Tax Return 1989

WILLIAM J — CLINTON — 429-92-9947
HILLARY — RODHAM — 353-40-2536
1800 CENTER
LITTLE ROCK — ARKANSAS — 72206

Filing Status
1 Single
2 X Married filing joint return (even if only one had income)
3 Married filing separate return. Enter spouse's social security number above and full name here.
4 Head of household (with qualifying person). (See page 7 of Instructions.) If the qualifying person is your child but not your dependent, enter child's name here.
5 Qualifying widow(er) with dependent child (year spouse died ▶ 19). (See page 7 of Instructions.)

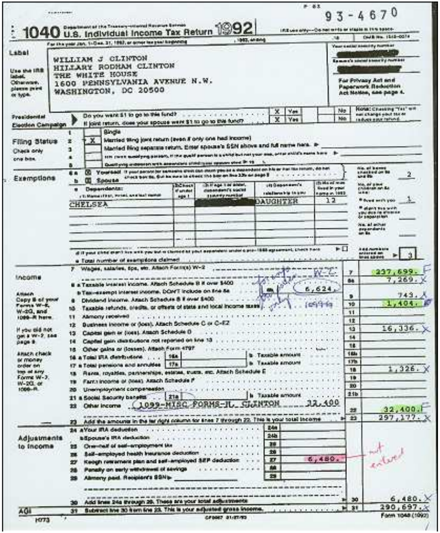
Exemptions
6a X Yourself
b X Spouse
c Dependents: CHELSEA — 431-43-0195 — DAUGHTER — 12
e Total number of exemptions claimed

Income
7 Wages, salaries, tips, etc. — SEE STMT 1 — 146,444.
8a Taxable interest income — 12,446.
b Tax-exempt interest income — 3,181
9 Dividend income — 326.
10 Taxable refunds of state and local income taxes — 1,153.
11 Alimony received
12 Business income or (loss)
13 Capital gain or (loss) — 11,036.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses) — -1,423.
16a Total IRA distributions
17a Total pensions and annuities
18 Rents, royalties, partnerships, estates, trusts, etc. — 1,269.
19 Farm income or (loss)
20 Unemployment compensation (insurance)
21a Social security benefits
22 Other income — SEE STATEMENT 8 — 26,752.
23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income — 197,651.

Adjustments to Income
24 Your IRA deduction
25 Spouse's IRA deduction
26 Self-employed health insurance deduction
27 Keogh retirement plan and self-employed SEP deduction — 3,483.
28 Penalty on early withdrawal of savings
29 Alimony paid
30 Add lines 24 through 29 — 3,483.

Adjusted Gross Income
31 Subtract line 30 from line 23. This is your adjusted gross income. — 194,168.

Applications for the IRS: Detecting Fraud


93-4670

1040 U.S. Individual Income Tax Return 1992

Label

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Your social security number [redacted]

Spouse's social security number [redacted]

For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Presidential Election Campaign — Do you want $1 to go to this fund? X Yes — If joint return, does your spouse want $1 to go to this fund? X Yes

Filing Status — 2 X Married filing joint return (even if only one had income)

Exemptions — 6a Yourself — b Spouse — c Dependents: CHELSEA [redacted] DAUGHTER 12

Income

7	Wages, salaries, tips, etc. Attach Form(s) W-2	237,699.
8a	Taxable interest income. Attach Schedule B if over $400	7,269.
9	Dividend income. Attach Schedule B if over $400	743.
10	Taxable refunds, credits, or offsets of state and local income taxes	1,404.
13	Capital gain or (loss). Attach Schedule D	16,336.
18	Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E	1,326.
22	Other income (1099-MISC FORMS-H. CLINTON 32,400)	32,400.
23	Add the amounts in the far right column for lines 7 through 22. This is your total income	297,177.

Adjustments to Income

27	Keogh retirement plan and self-employed SEP deduction	6,480.
30	Add lines 24a through 29. These are your total adjustments	6,480.
31	Subtract line 30 from line 23. This is your adjusted gross income	290,697.

Form 1040 (1992)

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers

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- Audit of a bank revealed huge spike of numbers starting with 4

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- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

Benford Good Processes

Poisson Summation and Benford's Law: Definitions

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- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\vec{Y}_{T,B}}(y) = \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) \ := \ G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

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- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b+\ell) - E_T(a+\ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

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- Examples
  - ◇ L-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x + 1$ problem
  - ◇ geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation

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  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right)$$

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$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right)$$

$$= \sum_{|\ell| \leq Th(T)} \left[G_T(b+\ell) - G_T(a+\ell)\right] + o(1)$$

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$$= \int_a^b \sum_{|\ell| \leq Th(T)} \frac{1}{T} f\left(\frac{t}{T}\right) dt + \mathcal{E}(a,b,T) + o(1)$$

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$$= \int_a^b \sum_{|\ell|\leq Th(T)} \frac{1}{T} f\left(\frac{t}{T}\right) dt + \mathcal{E}(a,b,T) + o(1)$$

$$= \widehat{f}(0)\cdot(b-a) + \sum_{\ell\neq 0} \widehat{f}(T\ell)\frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i\ell} + o(1).$$

L-functions and Random Matrix Theory

Good L-functions ($\zeta(s)$, full level cusp form)

L-function is good if:

- Euler product:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} \prod_{j=1}^{d} \left(1 - \alpha_{f,j}(p) p^{-s}\right)^{-1}.$$

- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely many poles (all on the line $\mathrm{Re}(s) = 1$).
- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ prod Γ-fns:

$$e^{\mathrm{i}\omega} G(s) L(s,f) = e^{-\mathrm{i}\omega} \overline{G(1-\overline{s}) L(1-\overline{s})}.$$

Good L-functions

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$:

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} = \aleph \log\log x + c + O\left(\frac{1}{\log x}\right).$$

- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- $N(\sigma, T) = \#\{\rho : L(\rho, f) = 0, \mathrm{Re}(\rho) \geq \sigma,$ $\mathrm{Im}(\rho) \in [0, T]\}$. $\exists \beta > 0$

$$N(\sigma, T) = O\left(T^{1-\beta\left(\sigma-\frac{1}{2}\right)} \log T\right).$$

Log-Normal Law (Hejhal, Laurinčikas, Selberg)

Log-Normal Law

$$\frac{\mu\left(\{t \in [T, 2T] : \log\left|L\left(\sigma + it, f\right)\right| \in [a, b]\}\right)}{T} =$$

$$\frac{1}{\sqrt{\psi(\sigma, T)}} \int_a^b e^{-\pi u^2/\psi(\sigma,T)} du + \mathrm{Error}$$

$$\psi(\sigma, T) = \aleph \log\left[\min\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right)\right] + O(1)$$

$$\frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log^{\delta} T}, \quad \delta \in (0, 1).$$

Values of L-functions and Benford's Law

Theorem (Kontorovich and M–, 2005)

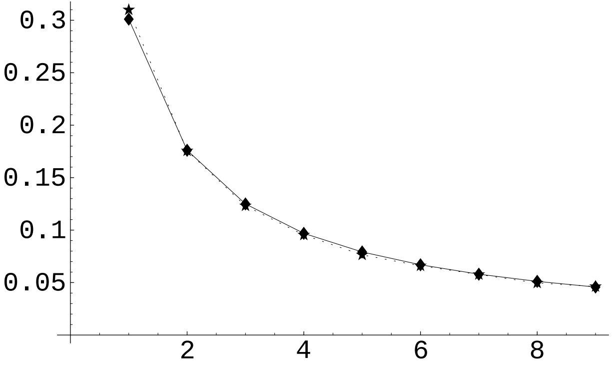
$L(s, f)$ a good L-function, as $T \to \infty$, $L(\sigma_T + it, f)$ is Benford.

Ingredients

- Approximate $\log L(\sigma_T + it, f)$ with $\sum_{n \le x} \frac{c(n)\Lambda(n)}{\log n} \frac{1}{n^{\sigma_T + it}}$.
- study moments $\int_T^{2T} |\cdot|$, $k \le \log^{1-\delta} T$.
- Montgomery-Vaughan: $\int_T^{2T} \sum a_n n^{-it} \overline{\sum b_m m^{-it}} dt = H \sum a_n \overline{b}_n + O(1)\sqrt{\sum n|a_n|^2 \sum n|b_n|^2}$.

Riemann Zeta Function

$$\left|\zeta\left(\frac{1}{2}+i\frac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$



Random Matrices: Preliminaries

- $N \times N$ unitary matrices U (Haar measure):

$$p_N(U) = \frac{1}{(2\pi)^N N!} \prod_{1 \le j < m \le N} \left| e^{i\theta_j} - e^{i\theta_m} \right|.$$

- characteristic polynomial:

$$Z(U,\theta) = \det(I - Ue^{-i\theta}) = \prod \left(1 - e^{i(\theta_n - \theta)}\right).$$

- $\rho_N(x)$ the probability density for $\log|Z(U,\theta)|$:

$$\widetilde{\rho}_N(x) \; = \; \sqrt{Q_2(N)}\, \rho_N(\sqrt{Q_2(N)}\, x),$$

variance $Q_2(N) \sim (\log N)/2$.

Random Matrices and Benford's Law

Theorem (Kontorovich and M–, 2005)

As $N \to \infty$, the distribution of digits of the absolute values of the characteristic polynomials of $N \times N$ unitary matrices (with respect to Haar measure) converges to the Benford probabilities.

- Key Ingredient: Keating-Snaith:

$$\widetilde{\rho}_N(x)dx \;=\; \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx + O\left((\log N)^{-3/2}dx\right).$$

The $3x + 1$ Problem and Benford's Law

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- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x+1$.

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

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\begin{aligned}
a_{n+1} &= T(a_n) \\
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Geometric Brownian Motion, drift $\log(3/4) < 1$.

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n \le N : n \equiv 1,5 \bmod 6, n \in A\}}{\#\{n \le N : n \equiv 1,5 \bmod 6\}}.$$

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$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

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$C = \log_B 2$ of irrationality type $\kappa < \infty$:
$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa})$.

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$\left|\log_{10} 2 - p/q\right| = \left|q \log 2 - p \log 10\right| / q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} \left|N(b-a) - \#\{n \le N : x_n \in [a,b]\}\right|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\le \frac{1}{|\sin(\pi h\alpha)|} \le \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x + 1$ Data: random 10,000 digit number, $2^k || 3x + 1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

Conclusions

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.