

Why the IRS cares about the Riemann Zeta Function and Number Theory (and why you should too!)

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[http://web.williams.edu/Mathematics/
sjmiller/public_html/](http://web.williams.edu/Mathematics/sjmiller/public_html/)

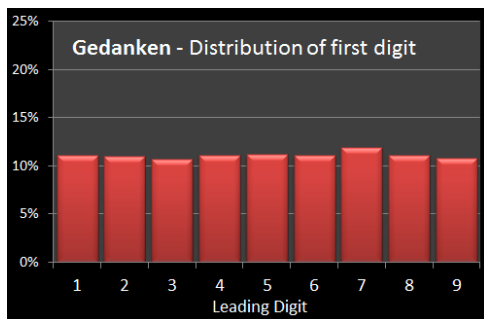
The College of the Holy Cross, April 20, 2016

Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?

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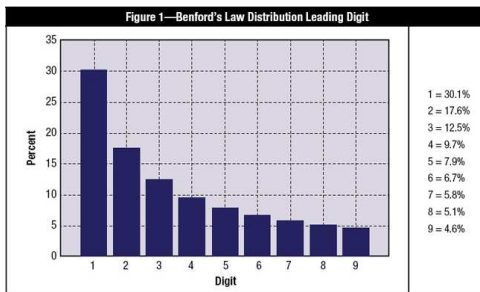
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Natural guess: 10% (but immediately correct to 11%!).

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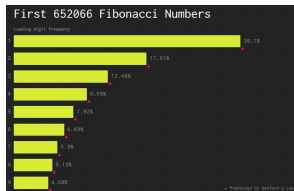
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Answer: Benford's law!

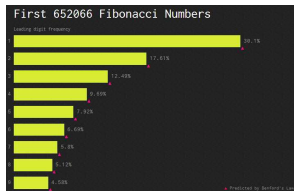
Examples with First Digit Bias

Fibonacci numbers

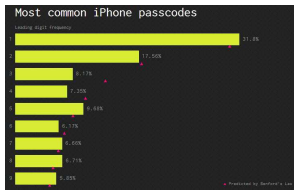


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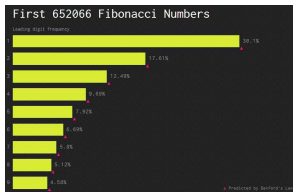


Most common iPhone passcodes

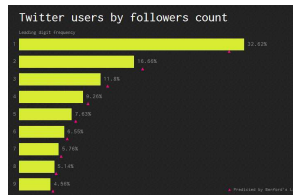


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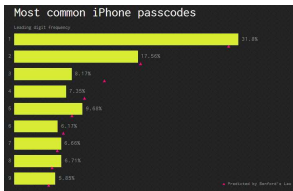
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Twitter users by # followers

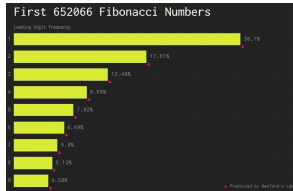


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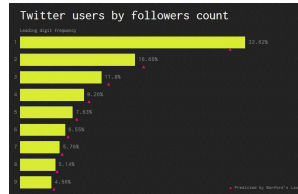


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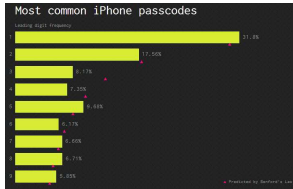
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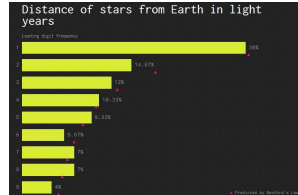
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Distance of stars from Earth



Summary

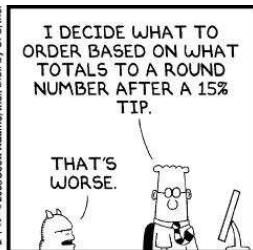
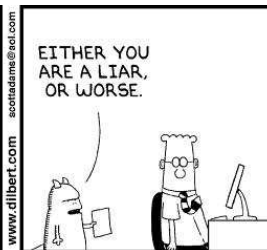
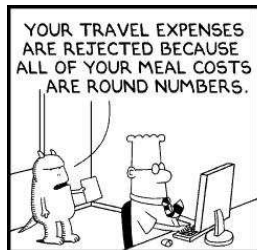
- Explain Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

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Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L -functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

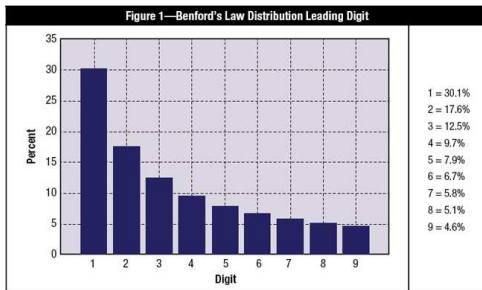
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

General Theory

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B \left(\frac{d+1}{d} \right)$; base 10 about 30% are 1s.



Benford's Law (probabilities)

Background Material

- Modulo: $a = b \bmod c$ if $a - b$ is an integer times c ; thus $17 = 5 \bmod 12$, and $4.5 = .5 \bmod 1$.

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- Key observation:** $\log_{10}(x) = \log_{10}(\tilde{x}) \bmod 1$ if and only if x and $\tilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \bmod 1$.

Advanced: $e^{2\pi i u} = e^{2\pi i (u \bmod 1)}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

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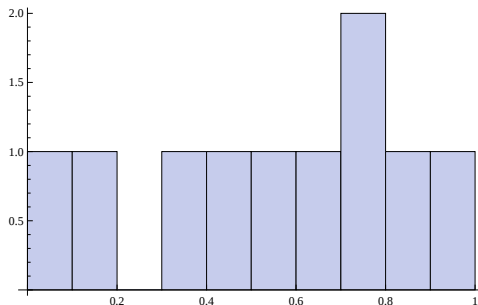
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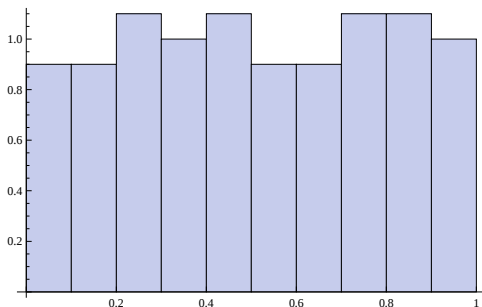
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 Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



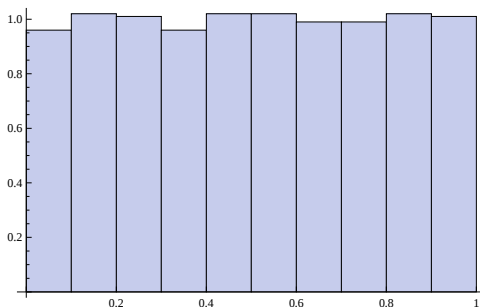
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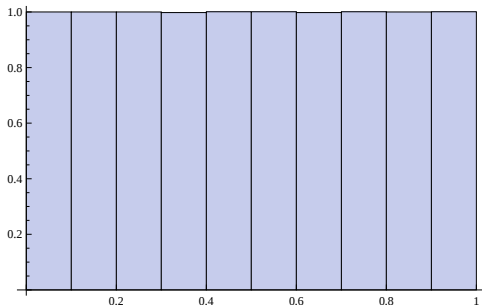
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$n\sqrt{\pi} \bmod 1$ for $n \leq 1000$

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10,000$

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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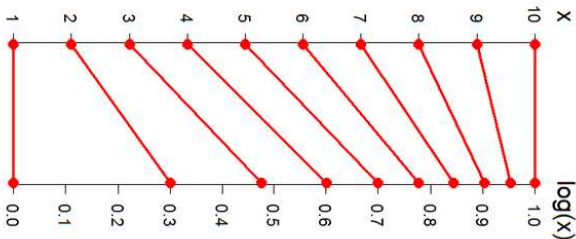
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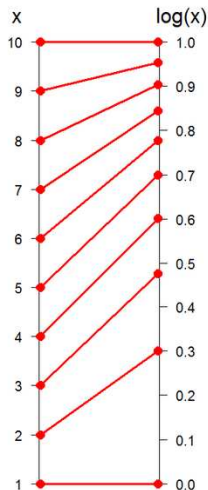
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Logarithms and Benford's Law

$$\begin{aligned}
 &\text{Prob}(\text{leading digit } d) \\
 &= \log_{10}(d+1) - \log_{10}(d) \\
 &= \log_{10}\left(\frac{d+1}{d}\right) \\
 &= \log_{10}\left(1 + \frac{1}{d}\right).
 \end{aligned}$$

Have Benford's law $\leftrightarrow$
 mantissa of logarithms
 of data are uniformly
 distributed



Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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$$\diamond a_{n+1} = 2a_n - a_{n-1}$$

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- **Most linear recurrence relations Benford:**

$$\diamond a_{n+1} = 2a_n - a_{n-1}$$

$\diamond$ take $a_0 = a_1 = 1$ or $a_0 = 0, a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576				
2	2048	2097152	1	18	.300	.301
4	4096	4194304	2	12	.200	.176
8	8192	8388608	3	6	.100	.125
16	16384	16777216	4	6	.100	.097
32	32768	33554432	5	6	.100	.079
64	65536	67108864	6	4	.067	.067
128	131072	134217728	7	2	.033	.058
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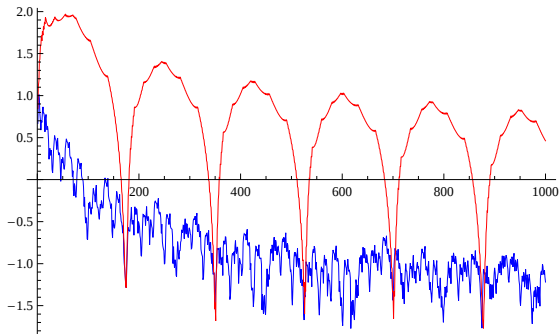
Logarithms and Benford's Law

χ^2 values for α^n , $1 \leq n \leq N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

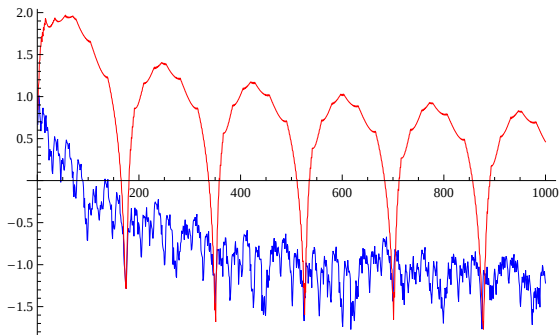
Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue),
 $n \in \{1, \dots, N\}$.



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$\log(\chi^2)$ vs N for π^n (red) and e^n (blue),
 $n \in \{1, \dots, N\}$. **Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$.**



Why Benford's Law?

Streets

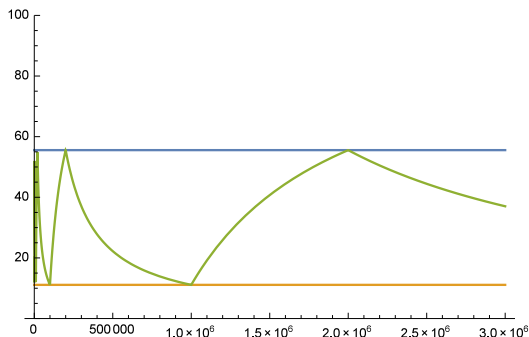
Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
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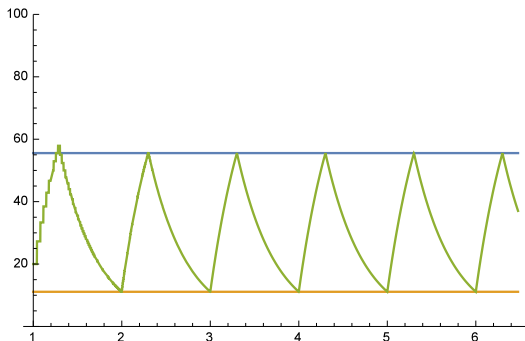


Probability first digit 1 versus street length L .

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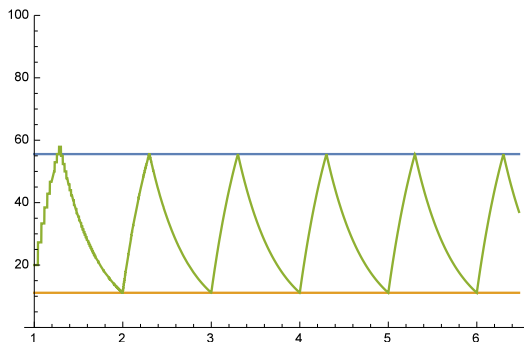


Probability first digit 1 versus $\log(\text{street length } L)$.

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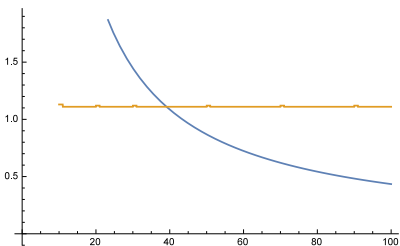
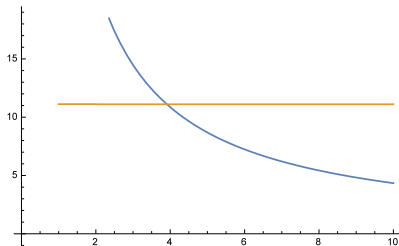


Probability first digit 1 versus $\log(\text{street length } L)$.

What if we have many streets of different lengths?

Amalgamating Streets

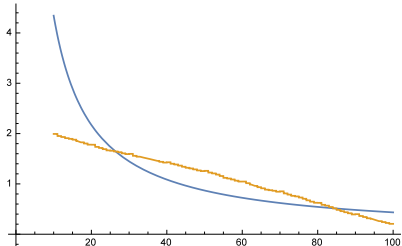
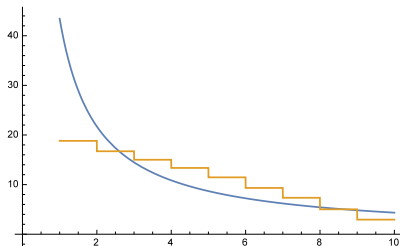
All houses: 1000 Streets,
each from 1 to 10000.



First digit and first two digits vs Benford.

Amalgamating Streets

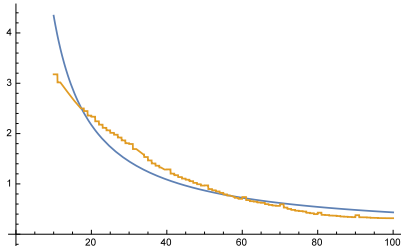
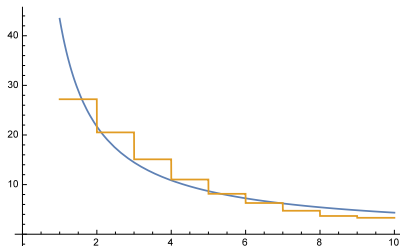
All houses: 1000 Streets,
each from 1 to rand(10000).



First digit and first two digits vs Benford.

Amalgamating Streets

All houses: 1000 Streets,
each 1 to $\text{rand}(\text{rand}(10000))$.

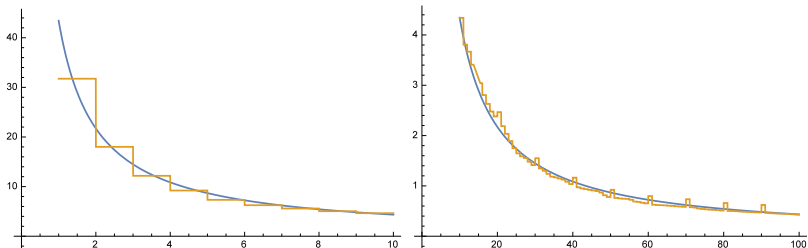


First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

Amalgamating Streets

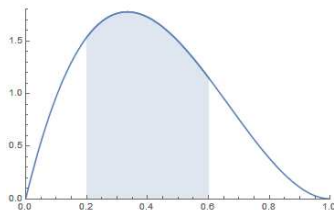
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First digit and first two digits vs Benford.

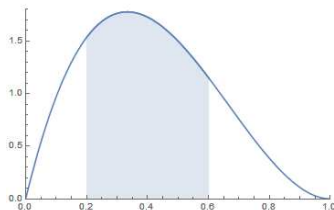
Conclusion: More processes, closer to Benford.

Probability Review



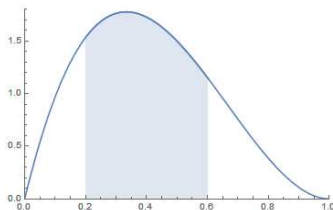
- Let X be random variable with density $p(x)$:
 - ◇ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)dx = 1$;
 - ◇ $\text{Prob}(a \leq X \leq b) = \int_a^b p(x)dx$.

Probability Review



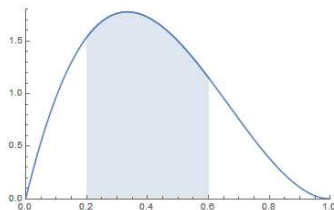
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- Mean $\mu = \int_{-\infty}^{\infty} xp(x)dx$.

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- Mean $\mu = \int_{-\infty}^{\infty} xp(x)dx$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx$.

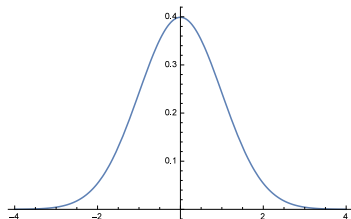
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- Mean $\mu = \int_{-\infty}^{\infty} xp(x)dx$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx$.
- Independence: knowledge of one random variable gives no knowledge of the other.

Central Limit Theorem

Normal $N(\mu, \sigma^2) : p(x) = e^{-(x-\mu)^2/2\sigma^2} / \sqrt{2\pi\sigma^2}$.



Theorem

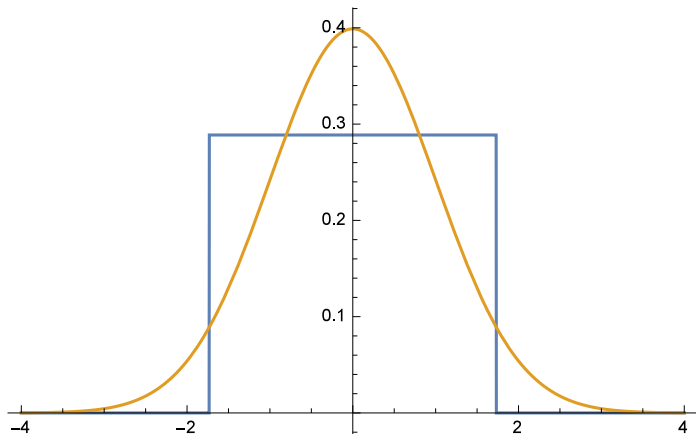
If $X_1, X_2, \dots$ independent, identically distributed random variables (mean μ , variance σ^2 , finite moments) then

$$S_N := \frac{X_1 + \dots + X_N - N\mu}{\sigma\sqrt{N}} \text{ converges to } N(0, 1).$$

Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \text{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

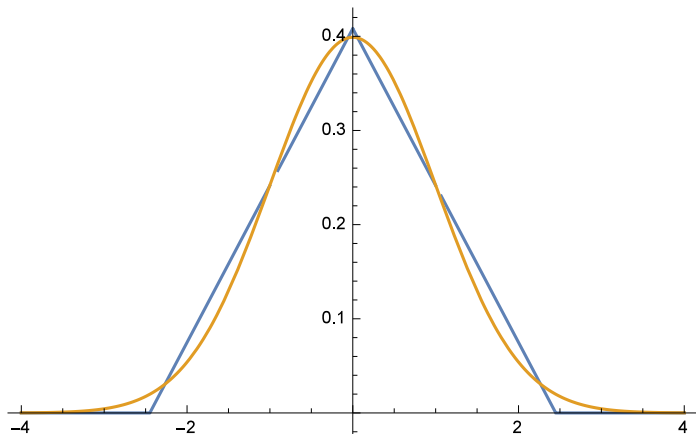
$Y_1 = X_1/\sigma_{X_1}$ vs $N(0, 1)$.



Central Limit Theorem: Sums of Uniform Random Variables

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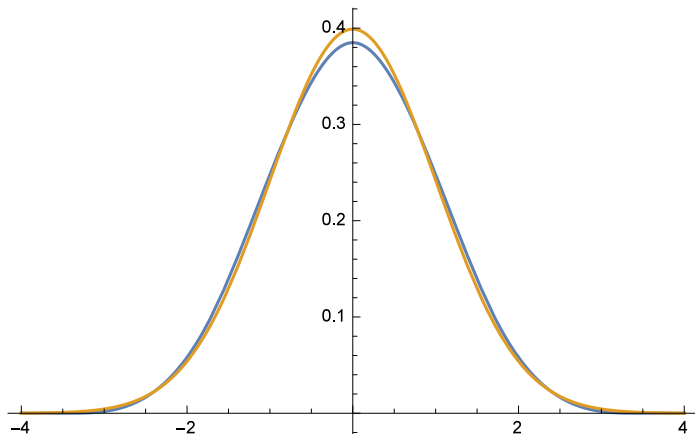
$$Y_2 = (X_1 + X_2)/\sigma_{X_1+X_2} \text{ vs } N(0, 1).$$



Central Limit Theorem: Sums of Uniform Random Variables

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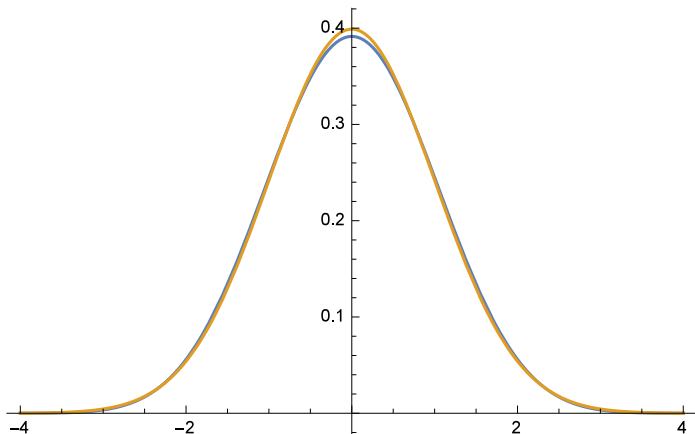
$$Y_4 = (X_1 + X_2 + X_3 + X_4) / \sigma_{X_1 + X_2 + X_3 + X_4} \text{ vs } N(0, 1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \text{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_8 = (X_1 + \cdots + X_8) / \sigma_{X_1 + \cdots + X_8} \text{ vs } N(0, 1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \text{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

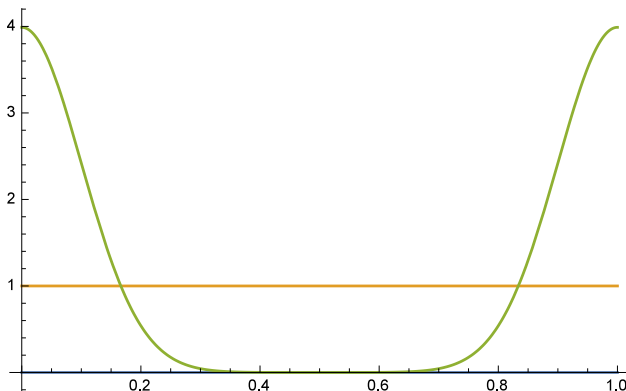
$$\left\{ \begin{array}{ll} \frac{1}{27} (18 + 9\sqrt{3}y - \sqrt{3}y^3) & y = 0 \\ \frac{1}{18} (12 - 6y^2 - \sqrt{3}y^3) & -\sqrt{3} < y < 0 \\ \frac{1}{54} (72 - 36\sqrt{3}y + 18y^2 - \sqrt{3}y^3) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54} (18\sqrt{3}y - 18y^2 + \sqrt{3}y^3) & y = \sqrt{3} \\ \frac{1}{18} (12 - 6y^2 + \sqrt{3}y^3) & 0 < y < \sqrt{3} \\ \frac{1}{54} (72 + 36\sqrt{3}y + 18y^2 + \sqrt{3}y^3) & -2\sqrt{3} < y \leq -\sqrt{3} \\ 0 & \text{True} \end{array} \right.$$

$\sqrt{3}$

(Don't even think of asking to see Y_8 's!)

Normal Distributions Mod 1

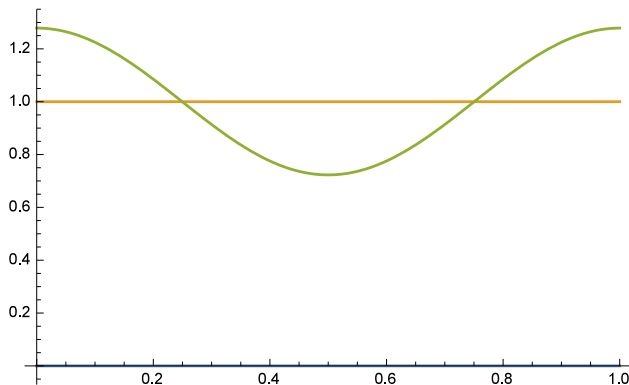
As $\sigma \rightarrow \infty$, $N(0, \sigma^2) \bmod 1 \rightarrow \text{Unif}(0, 1)$.



Variance is .01.

Normal Distributions Mod 1

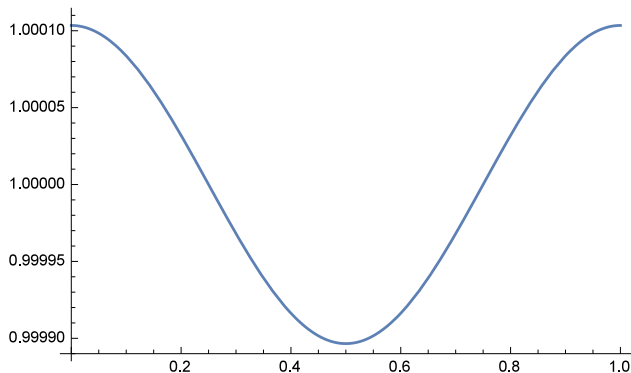
As $\sigma \rightarrow \infty$, $N(0, \sigma^2) \bmod 1 \rightarrow \text{Unif}(0, 1)$.



Variance is .1.

Normal Distributions Mod 1

As $\sigma \rightarrow \infty$, $N(0, \sigma^2) \bmod 1 \rightarrow \text{Unif}(0, 1)$.



Variance is .5.

Products and Benford's Law

Pavlovian Response: See a product, take a logarithm.

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$$\begin{aligned}
 V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\
 &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N
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Products and Benford's Law

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$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \\ &= Y_1 + Y_2 + \cdots + Y_N. \end{aligned}$$

Need distribution of $V_N \bmod 1$, which by CLT becomes uniform,
implying Benfordness!

Applications

Applications for the IRS: Detecting Fraud



A Tale of Two Steve Millers....

Applications for the IRS: Detecting Fraud

93-4670

1040 Department of the Treasury—Internal Revenue Service
U.S. Individual Income Tax Return 1992
 For the year 1992, 1-1-92 to 12-31-92, or other tax year beginning 1992, ending

Label **WILLIAM J. CLINTON**
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Use the IRS label. Otherwise, place name in type.

For Privacy Act and Paperwork Reduction Act Notice, see page 6.

Do you want \$1 to go to this fund? ☒ Yes ☐ No
 If you return, does your spouse want \$1 to go to the fund? ☒ Yes ☐ No

Filing Status
 1 ☒ Single
 2 ☐ Married filing separate returns (even if only one had income)
 3 ☐ Married filing separate returns. Do not attach a SSN above and all other forms. If you have a separate return, attach a separate return for each spouse.
 4 ☐ Head of household (see instructions on page 1)

Exemptions
 5 ☒ Yourself. If you are not a dependent on another person's return, check this box. If you are a dependent on another person's return, check this box on their return.
 6 ☐ Other. If you are a dependent on another person's return, check this box on their return. If you are a dependent on another person's return, check this box on their return.

Income
 7 ☐ Wages, salaries, tips, etc. Attach Form W-2.
 8 ☐ Taxable interest income. Attach Schedule B if over \$400.
 9 ☐ Tax-exempt interest income. Do not include on this line.
 10 ☐ Dividend income. Attach Schedule B if over \$400.
 11 ☐ Taxable refunds, credits, or offsets of state and local income taxes.
 12 ☐ Alimony received.
 13 ☐ Business income or loss. Attach Schedule C or C-EZ.
 14 ☐ Capital gain or loss. Attach Schedule D.
 15 ☐ Other gains or losses. Attach Form 4797.
 16 ☐ Total IRA distributions.
 17 ☐ Total pensions and annuities.
 18 ☐ Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E.
 19 ☐ Farm income or loss. Attach Schedule F.
 20 ☐ Unemployment compensation.
 21 ☐ Social Security benefits.
 22 ☐ Other income. **1099-MISC FORMS-IL. ECHENSON**
 23 ☐ Add the amounts in the far right column for lines 7 through 22. This is your total income.

Adjustments to income
 24 ☐ Your IRA deduction.
 25 ☐ Spouse's IRA deduction.
 26 ☐ Overhead of self-employment tax.
 27 ☐ Self-employed health insurance deduction.
 28 ☐ Keogh retirement plan and self-employed SEP deduction.
 29 ☐ Penalty on early withdrawal of savings.
 30 ☐ Alimony paid. Recipient's SSN.

AGI
 31 ☐ Subtract line 30 from line 23. This is your adjusted gross income.

Totals
 7 **237,659**
 8 **7,269**
 9 **743**
 10 **1,401**
 11 **16,336**
 12 **3,328**
 13 **22,400**
 14 **297,172**
 15 **6,480**
 16 **290,692**
 17 **290,692**

1073

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 4

Detecting Fraud

Bank Fraud

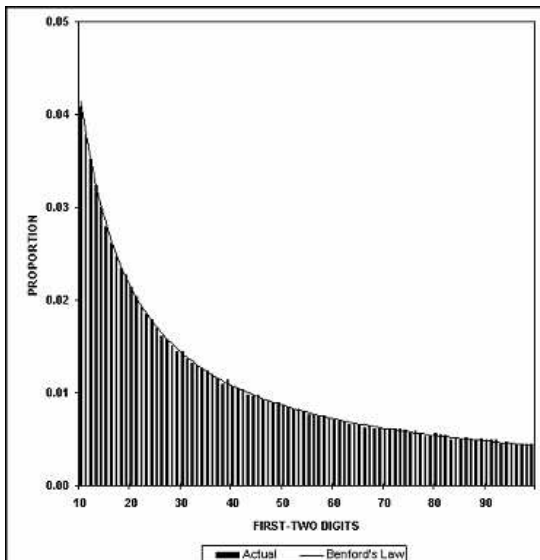
- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Election Fraud: Iran 2009

Numerous questions over Iran's 2009 elections.

Lot of analysis; data moderately suspicious:

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: enough tests, even if nothing wrong will find a suspicious result (but when all tests are on the boundary...).

Stick Decomposition

Fixed Proportion Decomposition Process

Decomposition Process

- 1 Consider a stick of length $\mathcal{L}$.

Fixed Proportion Decomposition Process

Decomposition Process

- 1 Consider a stick of length $\mathcal{L}$.
- 2 Uniformly choose a proportion $p \in (0, 1)$.

Fixed Proportion Decomposition Process

Decomposition Process

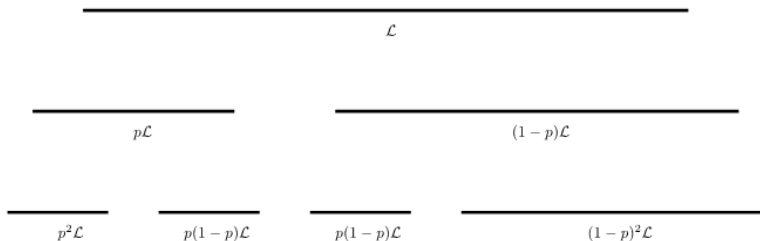
- 1 Consider a stick of length $\mathcal{L}$.
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- 3 Break the stick into two pieces—lengths $p\mathcal{L}$ and $(1 - p)\mathcal{L}$.

Fixed Proportion Decomposition Process

Decomposition Process

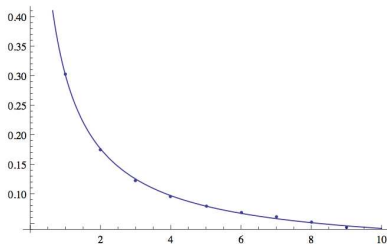
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- 2 Uniformly choose a proportion $p \in (0, 1)$.
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- 4 Repeat N times (using the same proportion).

Fixed Proportion Decomposition Process

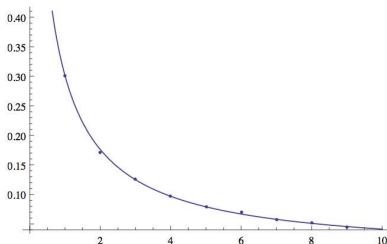


Fixed Proportion Conjecture (Joy Jing '13)

Conjecture: The above decomposition process is Benford as $N \rightarrow \infty$ for any $p \in (0, 1)$, $p \neq \frac{1}{2}$.



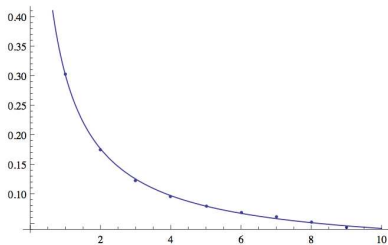
(B) $p = 0.51$ and $N = 10000$.



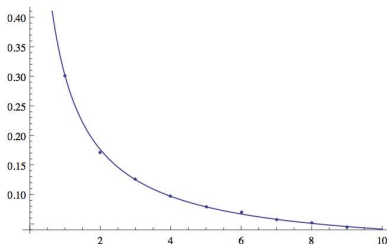
(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

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(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

Counterexample (SMALL '13): $p = \frac{1}{11}$, $1 - p = \frac{10}{11}$.

Benford Analysis

At N^{th} level,

- 2^N sticks
- $N + 1$ distinct lengths:

$p^N \left(\frac{1-p}{p} \right)^j$, $j \in \{0, \dots, N\}$, have $\binom{N}{j}$ times.

Benford Analysis

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(Weighted) Geometric with ratio $\frac{1-p}{p} = 10^y$;
behavior depends on irrationality of y !

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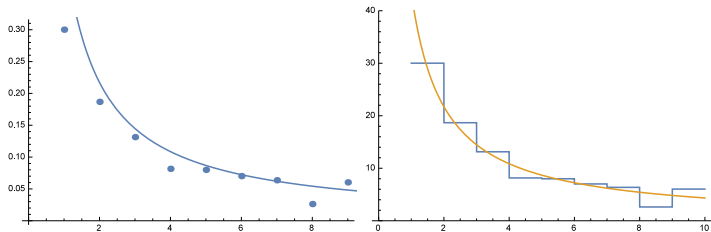
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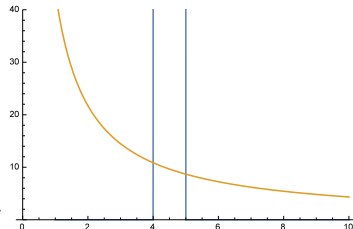
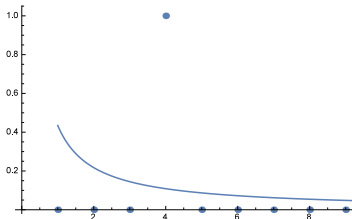
Theorem: Benford if and only if y irrational.

Examples



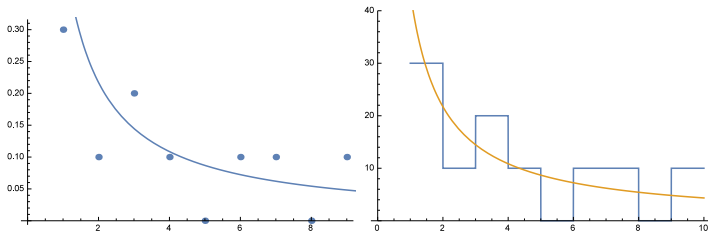
$p = 3/11$, 1000 levels; $y = \log_{10}(8/3) \notin \mathbb{Q}$
(irrational)

Examples



$p = 1/11$, 1000 levels; $y = 1 \in \mathbb{Q}$
(rational)

Examples



$$p = 1/(1 + 10^{33/10}), \text{ 1000 levels; } y = 33/10 \in \mathbb{Q} \\ \text{(rational)}$$

Conclusions

Current / Future Investigations

- Develop more sophisticated tests for fraud.
- Study digits of other systems.
 - ◇ Break rods of variable integer length, each piece breaks until is a prime, or a square,
 - ◇ Fragmentation models in higher dimensions.

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

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The Riemann Zeta Function $\zeta(s)$ and Benford's Law

Riemann Zeta Function (for real part of s greater than 1)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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Geometric Series Formula: $(1 - x)^{-1} = 1 + x + x^2 + \dots$

Unique Factorization: $n = p_1^{r_1} \dots p_m^{r_m}$.

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$$\begin{aligned} \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} &= \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \dots\right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \dots\right] \dots \\ &= \sum_n \frac{1}{n^s}. \end{aligned}$$

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$

$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \rightarrow 1^+} \zeta(s) = \infty, \pi(x) \rightarrow \infty.$

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$

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Properties of $\zeta(s)$ and Primes:

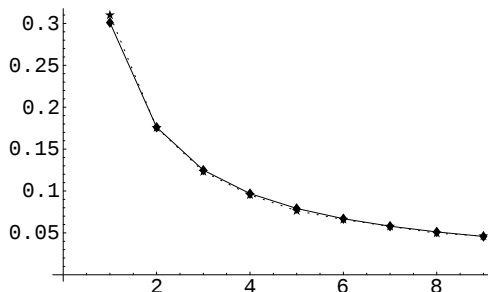
- $\lim_{s \rightarrow 1^+} \zeta(s) = \infty, \pi(x) \rightarrow \infty.$
- $\zeta(2) = \frac{\pi^2}{6}, \pi(x) \rightarrow \infty.$

The Riemann Zeta Function and Benford's Law

$$\left| \zeta \left(\frac{1}{2} + i \frac{k}{4} \right) \right|, k \in \{0, 1, \dots, 65535\}.$$

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First digits of $\left| \zeta \left(\frac{1}{2} + i \frac{k}{4} \right) \right|$ versus Benford's law.

Proof Sketch: 'Good' L -Functions

We say an L -function is *good* if:

- Euler product:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_p \prod_{j=1}^d (1 - \alpha_{f,j}(p)p^{-s})^{-1}.$$

- $L(s, f)$ has a meromorphic continuation to $\mathbb{C}$, is of finite order, and has at most finitely many poles (all on the line $\operatorname{Re}(s) = 1$).
- Functional equation:

$$e^{i\omega} G(s) L(s, f) = e^{-i\omega} \overline{G(1 - \bar{s}) L(1 - \bar{s})},$$

where $\omega \in \mathbb{R}$ and

$$G(s) = Q^s \prod_{i=1}^h \Gamma(\lambda_i s + \mu_i)$$

with $Q, \lambda_i > 0$ and $\operatorname{Re}(\mu_i) \geq 0$.

Proof Sketch: 'Good' L -Functions (cont)

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$ we have

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} = \aleph \log \log x + c + O\left(\frac{1}{\log x}\right).$$

- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- If $N(\sigma, T)$ is the number of zeros ρ of $L(s)$ with $\operatorname{Re}(\rho) \geq \sigma$ and $\operatorname{Im}(\rho) \in [0, T]$, then for some $\beta > 0$ we have

$$N(\sigma, T) = O\left(T^{1-\beta\left(\sigma-\frac{1}{2}\right)} \log T\right).$$

Known in some cases, such as $\zeta(s)$ and Hecke cuspidal forms of full level and even weight $k > 0$.

Log-Normal Law (Hejhal, Laurinćikas, Selberg)

Log-Normal Law

$$\begin{aligned}
 & \frac{\mu(\{t \in [T, 2T] : \log |L(\sigma + it, f)| \in [a, b]\})}{T} = \\
 & \frac{1}{\sqrt{\psi(\sigma, T)}} \int_a^b e^{-\pi u^2 / \psi(\sigma, T)} du + \text{Error} \\
 & \psi(\sigma, T) = \aleph \log \left[\min \left(\log T, \frac{1}{\sigma - \frac{1}{2}} \right) \right] + O(1) \\
 & \frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log^\delta T}, \quad \delta \in (0, 1).
 \end{aligned}$$

Result: Values of L -functions and Benford's Law

Theorem (Kontorovich and M–, 2005)

$L(s, f)$ a good L -function, as $T \rightarrow \infty$,
 $L(\sigma_T + it, f)$ is Benford.

Ingredients

- Approximate $\log L(\sigma_T + it, f)$ with $\sum_{n \leq x} \frac{c(n)\Lambda(n)}{\log n} \frac{1}{n^{\sigma_T + it}}$.
- study moments $\int_T^{2T} |\cdot|^k, k \leq \log^{1-\delta} T$.
- Montgomery-Vaughan: $\int_T^{2T} \sum a_n n^{-it} \overline{\sum b_m m^{-it}} dt = H \sum a_n \overline{b_n} + O(1) \sqrt{\sum n |a_n|^2 \sum n |b_n|^2}$.

Results: Explicit L -Function Statement

Theorem (Kontorovich-Miller '05)

Let $L(s, f)$ be a good L -function. Fix a $\delta \in (0, 1)$. For each T , let $\sigma_T = \frac{1}{2} + \frac{1}{\log^\delta T}$. Then as $T \rightarrow \infty$

$$\frac{\mu \{t \in [T, 2T] : M_B(|L(\sigma_T + it, f)|) \leq \tau\}}{T} \rightarrow \log_B \tau$$

Thus the values of the L -function satisfy Benford's Law in the limit for any base B .

The $3x + 1$ Problem and Benford's Law

$3x + 1$ Problem

- Kakutani (conspiracy), Erdős (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.

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- $7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \rightarrow_4 1 \rightarrow_2 1$,
 2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
 m -path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

$$\begin{aligned}
 a_{n+1} &= T(a_n) \\
 \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left(\frac{3a_n}{2^k} \right) \\
 &= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\
 &= \log a_n + \log \left(\frac{3}{4} \right).
 \end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \rightarrow \infty$, $x_m / (3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan, 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x + 1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N \rightarrow \infty} \frac{\#\{n \leq N : n \equiv 1, 5 \pmod{6}, n \in A\}}{\#\{n \leq N : n \equiv 1, 5 \pmod{6}\}}.$$

$(k_1, \dots, k_m)$: two full arithm progressions:

$$6 \cdot 2^{k_1 + \dots + k_m} p + q.$$

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Sketch of the proof of Benfordness

- Failed Proof: lattices, bad errors.

- CLT: $(S_m - 2m)/\sqrt{2m} \rightarrow N(0, 1)$:

$$\mathbb{P}(S_m - 2m = k) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:

$$I_\ell = \{\ell M, \dots, (\ell + 1)M - 1\}, M = m^c, c < 1/2$$

$$k_1, k_2 \in I_\ell: \left| \eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right) \right| \text{ small}$$

$$C = \log_B 2 \text{ of irrationality type } \kappa < \infty:$$

$$\#\{k \in I_\ell : \overline{kC} \in [a, b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\lim_{q \rightarrow \infty} q^{\gamma+1} \min_p \left| \alpha - \frac{p}{q} \right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$,
 $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C \Omega \log \Omega'}$, with
 $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$,
 $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| / q \log 10.$$

Quantified Equidistribution

Theorem (Erdős-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \leq N : x_n \in [a,b]\}|}{N}$$

There is a C such that for all m :

$$D_N \leq C \cdot \left(\frac{1}{m} + \sum_{h=1}^m \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h x_n} \right| \right)$$

Proof of Erdős-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^m \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ , $\sum_{h=1}^m \frac{1}{h||h\alpha||} = O(m^{\kappa-1+\epsilon})$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x + 1$ Data: random 10,000 digit number, $2^k || 3x + 1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319);
 $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x + 1$ Data: random 10,000 digit number, $2|3x + 1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x + 1$ Data: random 10,000 digit number, $2^k || 5x + 1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

5x + 1 Data: random 10,000 digit number, 2|5x + 1

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046