

Benford's Law: Why the IRS might care about the $3x + 1$ problem and $\zeta(s)$.

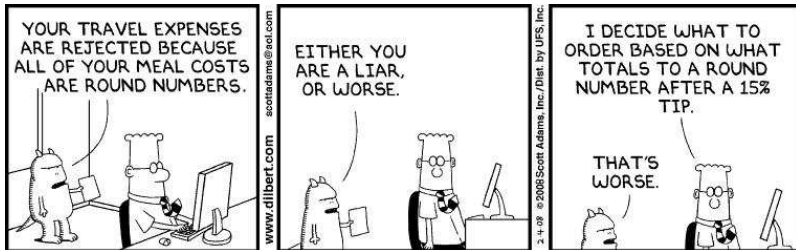
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Western New England College, February 11th, 2008

Caveats!

- Not all fraud can be detected by Benford's Law.
- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.



Notation

- **Logarithms:** $\log_B x = y$ means $x = B^y$.
 - ◇ Example: $\log_{10} 100 = 2$ as $100 = 10^2$.
 - ◇ $\log_B(uv) = \log_B u + \log_B v$.
 - ◇ $\log_{10}(100 \cdot 1000) = \log_{10}(100) + \log_{10}(1000)$.
- **Set Theory:**
 - ◇ $\mathbb{Q}$ = rational numbers = $\{p/q : p, q \text{ integers}\}$.
 - ◇ $x \in S$ means x belongs to S .
 - ◇ $[a, b] = \{x : a \leq x \leq b\}$.
- **Modulo 1:**
 - ◇ Any x can be written as integer + fraction.
 - ◇ $x \bmod 1$ means just the fractional part.
 - ◇ Example: $\pi \bmod 1$ is about .14159.

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B \left(\frac{d+1}{d} \right)$; base 10 about 30% are 1s.

- Not all data sets satisfy Benford's Law.
 - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
 - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
 - ◇ **Many streets of different sizes: close to Benford.**

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L -functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

Mantissas

Mantissa: $x = M_{10}(x) \cdot 10^k$, k integer.

$M_{10}(x) = M_{10}(\tilde{x})$ if and only if x and $\tilde{x}$ have the same leading digits.

Key observation: $\log_{10}(x) = \log_{10}(\tilde{x}) \bmod 1$ if and only if x and $\tilde{x}$ have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

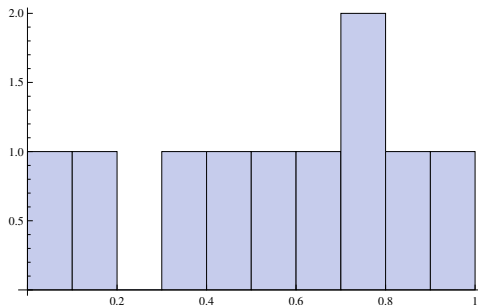
Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

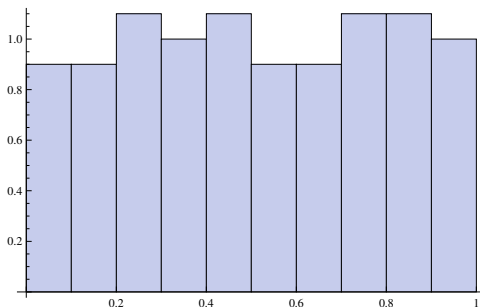
- Thm: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
- Examples: $\log_{10} 2, \log_{10} \left(\frac{1+\sqrt{5}}{2} \right) \notin \mathbb{Q}$.
Proof: if rational: $2 = 10^{p/q}$.
 Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



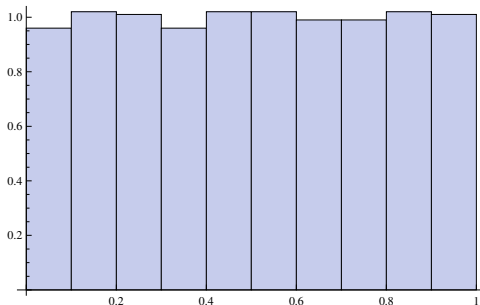
$n\sqrt{\pi} \bmod 1$ for $n \leq 10$

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



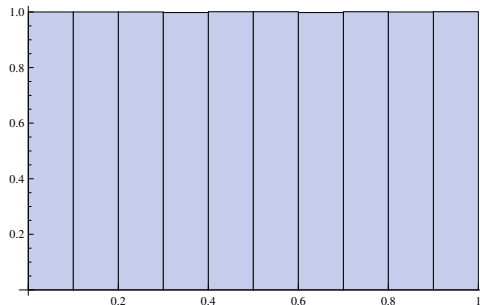
$n\sqrt{\pi} \bmod 1$ for $n \leq 100$

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 1000$

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10,000$

Denseness

Dense

A sequence $\{z_n\}_{n=1}^{\infty}$ of numbers in $[0, 1]$ is dense if for any interval $[a, b]$ there are infinitely many z_n in $[a, b]$.

- **Dirichlet's Box (or Pidgeonhole) Principle:**

If $n + 1$ objects are placed in n boxes, at least one box has two objects.

- **Denseness of $n\alpha$:**

Thm: If $\alpha \notin \mathbb{Q}$ then $z_n = n\alpha \bmod 1$ is dense.

Proof $n\alpha \bmod 1$ dense if $\alpha \notin \mathbb{Q}$

- Enough to show in $[0, b]$ infinitely often for any b .
- Choose any integer $Q > 1/b$.
- Q bins: $[0, \frac{1}{Q}]$, $[\frac{1}{Q}, \frac{2}{Q}]$, $\dots$, $[\frac{Q-1}{Q}, Q]$.
- $Q + 1$ objects:
 $\{\alpha \bmod 1, 2\alpha \bmod 1, \dots, (Q + 1)\alpha \bmod 1\}$.
- Two in same bin, say $q_1\alpha \bmod 1$ and $q_2\alpha \bmod 1$.
- Exists integer p with $0 < q_2\alpha - q_1\alpha - p < \frac{1}{Q}$.
- Get $(q_2 - q_1)\alpha \bmod 1 \in [0, b]$.

Logarithms and Benford's Law

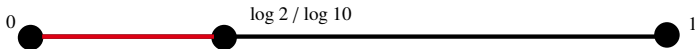
Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

Logarithms and Benford's Law

Fundamental Equivalence

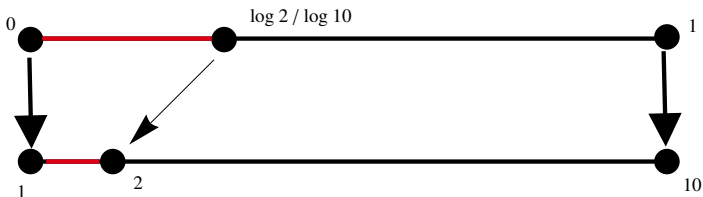
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Logarithms and Benford's Law

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Proof:

- $x = M_B(x) \cdot B^k$ for some $k \in \mathbb{Z}$.
- $\text{FD}_B(x) = d$ iff $d \leq M_B(x) < d + 1$.
- $\log_B d \leq y < \log_B(d + 1)$, $y = \log_B x \bmod 1$.
- If $Y \sim \text{Unif}(0, 1)$ then above probability is $\log_B \left(\frac{d+1}{d} \right)$.

Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.
- Fibonacci numbers are Benford base 10.

$$a_{n+1} = a_n + a_{n-1}.$$

Guess $a_n = n^r$: $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$.

Roots $r = (1 \pm \sqrt{5})/2$.

General solution: $a_n = c_1 r_1^n + c_2 r_2^n$.

$$\text{Binet: } a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

- Most linear recurrence relations Benford:

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$$\diamond a_{n+1} = 2a_n$$

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- Most linear recurrence relations Benford:

$$\diamond a_{n+1} = 2a_n - a_{n-1}$$

Examples

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- Most linear recurrence relations Benford:

$$\diamond a_{n+1} = 2a_n - a_{n-1}$$

$\diamond$ take $a_0 = a_1 = 1$ or $a_0 = 0, a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576				
2	2048	2097152	1	18	.300	.301
4	4096	4194304	2	12	.200	.176
8	8192	8388608	3	6	.100	.125
16	16384	16777216	4	6	.100	.097
32	32768	33554432	5	6	.100	.079
64	65536	67108864	6	4	.067	.067
128	131072	134217728	7	2	.033	.058
256	262144	268435456	8	5	.083	.051
512	524288	536870912	9	1	.017	.046

Data Analysis

- **χ^2 -Tests:** Test if theory describes data
 - ◇ Expected probability: $p_d = \log_{10} \left(\frac{d+1}{d} \right)$.
 - ◇ Expect about Np_d will have first digit d .
 - ◇ Observe $\text{Obs}(d)$ with first digit d .
 - ◇ $\chi^2 = \sum_{d=1}^9 \frac{(\text{Obs}(d) - Np_d)^2}{Np_d}$.
 - ◇ Smaller χ^2 , more likely correct model.

- Will study γ^n , e^n , π^n .

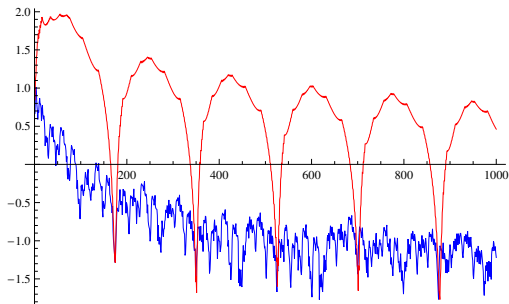
Logarithms and Benford's Law

χ^2 values for α^n , $1 \leq n \leq N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

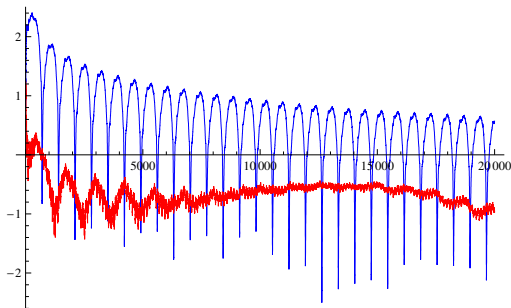
Logarithms and Benford's Law: Base 10

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue),
 $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5%,
 $\log(\chi^2) \approx 2.74$).



Logarithms and Benford's Law: Base 20

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue),
 $n \in \{1, \dots, N\}$. Note $e^3 \approx 20.0855$, (5%,
 $\log(\chi^2) \approx 2.74$).



Applications

Stock Market

Milestone	Date	Effective Rate from last milestone
108.35	Jan 12, 1906	
500.24	Mar 12, 1956	3.0%
1003.16	Nov 14, 1972	4.2%
2002.25	Jan 8, 1987	4.9%
3004.46	Apr 17, 1991	9.5%
4003.33	Feb 23, 1995	7.4%
5023.55	Nov 21, 1995	30.6%
6010.00	Oct 14, 1996	20.0%
7022.44	Feb 13, 1997	46.6%
8038.88	Jul 16, 1997	32.3%
9033.23	Apr 6, 1998	16.1%
10006.78	Mar 29, 1999	10.5%
11209.84	Jul 16, 1999	38.0%
12011.73	Oct 19, 2006	1.0%
13089.89	Apr 25, 2007	16.7%
14000.41	Jul 19, 2007	28.9%

Applications for the IRS: Detecting Fraud

U.S. Individual Income Tax Return 1989

For the year beginning January 1, 1989, or other tax year beginning 1989 ending 19 1989 ending 19

First name and initial **WILLIAM J** Last name **CLINTON** Your social security number **429-92-6947**

If a joint return, enter the first name and initial **HILJARY** Last name **RODHAM** Enter a second social security no. **353-40-2526**

Home address (street and street, R.F.D. box, or post office box) **1800 CENTER** Apt. no. **1800** Enter a second address (if you live at a large address, see page 1)

City, town, or village, state, and ZIP code **LITTLE ROCK ARKANSAS 72206**

CLIN President/Presiding Officer/General Do you expect \$1 to go to this fund? Yes No Do you expect \$1 to go to this fund? Yes No

Filing Status 1 Single 2 Married filing joint return (even if only one had income) 3 Married filing separate return (enter spouse's social security number above and full name here) 4 Head of household (only qualifying person) (See page 7 of instructions.) 5 Qualifying widow(er) with dependent child (See page 7 of instructions.)

Exemptions 6a X Spouse 6b Dependents (See instructions on page 6.) 6c Other (See instructions on page 6.)

Income 7 Wages, salaries, tips, etc. (attach Schedule W) 8a Taxable interest income (attach Schedule B) 8b Tax-exempt interest income (attach Schedule B) 9 Dividend income (attach Schedule D) 10 Taxable refunds of state and local income taxes (attach Schedule D) 11 Alimony received 12 Business income or loss (attach Schedule C) 13 Capital gain or loss (attach Schedule D) 14 Capital gain distributions (attach Schedule D) 15 Other gains or losses (attach Schedule D) 16a Total IRA distributions 16b Rollover distributions 17a Total pensions and annuities 17b Rollover distributions 18 Rents, royalties, partnerships, estates, trusts, etc. (attach Schedule E) 19 Farm income or loss (attach Schedule F) 20 Unemployment compensation (attach Schedule F) 21 Social security benefits 22 Other income (attach Schedule F) 23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income.

Adjustments 24 Your IRA deduction (attach worksheet on page 14 or 15) 25 Savings IRA deduction (attach worksheet on page 14 or 15) 26 Self-employed health insurance deduction (attach worksheet on page 16) 27 Raffle, sweepstakes, and other prizes (attach Schedule E) 28 Penalty on early withdrawal of savings 29 Alimony paid (attach worksheet on page 14 or 15)

Gross Income 30 Adjusted Gross Income 31

Handwritten notes: "I've never filed a negative deduction!" "How much have I lost?" "not entered"

Applications for the IRS: Detecting Fraud

93-4670

1040 Department of the Treasury Internal Revenue Service
U.S. Individual Income Tax Return 1992
 For the year 1992, 1-12-92, or other taxable year beginning 1992, ending

Label: **WILLIAM J. CLINTON**
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Use the IRS label. Otherwise, please print as type.

Your social security number: **XXXXXXXX-XX**
 Researcher's record number: **XXXXXXXXXX**
 For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Do you want \$1 to go to the fund? ☒ Yes ☐ No
 If joint return, does your spouse want \$1 to go to the fund? ☒ Yes ☐ No

Filing Status
 1 ☒ Single
 2 ☐ Married filing joint return (even if only one had income)
 3 ☐ Married filing separate return. Enter spouse's SSN above and full name here. If you were under 18, 19, or 20, you must be under 21, your spouse must be 21 or over, and you must have been married for at least one year.
 4 ☐ Qualifying surviving spouse. Attach Schedule D-1.

Exemptions
 6 a ☒ Yourself b ☐ Spouse
 c ☐ Dependents: (1) Name (Last, first, and last name) (2) Relationship to you (3) Date of birth (Month, day, and year) (4) Date of death (Month, day, and year) (5) Social Security number (6) Name of your spouse in 1992
 CHILSEA DAUGHTER 12

Income
 7 Wages, salaries, tips, etc. (Attach Form W-2) **237,698.F**
 8 Taxable interest income. Attach Schedule B if over \$400 **6,624.F**
 9 Tax-exempt interest income. (Do not include on this line.) **7,269.X**
 10 Dividend income. Attach Schedule B if over \$400 **1,404.F**
 11 Taxable refunds, credits, or offsets of state and local income taxes **16,336.X**
 12 Alimony received **16,336.X**
 13 Business income or loss. Attach Schedule C or C-EZ **22,400.F**
 14 Capital gain or loss. Attach Schedule D **22,400.F**
 15 Capital gain distributions not reported on line 13 **22,400.F**
 16 Other gains or losses. Attach Form 4797 **22,400.F**
 17 a Total IRA distributions **22,400.F** b Taxable amount **22,400.F**
 18 a Total pensions and annuities **22,400.F** b Taxable amount **22,400.F**
 19 Pensions, royalties, partnerships, estates, trusts, etc. Attach Schedule E **22,400.F**
 20 Farm income or loss. Attach Schedule F **22,400.F**
 21 Unemployment compensation **22,400.F**
 22 Social Security benefits **22,400.F**
 23 Other income (List on Schedule 1) **22,400.F**
 24 Add the amounts in the far right column for lines 7 through 23. This is your total income **299,657.F**

Adjustments to income
 25 a Your IRA deduction **250**
 26 b Employer's IRA deduction **250**
 27 c Other of self-employment tax **250**
 28 d Self-employed health insurance deduction **250**
 29 e Rollover investment plan and self-employed SEP deduction **250**
 30 f Penalty on early withdrawal of savings **250**
 31 g Alimony paid. Recipient's SSN: **250**
 32 Add lines 25 through 31. These are your total adjustments **250**
 33 Subtract line 32 from line 24. This is your adjusted gross income **299,407.F**

AGI **299,407.F**

1073

Applications for the IRS: Detecting Fraud

Exhibit 3: Check Fraud in Arizona

The table lists the checks that a manager in the office of the Arizona State Treasurer wrote to divert funds for his own use. The vendors to whom the checks were issued were fictitious.

Date of Check	Amount
October 9, 1992	\$ 1,927.48
↓	27,902.31
October 14, 1992	86,241.90
↓	72,117.46
	81,321.75
	97,473.96
October 19, 1992	93,249.11
↓	89,658.17
	87,776.89
	92,105.83
	79,949.16
	87,602.93
	96,879.27
	91,806.47
	84,991.67
	90,831.83
	93,766.67
	88,338.72
	94,639.49
	83,709.28
	96,412.21
	88,432.86
	71,552.16
TOTAL	\$ 1,878,687.58

Applications for the IRS: Detecting Fraud (cont)

- Embezzler started small and then increased dollar amounts.
- Most amounts below \$100,000 (critical threshold for data requiring additional scrutiny).
- Over 90% had first digit of 7, 8 or 9.

Detecting Fraud

Bank Fraud

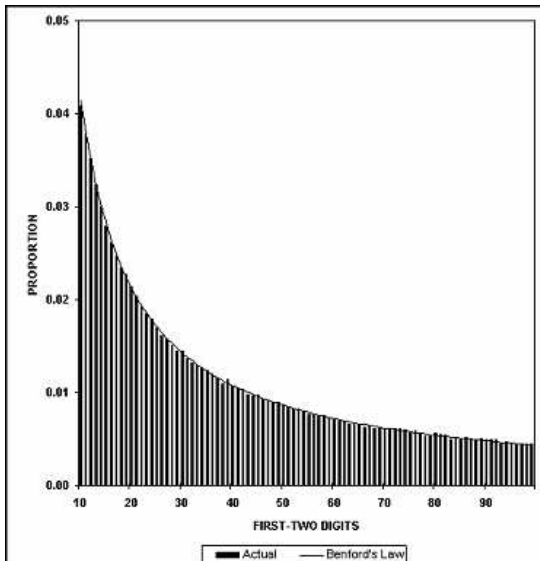
- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off.

Detecting Fraud

Enron

- Benford's Law detected manipulation of revenue numbers.
- Results showed a tendency towards round Earnings Per Share (0.10, 0.20, etc.).
Consistent with a small but noticeable increase in earnings management in 2002.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Benford Good Processes

Poisson Summation and Benford's Law: Definitions

- Feller, Pinkham (often exact processes)
- data $Y_{T,B} = \log_B \vec{X}_T$ (discrete/continuous):

$$\mathbb{P}(A) = \lim_{T \rightarrow \infty} \frac{\#\{n \in A : n \leq T\}}{T}$$

- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\ell),$$

$$\text{Fourier transform } \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Benford Good Process

X_T is **Benford Good** if there is a nice f st

$$\text{CDF}_{\vec{Y}_{T,B}}(y) = \int_{-\infty}^y \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) := G_T(y)$$

and monotonically increasing h ($h(|T|) \rightarrow \infty$):

- **Small tails:** $G_T(\infty) - G_T(Th(T)) = o(1)$,
 $G_T(-Th(T)) - G_T(-\infty) = o(1)$.
- **Decay of the Fourier Transform:** $\sum_{\ell \neq 0} \left| \frac{\hat{f}(T\ell)}{\ell} \right| = o(1)$.
- **Small translated error:** $\mathcal{E}(a, b, T) =$
 $\sum_{|\ell| \leq Th(T)} [E_T(b + \ell) - E_T(a + \ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \rightarrow \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

- **Examples**

- ◇ L -functions
- ◇ characteristic polynomials (RMT)
- ◇ $3x + 1$ problem
- ◇ geometric Brownian motion.

Sketch of the proof

- **Structure Theorem:**
 - ◇ main term is something nice spreading out
 - ◇ apply Poisson summation
- **Control translated errors:**
 - ◇ hardest step
 - ◇ techniques problem specific

Sketch of the proof (continued)

$$\begin{aligned}
 & \sum_{\ell=-\infty}^{\infty} \mathbb{P} \left(a + \ell \leq \overrightarrow{Y}_{T,B} \leq b + \ell \right) \\
 &= \sum_{|\ell| \leq Th(T)} [G_T(b + \ell) - G_T(a + \ell)] + o(1) \\
 &= \int_a^b \sum_{|\ell| \leq Th(T)} \frac{1}{T} f \left(\frac{t}{T} \right) dt + \mathcal{E}(a, b, T) + o(1) \\
 &= \widehat{f}(0) \cdot (b - a) + \sum_{\ell \neq 0} \widehat{f}(T\ell) \frac{e^{2\pi i b \ell} - e^{2\pi i a \ell}}{2\pi i \ell} + o(1).
 \end{aligned}$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Riemann Zeta Function

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$$\begin{aligned} \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) \\ &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right) \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \dots\right) \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{(2 \cdot 3)^s} + \dots \end{aligned}$$

Riemann Zeta Function

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Riemann Zeta Function

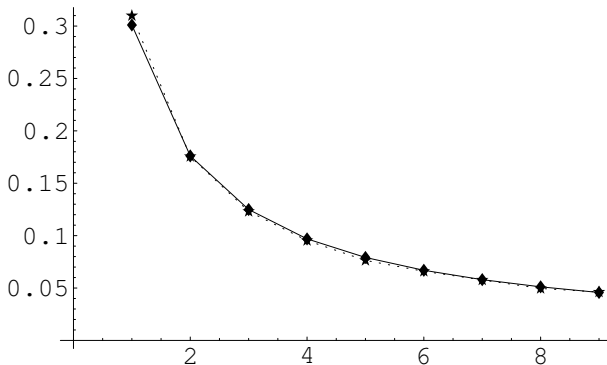
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$\lim_{s \rightarrow 1+} \zeta(s) = \infty$ implies infinitely many primes.

$\zeta(2) = \pi^2/6$ implies infinitely many primes.

Riemann Zeta Function

$$\left| \zeta \left(\frac{1}{2} + i \frac{k}{4} \right) \right|, k \in \{0, 1, \dots, 65535\}.$$



The $3x + 1$ Problem and Benford's Law

3x + 1 Problem

- Kakutani (conspiracy), Erdős (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.
- $7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \rightarrow_4 1 \rightarrow_2 1$,
 2-path (1, 1), 5-path (1, 1, 2, 3, 4).
 m -path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

$$\begin{aligned}
 a_{n+1} &= T(a_n) \\
 \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left(\frac{3a_n}{2^k} \right) \\
 &= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\
 &= \log a_n + \log \left(\frac{3}{4} \right).
 \end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N \rightarrow \infty} \frac{\#\{n \leq N: n \equiv 1, 5 \pmod{6}, n \in A\}}{\#\{n \leq N: n \equiv 1, 5 \pmod{6}\}}.$$

$(k_1, \dots, k_m)$: two full arithm progressions:

$$6 \cdot 2^{k_1 + \dots + k_m} p + q.$$

Theorem (Sinai, Kontorovich-Sinai)

k_i -values are i.i.d.r.v. (geometric, $1/2$):

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$$\mathbb{P} \left(\frac{\log_2 \left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0} \right]}{\sqrt{2m}} \leq a \right) = \mathbb{P} \left(\frac{S_m - 2m}{\sqrt{2m}} \leq a \right)$$

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$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \rightarrow \infty$, $x_m / (3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}$ X initial seeds the distribution of the first N iterates of the $3x + 1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Sketch of the proof

- Failed Proof: lattices, bad errors.

- CLT: $(S_m - 2m)/\sqrt{2m} \rightarrow N(0, 1)$:

$$\mathbb{P}(S_m - 2m = k) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:

$$I_\ell = \{\ell M, \dots, (\ell + 1)M - 1\}, \quad M = m^c, \quad c < 1/2$$

$$k_1, k_2 \in I_\ell: \left| \eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right) \right| \text{ small}$$

$$C = \log_B 2 \text{ of irrationality type } \kappa < \infty:$$

$$\#\{k \in I_\ell : \overline{kC} \in [a, b]\} = M(b - a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\lim_{q \rightarrow \infty} q^{\gamma+1} \min_p \left| \alpha - \frac{p}{q} \right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$
 with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$,
 $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| / q \log 10.$$

Quantified Equidistribution

Theorem (Erdős-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \leq N : x_n \in [a,b]\}|}{N}$$

There is a C such that for all m :

$$D_N \leq C \cdot \left(\frac{1}{m} + \sum_{h=1}^m \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h x_n} \right| \right)$$

Proof of Erdős-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^m \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ , $\sum_{h=1}^m \frac{1}{h||h\alpha||} = O(m^{\kappa-1+\epsilon})$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x + 1$ Data: random 10,000 digit number, $2^k \parallel 3x + 1$

80,514 iterations ($((4/3)^n = a_0$ predicts 80,319);
 $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x + 1$ Data: random 10,000 digit number, $2|3x + 1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x + 1$ Data: random 10,000 digit number, $2^k \parallel 5x + 1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x + 1$ Data: random 10,000 digit number, $2|5x + 1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Products and Chains of Random Variables

Key Ingredients

- Mellin transform and Fourier transform related by **logarithmic** change of variable.
- Poisson summation from collapsing to modulo 1 random variables.

Preliminaries

- $\Xi_1, \dots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.
- Density $\Xi_1 \cdot \Xi_2$:

$$\int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$

◇ Proof: $\text{Prob}(\Xi_1 \cdot \Xi_2 \in [0, x])$:

$$\begin{aligned}
 & \int_{t=0}^\infty \text{Prob}\left(\Xi_2 \in \left[0, \frac{x}{t}\right]\right) f_1(t) dt \\
 &= \int_{t=0}^\infty F_2\left(\frac{x}{t}\right) f_1(t) dt,
 \end{aligned}$$

differentiate.

Mellin Transform

$$(\mathcal{M}f)(s) = \int_0^{\infty} f(x) x^s \frac{dx}{x}$$

$$(\mathcal{M}^{-1}g)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) x^{-s} ds$$

$$g(s) = (\mathcal{M}f)(s), f(x) = (\mathcal{M}^{-1}g)(x).$$

$$(f_1 \star f_2)(x) = \int_0^{\infty} f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$

$$(\mathcal{M}(f_1 \star f_2))(s) = (\mathcal{M}f_1)(s) \cdot (\mathcal{M}f_2)(s).$$

Mellin Transform Formulation: Products Random Variables

Theorem

X_i 's independent, densities f_i . $\Xi_n = X_1 \cdots X_n$,

$$h_n(x_n) = (f_1 \star \cdots \star f_n)(x_n)$$

$$(\mathcal{M}h_n)(s) = \prod_{m=1}^n (\mathcal{M}f_m)(s).$$

As $n \rightarrow \infty$, Ξ_n becomes Benford: $Y_n = \log_B \Xi_n$,

$$|\text{Prob}(Y_n \bmod 1 \in [a, b]) - (b - a)| \leq$$

$$(b - a) \cdot \sum_{\ell \neq 0, \ell = -\infty}^{\infty} \prod_{m=1}^n (\mathcal{M}f_i) \left(1 - \frac{2\pi i \ell}{\log B} \right).$$

Proof of Kossovsky's Chain Conjecture for certain densities

Conditions

- $\{\mathcal{D}_i(\theta)\}_{i \in I}$: one-parameter distributions, densities $f_{\mathcal{D}_i(\theta)}$ on $[0, \infty)$.
- $p: \mathbb{N} \rightarrow I$, $X_1 \sim \mathcal{D}_{p(1)}(1)$, $X_m \sim \mathcal{D}_{p(m)}(X_{m-1})$.
- $m \geq 2$,

$$f_m(x_m) = \int_0^\infty f_{\mathcal{D}_{p(m)}(1)} \left(\frac{x_m}{x_{m-1}} \right) f_{m-1}(x_{m-1}) \frac{dx_{m-1}}{x_{m-1}}$$

•

$$\lim_{n \rightarrow \infty} \sum_{\substack{\ell = -\infty \\ \ell \neq 0}}^{\infty} \prod_{m=1}^n (\mathcal{M} f_{\mathcal{D}_{p(m)}(1)}) \left(1 - \frac{2\pi i \ell}{\log B} \right) = 0$$

Proof of Kossovsky's Chain Conjecture for certain densities

Theorem (JKKKM)

- *If conditions hold, as $n \rightarrow \infty$ the distribution of leading digits of X_n tends to Benford's law.*
- *The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then*

$$\left| \text{Prob}(Y_n \bmod 1 \in [a, b]) - (b - a) \right| \leq \left| (b - a) \cdot \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} \prod_{m=1}^n (\mathcal{M} f_{\mathcal{D}_{\rho(m)}(1)}) \left(1 - \frac{2\pi i \ell}{\log B} \right) \right|$$

Example: All $X_i \sim \text{Exp}(1)$

- $X_i \sim \text{Exp}(1)$, $Y_n = \log_B \Xi_n$.
- Needed ingredients:
 - ◇ $\int_0^\infty \exp(-x) x^{s-1} dx = \Gamma(s)$.
 - ◇ $|\Gamma(1 + ix)| = \sqrt{\pi x / \sinh(\pi x)}$, $x \in \mathbb{R}$.
- $|P_n(s) - \log_{10}(s)| \leq$

$$\log_B s \sum_{\ell=1}^{\infty} \left(\frac{2\pi^2 \ell / \log B}{\sinh(2\pi^2 \ell / \log B)} \right)^{n/2}.$$

Example: All $X_i \sim \text{Exp}(1)$

Bounds on the error

- $|P_n(s) - \log_{10} s| \leq$
 - ◇ $3.3 \cdot 10^{-3} \log_B s$ if $n = 2$,
 - ◇ $1.9 \cdot 10^{-4} \log_B s$ if $n = 3$,
 - ◇ $1.1 \cdot 10^{-5} \log_B s$ if $n = 5$, and
 - ◇ $3.6 \cdot 10^{-13} \log_B s$ if $n = 10$.
- Error at most

$$\log_{10} s \sum_{\ell=1}^{\infty} \left(\frac{17.148\ell}{\exp(8.5726\ell)} \right)^{n/2} \leq .057^n \log_{10} s$$

Conclusions

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.
- **Future work:**
 - ◇ Study digits of other systems.
 - ◇ Develop more sophisticated tests for fraud.

References

-  A. K. Adhikari, *Some results on the distribution of the most significant digit*, Sankhyā: The Indian Journal of Statistics, Series B **31** (1969), 413–420.
-  A. K. Adhikari and B. P. Sarkar, *Distribution of most significant digit in certain functions whose arguments are random variables*, Sankhyā: The Indian Journal of Statistics, Series B **30** (1968), 47–58.
-  R. N. Bhattacharya, *Speed of convergence of the n -fold convolution of a probability measure on a compact group*, Z. Wahrscheinlichkeitstheorie verw. Geb. **25** (1972), 1–10.
-  F. Benford, *The law of anomalous numbers*, Proceedings of the American Philosophical Society **78** (1938), 551–572.
-  A. Berger, Leonid A. Bunimovich and T. Hill, *One-dimensional dynamical systems and Benford's Law*, Trans. Amer. Math. Soc. **357** (2005), no. 1, 197–219.

-  A. Berger and T. Hill, *Newton's method obeys Benford's law*, The Amer. Math. Monthly **114** (2007), no. 7, 588-601.
-  J. Boyle, *An application of Fourier series to the most significant digit problem* Amer. Math. Monthly **101** (1994), 879-886.
-  J. Brown and R. Duncan, *Modulo one uniform distribution of the sequence of logarithms of certain recursive sequences*, Fibonacci Quarterly **8** (1970) 482-486.
-  P. Diaconis, *The distribution of leading digits and uniform distribution mod 1*, Ann. Probab. **5** (1979), 72-81.
-  W. Feller, *An Introduction to Probability Theory and its Applications, Vol. II*, second edition, John Wiley & Sons, Inc., 1971.



R. W. Hamming, *On the distribution of numbers*, Bell Syst. Tech. J. **49** (1970), 1609-1625.



T. Hill, *The first-digit phenomenon*, American Scientist **86** (1996), 358–363.



T. Hill, *A statistical derivation of the significant-digit law*, Statistical Science **10** (1996), 354–363.



P. J. Holeyijn, *On the uniform distribuiton of sequences of random variables*, Z. Wahrscheinlichkeitstheorie verw. Geb. **14** (1969), 89–92.



W. Hurlimann, *Benford's Law from 1881 to 2006: a bibliography*, <http://arxiv.org/abs/math/0607168>.



D. Jang, J. U. Kang, A. Kruckman, J. Kudo and S. J. Miller, *Chains of distributions, hierarchical Bayesian models and Benford's Law*, preprint.

-  E. Janvresse and T. de la Rue, *From uniform distribution to Benford's law*, Journal of Applied Probability **41** (2004) no. 4, 1203–1210.
-  A. Kontorovich and S. J. Miller, *Benford's Law, Values of L-functions and the $3x + 1$ Problem*, Acta Arith. **120** (2005), 269–297.
-  D. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, Addison-Wesley, third edition, 1997.
-  J. Lagarias and K. Soundararajan, *Benford's Law for the $3x + 1$ Function*, J. London Math. Soc. (2) **74** (2006), no. 2, 289–303.
-  S. Lang, *Undergraduate Analysis*, 2nd edition, Springer-Verlag, New York, 1997.

-  P. Levy, *L'addition des variables aléatoires définies sur une circonférence*, Bull. de la S. M. F. **67** (1939), 1–41.
-  E. Ley, *On the peculiar distribution of the U.S. Stock Indices Digits*, The American Statistician **50** (1996), no. 4, 311–313.
-  R. M. Loynes, *Some results in the probabilistic theory of asymptotic uniform distributions modulo 1*, Z. Wahrscheinlichkeitstheorie verw. Geb. **26** (1973), 33–41.
-  S. J. Miller, *When the Cramér-Rao Inequality provides no information*, to appear in Communications in Information and Systems.
-  S. J. Miller and M. Nigrini, *The Modulo 1 Central Limit Theorem and Benford's Law for Products*, International Journal of Algebra **2** (2008), no. 3, 119–130.
-  S. J. Miller and M. Nigrini, *Differences between Independent Variables and Almost Benford Behavior*, preprint.
<http://arxiv.org/abs/math/0601344>

-  S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory*, Princeton University Press, Princeton, NJ, 2006.
-  S. Newcomb, *Note on the frequency of use of the different digits in natural numbers*, Amer. J. Math. **4** (1881), 39-40.
-  M. Nigrini, *Digital Analysis and the Reduction of Auditor Litigation Risk*. Pages 69–81 in *Proceedings of the 1996 Deloitte & Touche / University of Kansas Symposium on Auditing Problems*, ed. M. Ettredge, University of Kansas, Lawrence, KS, 1996.
-  M. Nigrini, *The Use of Benford's Law as an Aid in Analytical Procedures*, Auditing: A Journal of Practice & Theory, **16** (1997), no. 2, 52–67.
-  M. Nigrini and S. J. Miller, *Benford's Law applied to hydrology data – results and relevance to other geophysical data*, Mathematical Geology **39** (2007), no. 5, 469–490.

-  R. Pinkham, *On the Distribution of First Significant Digits*, The Annals of Mathematical Statistics **32**, no. 4 (1961), 1223–1230.
-  R. A. Raimi, *The first digit problem*, Amer. Math. Monthly **83** (1976), no. 7, 521–538.
-  H. Robbins, *On the equidistribution of sums of independent random variables*, Proc. Amer. Math. Soc. **4** (1953), 786–799.
-  H. Sakamoto, *On the distributions of the product and the quotient of the independent and uniformly distributed random variables*, Tôhoku Math. J. **49** (1943), 243–260.
-  P. Schatte, *On sums modulo 2π of independent random variables*, Math. Nachr. **110** (1983), 243–261.

-  P. Schatte, *On the asymptotic uniform distribution of sums reduced mod 1*, Math. Nachr. **115** (1984), 275–281.
-  P. Schatte, *On the asymptotic logarithmic distribution of the floating-point mantissas of sums*, Math. Nachr. **127** (1986), 7–20.
-  E. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.
-  M. D. Springer and W. E. Thompson, *The distribution of products of independent random variables*, SIAM J. Appl. Math. **14** (1966) 511–526.
-  K. Stromberg, *Probabilities on a compact group*, Trans. Amer. Math. Soc. **94** (1960), 295–309.
-  P. R. Turner, *The distribution of leading significant digits*, IMA J. Numer. Anal. **2** (1982), no. 4, 407–412.