Benford's Law: Why the IRS might care about the $3x + 1$ problem and $\zeta(s)$

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New York Number Theory Seminar: June 16, 2022

Introduction

- A. Berger and T. P. Hill, *An Introduction to Benford's Law*, Princeton University Press, Princeton, 2015. See also http://www.benfordonline.net/.

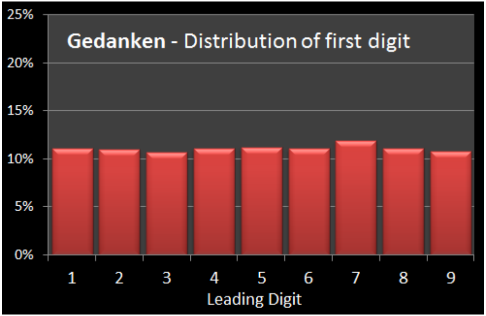
- A. E. Kossovsky, *Benford's Law: Theory, the General Law of Relative Quantities, and Forensic Fraud Detection Applications*, WSPC, 2014.

- S. J. Miller (editor), *Theory and Applications of Benford's Law*, Princeton University Press, 2015.

- M. Nigrini, *Benford's Law: Applications for Forensic Accounting, Auditing, and Fraud Detection*, 1st Edition, Wiley, 2014.

Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?



Natural guess: 10% (but immediately correct to 11%!).

Interesting Question

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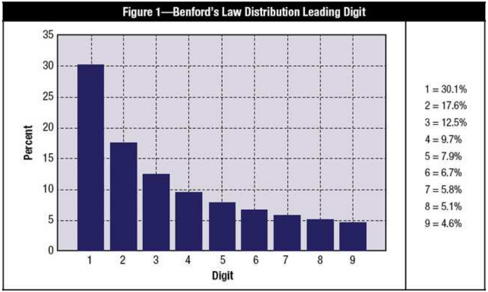
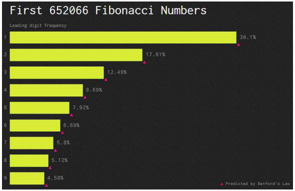


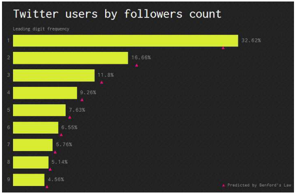
Answer: Benford's law!

Examples with First Digit Bias

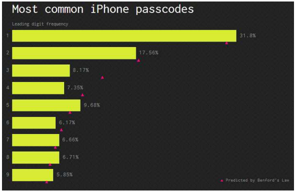
Fibonacci numbers



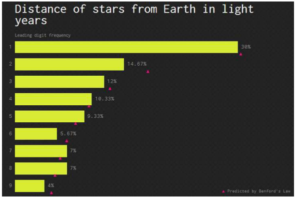
Twitter users by # followers



Most common iPhone passcodes



Distance of stars from Earth

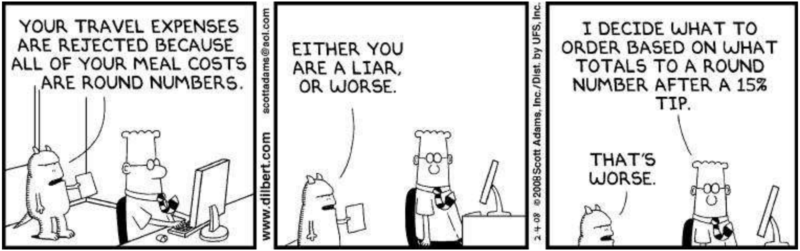


Summary

- Explain Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.



Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

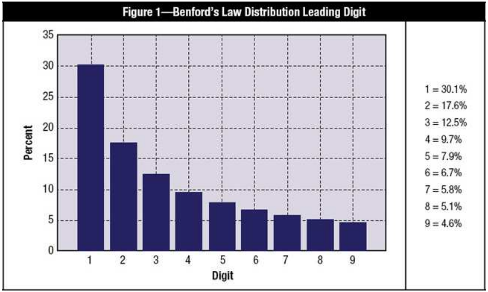
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

General Theory

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1's ($\log_{10}(2) \approx .3010\ldots$).



Benford's Law (probabilities)

Background Material

- Modulo: $a = b \mod c$ if $a - b$ is an integer times c; thus $17 = 5 \mod 12$, and $4.5 = .5 \mod 1$.
- Significand: $x = S_{10}(x) \cdot 10^k$, k integer, $1 \leq S_{10}(x) < 10$. Thus $2020.1701 = 2.0201701 \cdot 10^3$.
- $S_{10}(a) = S_{10}(b)$ if and only if a and b have the same leading digits. Note $\log_{10} a = \log_{10} S_{10}(b) + k$.
- Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \mod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \mod 1$.
Advanced: $e^{2\pi i u} = e^{2\pi i (u \bmod 1)}$.

Equidistribution and Benford's Law

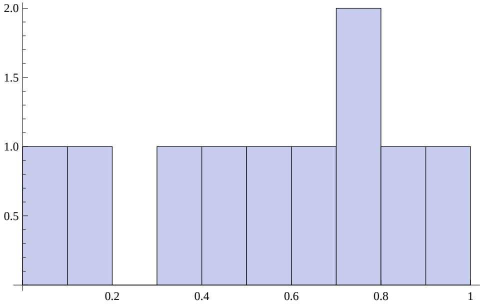
Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a, b]\}}{N} \to b - a.$$

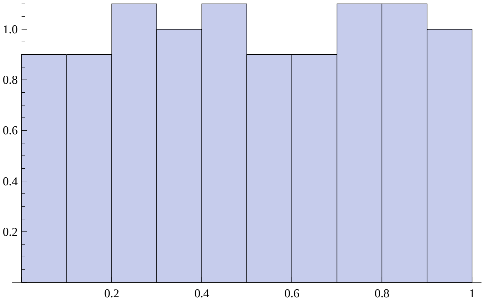
- Thm: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
- Examples: $\log_{10} 2, \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



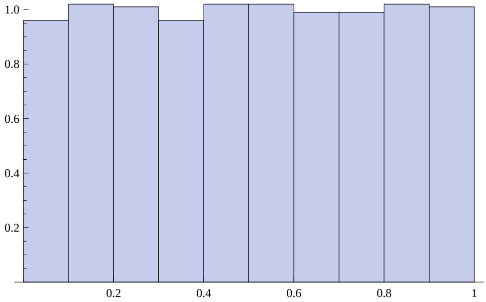
$n\sqrt{\pi}$ mod 1 for $n \leq 10$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



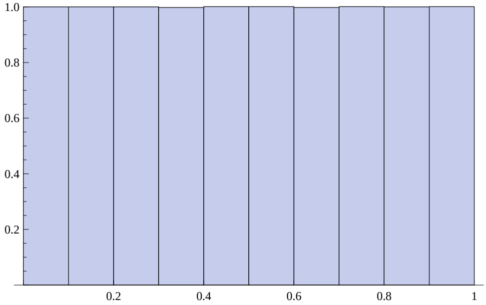
$n\sqrt{\pi}$ mod 1 for $n \leq 100$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 1000$

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10,000$

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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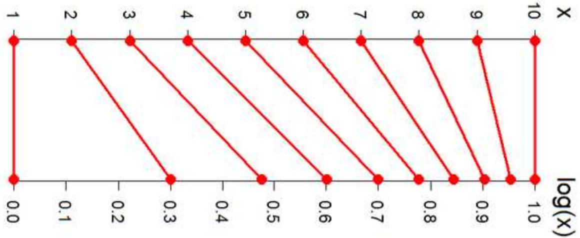
$$x = S_{10}(x) \cdot 10^k \text{ then}$$

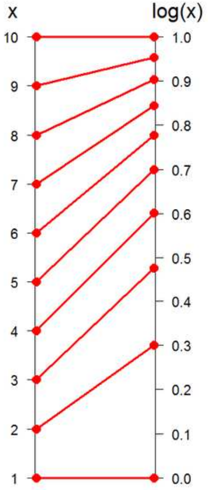
$$\log_{10} x = \log_{10} S_{10}(x) + k = \log_{10} S_{10} x \bmod 1.$$



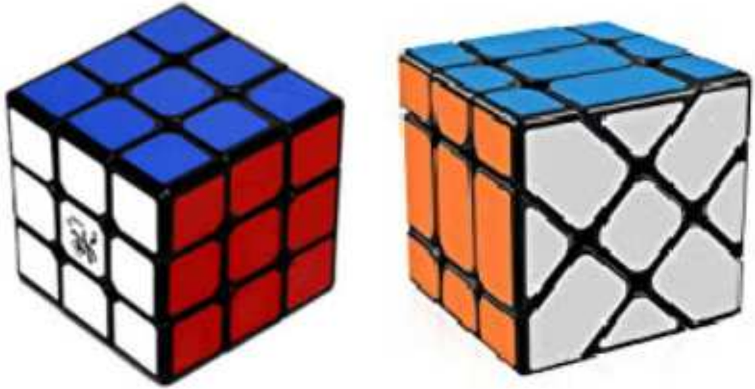
Logarithms and Benford's Law

$$\begin{aligned}
&\text{Prob}(\text{leading digit } d) \\
&= \log_{10}(d+1) - \log_{10}(d) \\
&= \log_{10}\left(\frac{d+1}{d}\right) \\
&= \log_{10}\left(1+\frac{1}{d}\right).
\end{aligned}$$

Have Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



The Power of the Right Perspective



Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

Examples

- Fibonacci numbers are Benford base 10.
  $F_{n+1} = F_n + F_{n-1}$.
  Guess $F_n = r^n$: $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$.
  Roots $r = (1 \pm \sqrt{5})/2$.
  General solution: $F_n = c_1 r_1^n + c_2 r_2^n$.
  Binet: $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$.

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  ◇ $a_{n+1} = 2a_n$

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- Most linear recurrence relations Benford:
  - ◇ $a_{n+1} = 2a_n - a_{n-1}$
  - ◇ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

1	1024	1048576	digit	#	Obs Prob	Benf Prob
2	2048	2097152	1	18	.300	.301
4	4096	4194304	2	12	.200	.176
8	8192	8388608	3	6	.100	.125
16	16384	16777216	4	6	.100	.097
32	32768	33554432	5	6	.100	.079
64	65536	67108864	6	4	.067	.067
128	131072	134217728	7	2	.033	.058
256	262144	268435456	8	5	.083	.051
512	524288	536870912	9	1	.017	.046

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Digits of 2^n

First 1000 values of 2^n (only displaying 30): $2^{10} = 1024 \approx 10^3$.

1	1024	1048576	digit	#	Obs Prob	Benf Prob
2	2048	2097152	1	301	0.301	.301
4	4096	4194304	2	176	0.176	.176
8	8192	8388608	3	125	0.125	.125
16	16384	16777216	4	97	0.097	.097
32	32768	33554432	5	79	0.079	.079
64	65536	67108864	6	69	0.069	.067
128	131072	134217728	7	56	0.056	.058
256	262144	268435456	8	52	0.052	.051
512	524288	536870912	9	45	0.045	.046

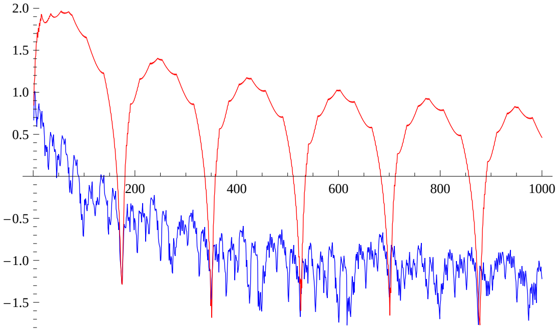
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

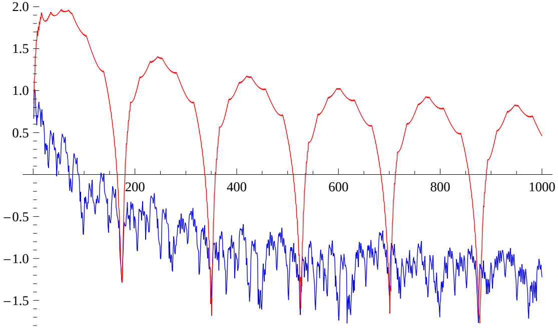
Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$.



Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$.

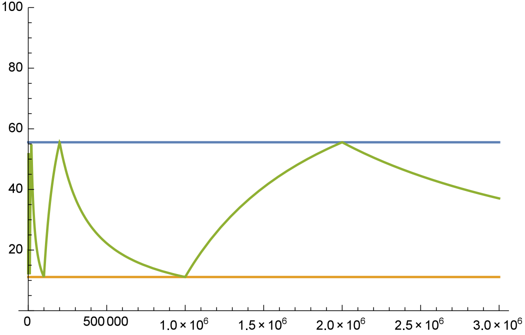


Why Benford's Law?

Streets

Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
- Oscillates b/w $1/9$ and $5/9$ with first digit 1.



Probability first digit 1 versus street length L.

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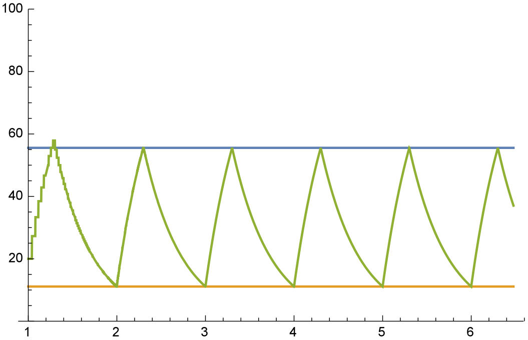

Probability first digit 1 versus log(street length L).

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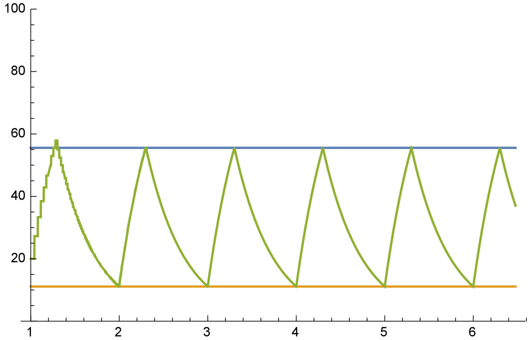


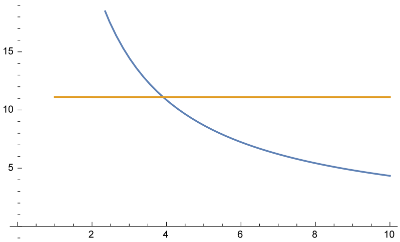
Probability first digit 1 versus log(street length L).

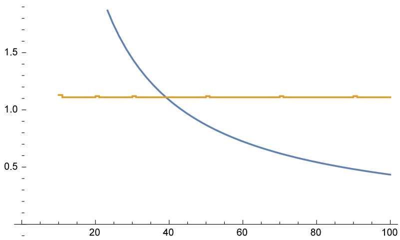
What if we have many streets of different lengths?

Amalgamating Streets

All houses: 1000 Streets,

each from 1 to 10000.





First digit and first two digits vs Benford.

Amalgamating Streets

All houses: 1000 Streets,

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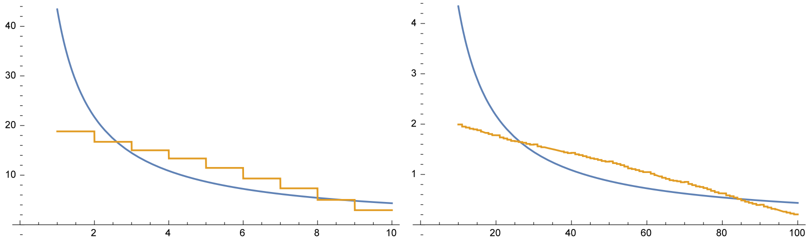

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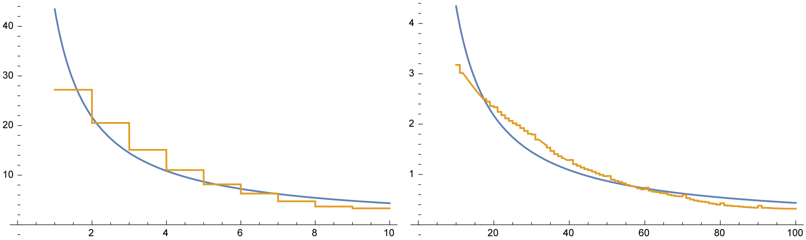

First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

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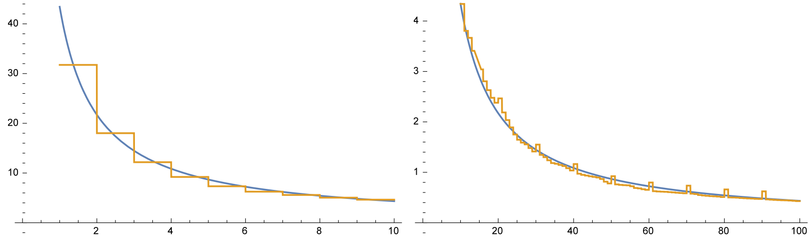
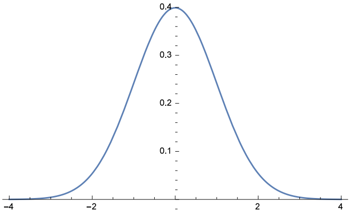


First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

Central Limit Theorem

$$\text{Normal } N(\mu,\sigma^2):\ p(x) \ = \ e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}.$$



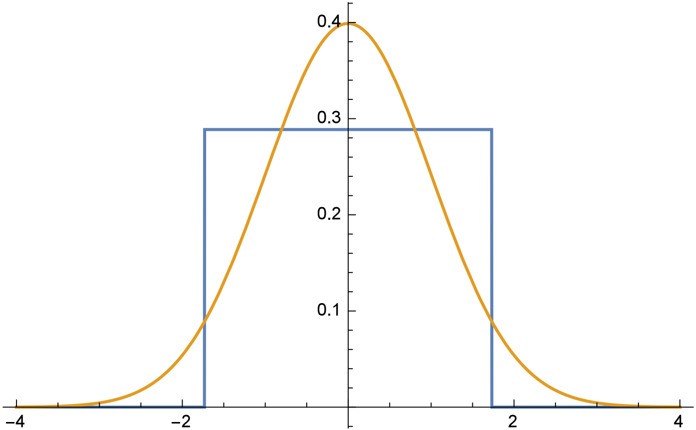
Theorem

If $X_1, X_2, \dots$ independent, identically distributed random variables (mean μ, variance σ^2, finite moments) then

$$S_N \ := \ \frac{X_1+\cdots+X_N - N\mu}{\sigma\sqrt{N}} \ \text{ converges to } \ N(0,1).$$

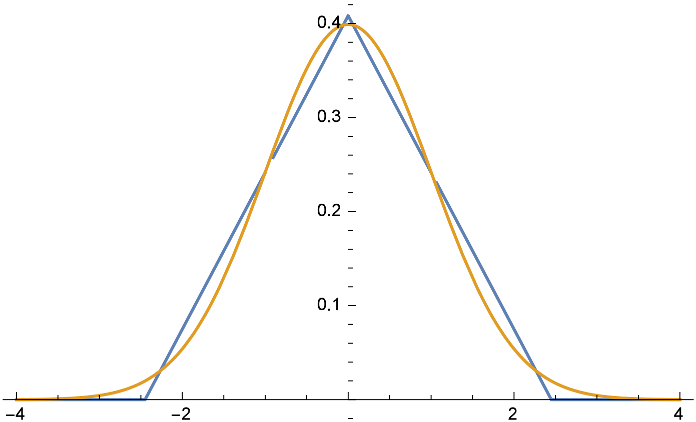
Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \text{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_1 = X_1/\sigma_{X_1} \text{ vs } N(0,1).$$



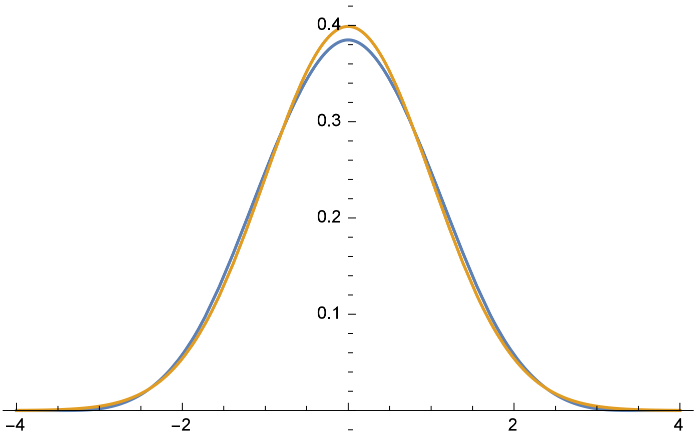
Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_2 = (X_1 + X_2)/\sigma_{X_1+X_2} \text{ vs } N(0,1).$$



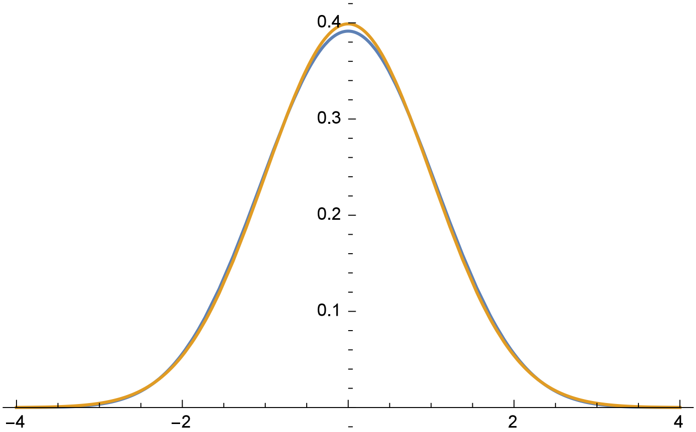
Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_4 = (X_1 + X_2 + X_3 + X_4)/\sigma_{X_1+X_2+X_3+X_4} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_8 = (X_1 + \cdots + X_8)/\sigma_{X_1+\cdots+X_8} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ **(adjusted to mean 0, variance 1)**

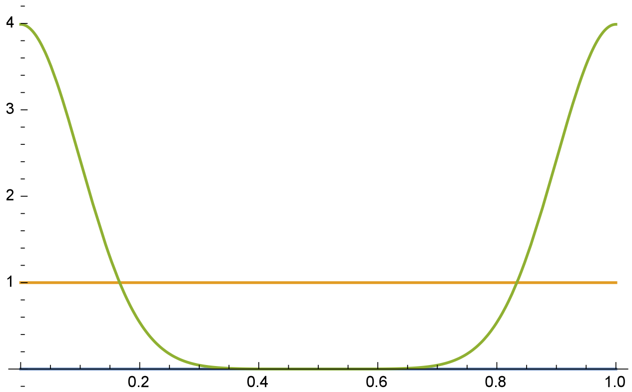
Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

$$\frac{\begin{cases} \frac{1}{27}\left(18 + 9\sqrt{3}\,y - \sqrt{3}\,y^3\right) & y == 0 \\ \frac{1}{18}\left(12 - 6y^2 - \sqrt{3}\,y^3\right) & -\sqrt{3} < y < 0 \\ \frac{1}{54}\left(72 - 36\sqrt{3}\,y + 18y^2 - \sqrt{3}\,y^3\right) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54}\left(18\sqrt{3}\,y - 18y^2 + \sqrt{3}\,y^3\right) & y == \sqrt{3} \\ \frac{1}{18}\left(12 - 6y^2 + \sqrt{3}\,y^3\right) & 0 < y < \sqrt{3} \\ \frac{1}{54}\left(72 + 36\sqrt{3}\,y + 18y^2 + \sqrt{3}\,y^3\right) & -2\sqrt{3} < y \le -\sqrt{3} \\ 0 & \mathrm{True} \end{cases}}{\sqrt{3}}$$

(Don't even think of asking to see Y_8's!)

Normal Distributions Mod 1

As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \mathrm{Unif}(0, 1)$.



Variance is .01.

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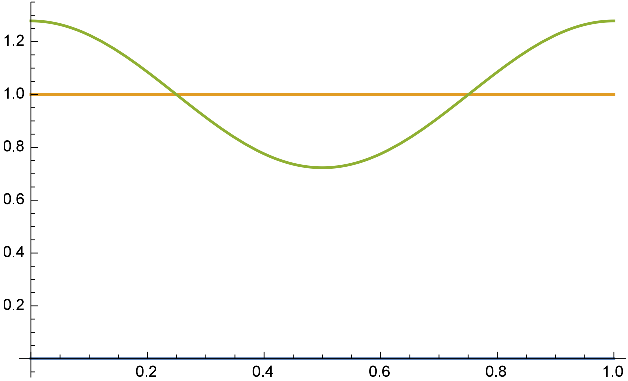

Variance is .1.

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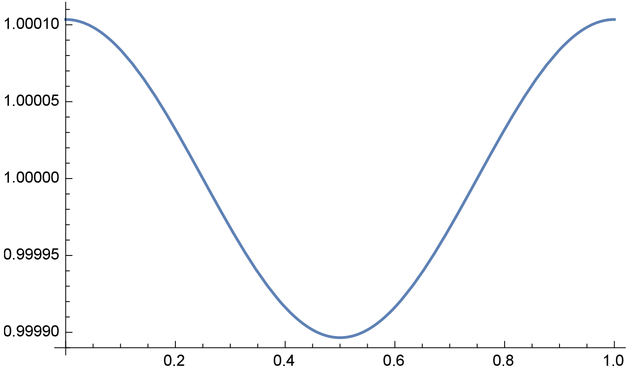

Variance is .5.

Products and Benford's Law

Pavlovian Response: See a product, take a logarithm.

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Need distribution of $V_N \bmod 1$, which by CLT becomes uniform, implying Benfordness!

Stick Decomposition

- T. Becker, D. Burt, T. C. Corcoran, A. Greaves-Tunnell, J. R. Iafrate, J. Jing, S. J. Miller, J. D. Porfilio, R. Ronan, J. Samranvedhya, F. W. Strauch and B. Talbut, *Benford's Law and Continuous Dependent Random Variables*, Annals of Physics **388** (2018), 350–381.
- J. Iafrate, S. J. Miller and F. W. Strauch, *Equipartitions and a distribution for numbers: A statistical model for Benford's law*, Physical Review E **91** (2015), no. 6, 062138 (6 pages).

Fixed Proportion Decomposition Process

Decomposition Process

1. Consider a stick of length $\mathcal{L}$.

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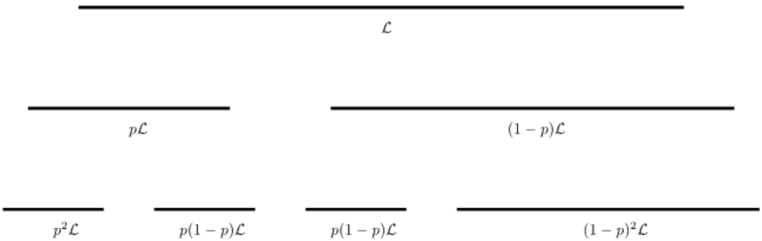
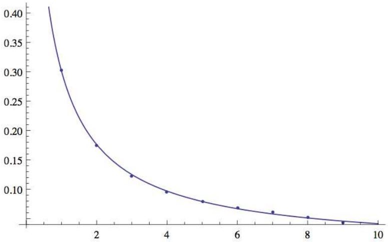
4. Repeat N times (using the same proportion).

Fixed Proportion Decomposition Process

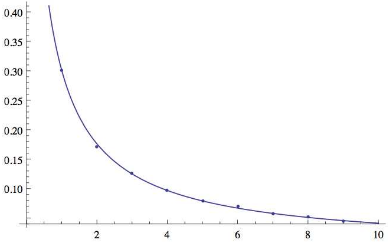


Fixed Proportion Conjecture (Joy Jing '13)

Conjecture: The above decomposition process is Benford as $N \to \infty$ for any $p \in (0,1)$, $p \neq \frac{1}{2}$.



(B) $p = 0.51$ and $N = 10000$.



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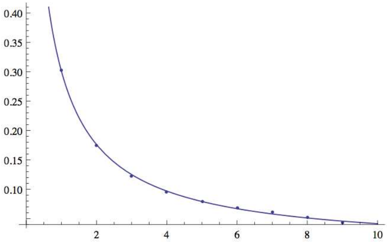

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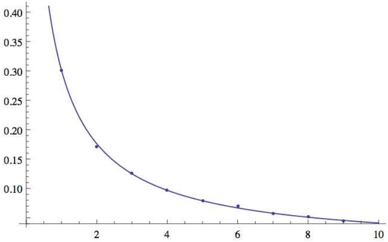

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Counterexample (SMALL REU '13): $p = \frac{1}{11}$, $1 - p = \frac{10}{11}$.

Benford Analysis

At N^{th} level,

- 2^N sticks
- $N+1$ distinct lengths: write $p^{N-j}(1-p)^j$ as

$$p^N\left(\frac{1-p}{p}\right)^j,\ \ j\in\{0,\dots,N\},\ \text{have}\ \binom{N}{j}\ \text{times.}$$

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Theorem: Benford if and only if y irrational.

Benford Analysis (cont)

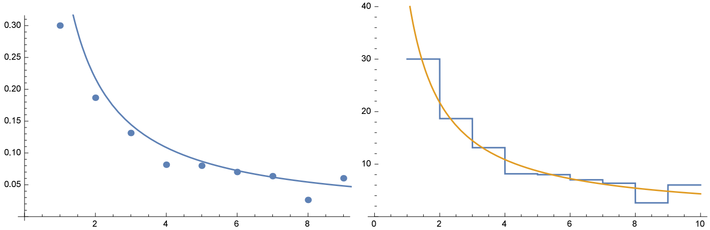
Say $\frac{1-p}{p} = 10^{r/q}$ for r, q integers.

All terms with index j mod q have same leading digit; probability index $j \bmod q$ is

$$\begin{aligned}
\frac{1}{2^N}\left[\binom{N}{j}+\binom{N}{j+q}+\binom{N}{j+2q}+\cdots\right] &= \frac{1}{q}\sum_{s=0}^{q-1}\left(\cos\frac{\pi s}{q}\right)^N \cos\frac{\pi(N-2j)s}{q} \\
&= \frac{1}{q}\left(1+\sum_{s=1}^{q-1}\left(\cos\frac{\pi s}{q}\right)^N \cos\frac{\pi(N-2j)s}{q}\right) \\
&= \frac{1}{q}\left(1+\mathrm{Err}\left[(q-1)\left(\cos\frac{\pi}{q}\right)^N\right]\right),
\end{aligned}$$

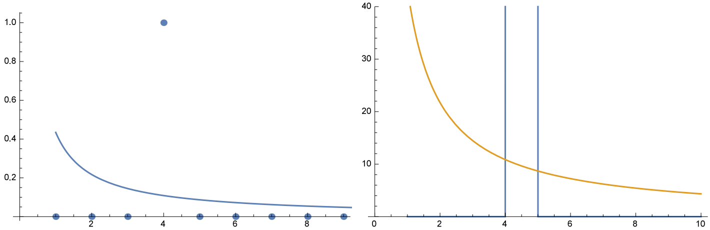
where $\mathrm{Err}[X]$ indicates an absolute error of size at most X

Examples



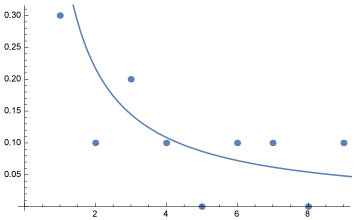
$p = 3/11$, 1000 levels; $y = \log_{10}(8/3) \notin \mathbb{Q}$ (irrational)

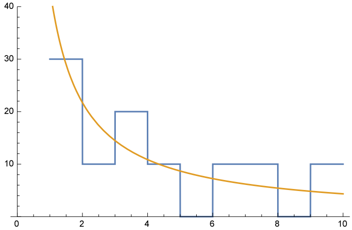
Examples



$p = 1/11$, 1000 levels; $y = 1 \in \mathbb{Q}$ (rational)

Examples





$p = 1/(1+10^{33/10})$, 1000 levels; $y = 33/10 \in \mathbb{Q}$ (rational)

Random Cuts

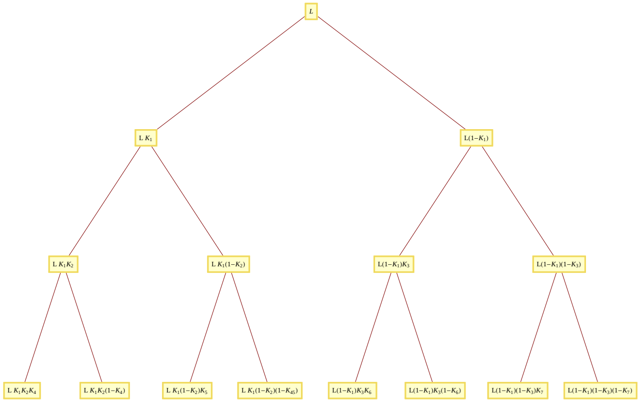


Figure: Unrestricted Decomposition: Breaking L into pieces, $N = 3$.

Benford Good Processes

- A. Kontorovich and S. J. Miller, *Benford's Law, values of L-functions and the* $3x + 1$ *problem*, Acta Arithmetica **120** (2005), no. 3, 269–297.

Poisson Summation and Benford's Law: Definitions

- Feller, Pinkham (often exact processes)
- data $Y_{T,B} = \log_B \overrightarrow{X}_T$ (discrete/continuous):

$$\mathbb{P}(A) \ = \ \lim_{T\to\infty} \frac{\#\{n \in A : n \le T\}}{T}$$

- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\overrightarrow{Y}_{T,B}}(y) \ = \ \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) \ := \ G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

- Small tails: $G_T(\infty) - G_T(Th(T)) = o(1)$, $G_T(-Th(T)) - G_T(-\infty) = o(1)$.
- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b + \ell) - E_T(a + \ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

- Examples
  - ◇ L-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x + 1$ problem
  - ◇ geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right)$$

$$= \sum_{|\ell| \leq Th(T)} [G_T(b+\ell) - G_T(a+\ell)] + o(1)$$

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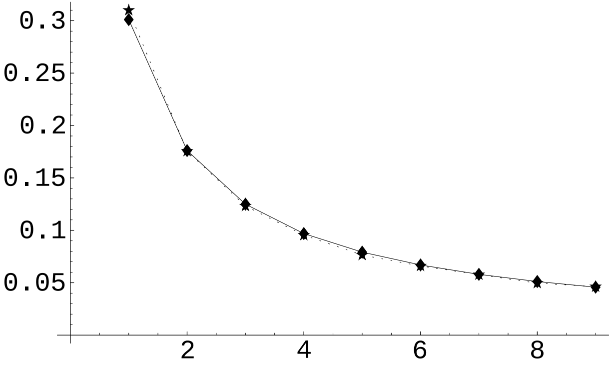
$$= \widehat{f}(0) \cdot (b-a) + \sum_{\ell \ne 0} \widehat{f}(T\ell) \frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i \ell} + o(1).$$

Riemann Zeta Function

$$\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$



The $3x+1$ Problem and Benford's Law

$3x + 1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1,1)$, 5-path $(1,1,2,3,4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

$$\begin{aligned} a_{n+1} &= T(a_n) \\ \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \\ &= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\ &= \log a_n + \log\left(\frac{3}{4}\right). \end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan, 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Structure Theorem: Sinai, Kontorovich-Sinai

$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\le N: n\equiv 1,5 \bmod 6, n\in A\}}{\#\{n\le N: n\equiv 1,5 \bmod 6\}}$.

$(k_1, \dots, k_m)$: two full arithm progressions:

$6 \cdot 2^{k_1+\cdots+k_m} p + q$.

Theorem (Sinai, Kontorovich-Sinai)

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Sketch of the proof of Benfordness

- Failed Proof: lattices, bad errors.

- CLT: $(S_m - 2m)/\sqrt{2m} \to N(0,1)$:

$$\mathbb{P}\left(S_m - 2m = k\right) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:
$I_\ell = \{\ell M, \dots, (\ell+1)M - 1\}$, $M = m^c$, $c < 1/2$
$k_1, k_2 \in I_\ell$: $\left|\eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$ small
$C = \log_B 2$ of irrationality type $\kappa < \infty$:
$$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ *algebraic numbers height* $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ *with height at most* $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ *then* $|\Lambda| > B^{-C\Omega \log \Omega'}$, *with* $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| \,/ q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} \left|N(b-a) - \#\{n \le N : x_n \in [a,b]\}\right|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left|\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n}\right|\right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\le \frac{1}{|\sin(\pi h\alpha)|} \le \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x + 1$ Data: random 10,000 digit number, $2^k || 3x + 1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

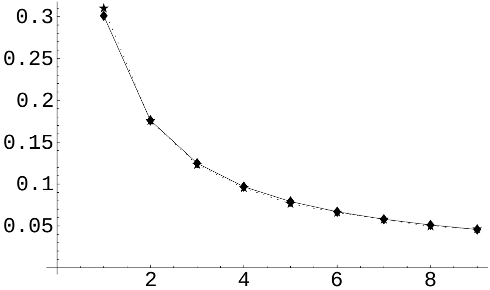
241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

The Riemann Zeta Function $\zeta(s)$ and Benford's Law

The Riemann Zeta Function and Benford's Law

$$\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}\tfrac{k}{4}\right)\right|,\ k\in\{0,1,\dots,65535\}.$$



First digits of $\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}\tfrac{k}{4}\right)\right|$ versus Benford's law.

Proof Sketch: ‘Good’ L-Functions

We say an L-function is *good* if:

- Euler product:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \alpha_{f,j}(p)p^{-s}\right)^{-1}.$$

- $L(s,f)$ has a meromorphic continuation to $\mathbb{C}$, is of finite order, and has at most finitely many poles (all on the line $\mathrm{Re}(s) = 1$).
- Functional equation:

$$e^{\mathrm{i}\omega} G(s) L(s,f) \;=\; e^{-\mathrm{i}\omega}\overline{G(1-\overline{s})L(1-\overline{s})},$$

where $\omega \in \mathbb{R}$ and

$$G(s) \;=\; Q^s \prod_{i=1}^{h} \Gamma(\lambda_i s + \mu_i)$$

with $Q, \lambda_i > 0$ and $\mathrm{Re}(\mu_i) \geq 0$.

Proof Sketch: 'Good' L-Functions (cont)

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$ we have

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} \;=\; \aleph \log\log x + c + O\left(\frac{1}{\log x}\right).$$

- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- If $N(\sigma, T)$ is the number of zeros ρ of $L(s)$ with $\mathrm{Re}(\rho) \geq \sigma$ and $\mathrm{Im}(\rho) \in [0, T]$, then for some $\beta > 0$ we have

$$N(\sigma, T) \;=\; O\left(T^{1-\beta\left(\sigma - \frac{1}{2}\right)} \log T\right).$$

Known in some cases, such as $\zeta(s)$ and Hecke cuspidal forms of full level and even weight $k > 0$.

Log-Normal Law (Hejhal, Laurinčikas, Selberg)

Log-Normal Law

$$\frac{\mu\left(\{t \in [T, 2T] : \log|L(\sigma + it, f)| \in [a, b]\}\right)}{T} =$$

$$\frac{1}{\sqrt{\psi(\sigma, T)}} \int_a^b e^{-\pi u^2/\psi(\sigma,T)} du + \mathrm{Error}$$

$$\psi(\sigma, T) = \aleph \log\left[\min\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right)\right] + O(1)$$

$$\frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log^{\delta} T}, \quad \delta \in (0, 1).$$

Result: Values of L-functions and Benford's Law

Theorem (Kontorovich and M–, 2005)

$L(s,f)$ a good L-function, as $T \to \infty$, $L(\sigma_T + it, f)$ is Benford.

Ingredients

- Approximate $\log L(\sigma_T + it, f)$ with $\sum_{n \le x} \frac{c(n)\Lambda(n)}{\log n} \frac{1}{n^{\sigma_T + it}}$.
- study moments $\int_T^{2T} |\cdot|$, $k \le \log^{1-\delta} T$.
- Montgomery-Vaughan: $\int_T^{2T} \sum a_n n^{-it} \overline{\sum b_m m^{-it}} dt = H \sum a_n \overline{b}_n + O(1)\sqrt{\sum n|a_n|^2 \sum n|b_n|^2}$.

Results: Explicit L-Function Statement

Theorem (Kontorovich-Miller '05)

Let $L(s,f)$ be a good L-function. Fix a $\delta \in (0,1)$. For each T, let $\sigma_T = \frac{1}{2} + \frac{1}{\log^\delta T}$. Then as $T \to \infty$

$$\frac{\mu\left\{t \in [T, 2T] : \ M_B\left(|L(\sigma_T + \mathrm{i}t, f)|\right) \leq \tau\right\}}{T} \to \log_B \tau$$

Thus the values of the L-function satisfy Benford's Law in the limit for any base B.

Conclusions and References

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

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