

Benford's law, or: Why the IRS cares about number theory!

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Interesting Question

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1. In the Beginning

This problem provides an opportunity to learn more about the initial digits of the first 10,000 powers of 2 than it might seem reasonable to know.

Our list will start with $2^0 = 1$ and end with 2^{9999} . In this list, let us group together the blocks—the powers which have the same number of digits:

(1), (2,4,8), (16,32,64), (128,256,512), ...

Each of the blocks contains either three or four terms (since $2^3 < 10 < 2^4$). The three blocks above exhibit the following sequences of initial digits: 1-2-4-8, 1-3-6, 1-2-5. Though they do not occur for a while, there are two other sequences for initial digits within a block. What are they?

$2^{10,000}$ is the first power of 2 not on our list; it has 3011 digits and begins with the digit 1. (This can be verified using logarithms.) Our list, therefore, has exactly 3010 complete blocks—beginning with the block of four 1-digit powers of 2 and ending with the block of (four) 3010-digit powers of 2.

How many numbers in each block begin with the digit 1? _____

How many numbers on our list of 10,000 powers of 2 begin with the digit 1? _____

How many numbers on our list begin with either a 2 or a 3? _____

How many begin with a 4, 5, 6, or 7? _____

How many begin with an 8 or a 9? _____ How many begin with a 4? _____

How many begin with a 5, 6, or 7? _____

How many of the 3010 blocks contain four numbers? _____

How many numbers on our list end with the digit 2? _____

Plausible answers:

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Plausible answers: 10%, 11%

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Plausible answers: 10%, 11%, about 30%.

Summary

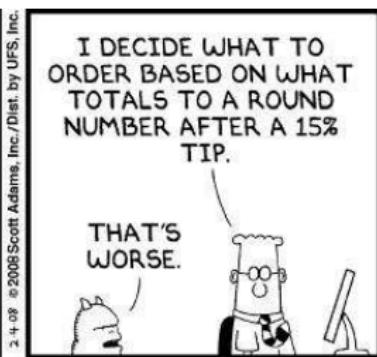
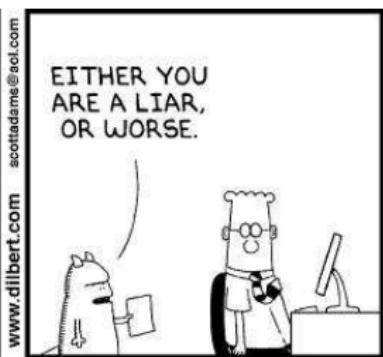
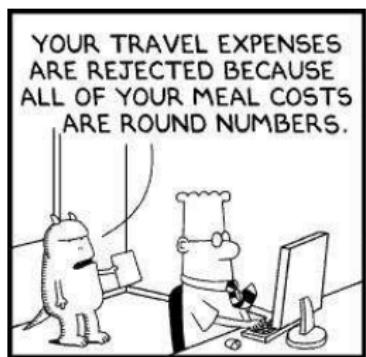
- State Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems..

Caveats!

- A math test indicating fraud is *not* proof of fraud:
unlikely events, alternate reasons.

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For many data sets, probability of observing a first digit of d base B is $\log_B \left(\frac{d+1}{d} \right)$; base 10 about 30% are 1s.

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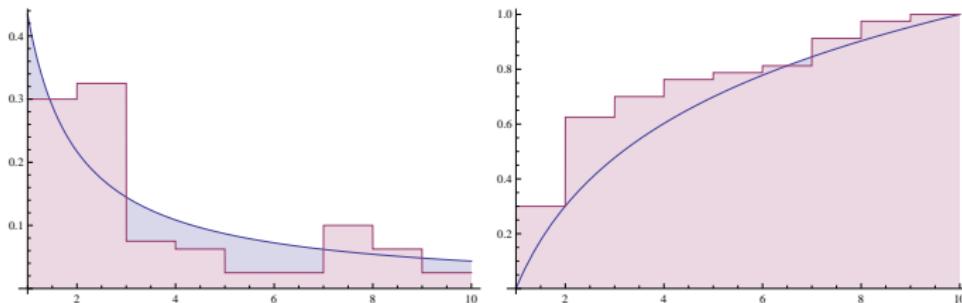
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 - Long street [1, L]: $L = 199$ versus $L = 999$.
 - Oscillates between 1/9 and 5/9 with first digit 1.
 - Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L -functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Bastille Day, Wikipedia and Google

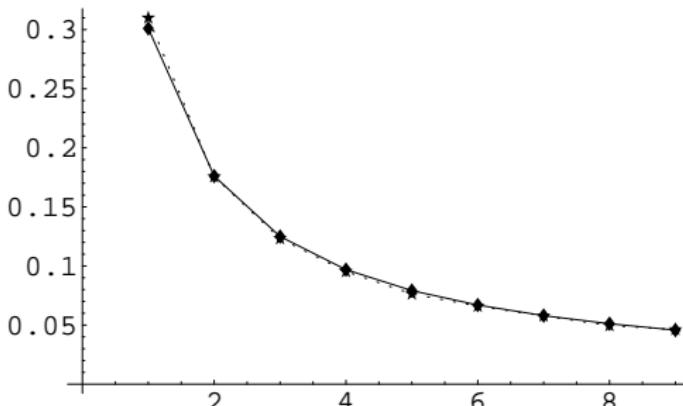


- First digit of number of hits Google returns for first 80 distinct words and numbers from Wikipedia's *Bastille Day* entry on July 10, 2011. Chisquare of 46.2, highly non-Benford (99% of time should be at most 20).

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\text{if } \operatorname{Re}(s) > 1).$$

$|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)|, k \in \{0, 1, \dots, 65535\}.$



Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

Mantissas

Mantissa: $x = M_{10}(x) \cdot 10^k$, k integer.

$M_{10}(x) = M_{10}(\tilde{x})$ if and only if x and \tilde{x} have the same leading digits.

Key observation: $\log_{10}(x) = \log_{10}(\tilde{x}) \bmod 1$ if and only if x and \tilde{x} have the same leading digits.
Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

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Proof: if rational: $2 = 10^{p/q}$.

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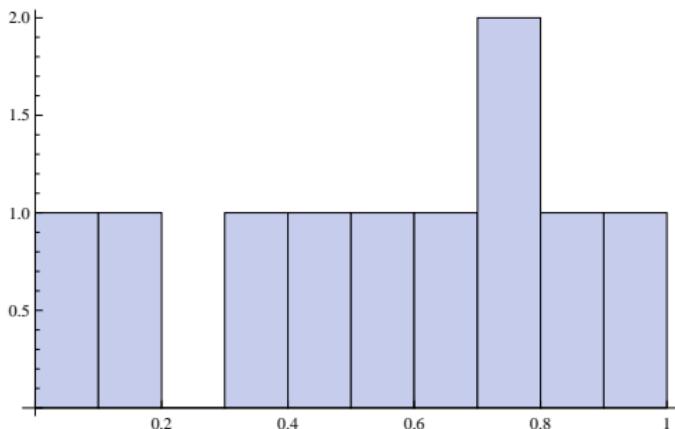
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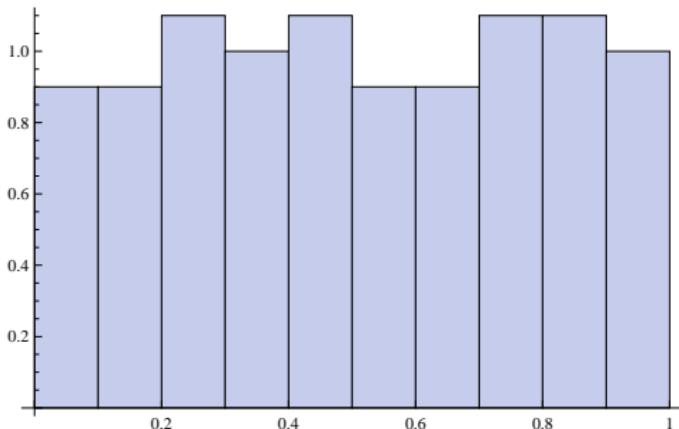
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Proof: if rational: $2 = 10^{p/q}$.
Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



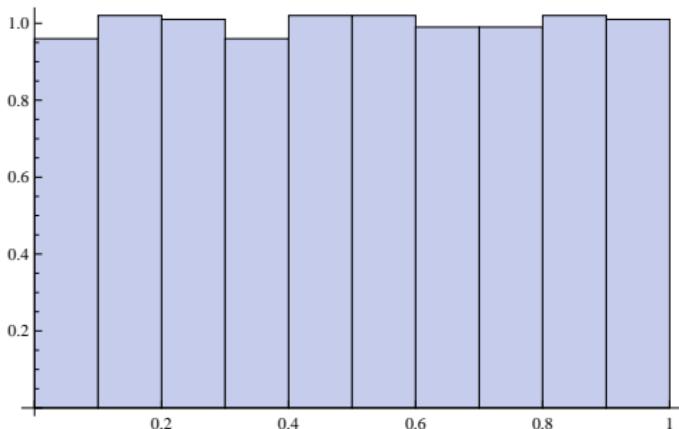
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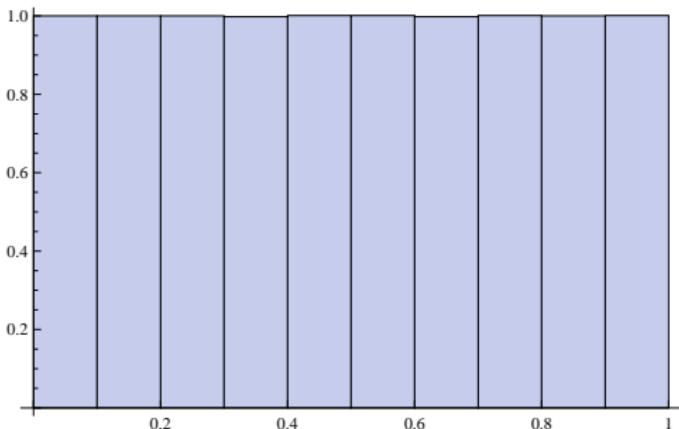
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$n\sqrt{\pi} \bmod 1$ for $n \leq 10,000$

Logarithms and Benford's Law

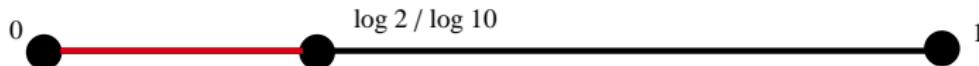
Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i..$

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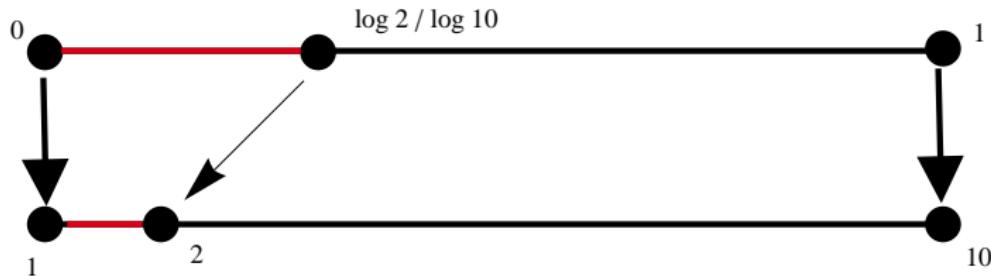
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Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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- Most linear recurrence relations Benford:

◊ $a_{n+1} = 2a_n - a_{n-1}$

◊ take $a_0 = a_1 = 1$ or $a_0 = 0, a_1 = 1$.

Digits of 2^n

• First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
512	524288	536870912				

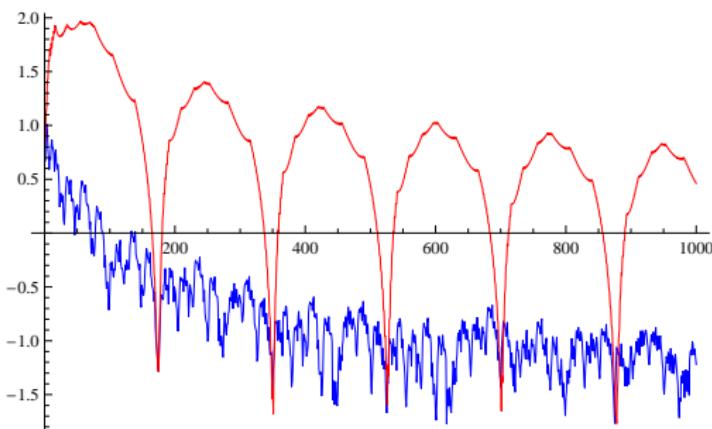
Logarithms and Benford's Law

χ^2 values for α^n , $1 \leq n \leq N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

Logarithms and Benford's Law: Base 10

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue),
 $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5%,
 $\log(\chi^2) \approx 2.74$).



Applications

Applications for the IRS: Detecting Fraud

Department of the Treasury-Internal Revenue Service		1040 U.S. Individual Income Tax Return 1989		OLIVER J. GUFF	
For the year ended December 31, 1988, or other tax year indicated		1988 ending	1988 ending	OMB No. 1545-0070	
Year taxes were last filed		Last name	Use name	Year model number issued	
WILLIAM J.		CLINTON		429-52-9947	
If a joint return, spouse's first name and initial				Community property no.	
HILARY		RODHEAN		351-20-2536	
Address where you resided at time of filing		St. & P.O. Box, see page 13	Line no.	For more information about Paperwork Reduction Act Notice, see instructions.	
1-800-CENTER		1-800-CENTER			
City, state or post office, box or ZIP code, if a foreign address, see page 13					
1-7100-6 ROCK		ARKANSAS 72205			
CLIN Presidential Campaign		Do you want \$1 to go to this fund?	Yes <input type="checkbox"/>	No <input type="checkbox"/> Charlie "T" or any other candidate or party you choose	
Do you want your spouse to do the same?		Yes <input type="checkbox"/>	No <input type="checkbox"/>		
Filing Status		Single			
1 Check only one box.		2 Married filing separate returns? If only one had income: Married filing separate returns. Enter spouse's social security number above. 3 Head of household with qualifying person. (See page 7 of instructions.) If the qualifying person is your child but not your dependent, enter child's name here. 4 Qualifying widow/widower with dependent child/tower son/daughter. Do not check if 1-5 (See page 7 of instructions.)			
Exemptions		6a <input checked="" type="checkbox"/> You're 65 or older in tax year prior to claim as a dependent or as claimed by another. Enter age for tax year 1988 6b <input checked="" type="checkbox"/> Spouse	7a If base income is less than \$6,000 and \$6,000 is not claimed as an ex- em- plo- yee or self- em- ployed in 1988 and you are not claiming any other ex- em- plo- yee or self- em- ployed as a de- pend- ent in 1988 then add this box to page 20 7b If base income is less than \$6,000 and \$6,000 is not claimed as an ex- em- plo- yee or self- em- ployed in 1988 and you are not claiming any other ex- em- plo- yee or self- em- ployed as a de- pend- ent in 1988 then add this box to page 20 8a <input checked="" type="checkbox"/> Spouse 8b <input checked="" type="checkbox"/> Dependents 9a <input checked="" type="checkbox"/> Head of household, tax year end 9b <input checked="" type="checkbox"/> Taxable income 10 <input checked="" type="checkbox"/> Qualifying widow/widower with dependent child/tower son/daughter 11 <input checked="" type="checkbox"/> Alimony received 12 <input checked="" type="checkbox"/> Capital gain or loss from sale of capital assets 13 <input checked="" type="checkbox"/> Capital gain distributions not reported on line 13 14 <input checked="" type="checkbox"/> Other gains or losses (except from Form 4797) 15 <input checked="" type="checkbox"/> Total IRA distributions 16a <input checked="" type="checkbox"/> Total pension and annuities 16b <input checked="" type="checkbox"/> Retirement savings plan contributions 17a <input checked="" type="checkbox"/> Rent, royalties, partnerships, estates, trusts, etc. (except Schedule A) 17b <input checked="" type="checkbox"/> Net farm income (except losses) 18 <input checked="" type="checkbox"/> Employment compensation (wages, salaries, tips, etc.) 19 <input checked="" type="checkbox"/> Social security benefits 20 <input checked="" type="checkbox"/> Other income (list type and amount) SEE STATEMENT B 21 <input checked="" type="checkbox"/> Add the amounts shown in lines 7 through 20. This is your total income 22 <input checked="" type="checkbox"/> Year 1988 deduction, from applicable worksheet on pages 14 or 15 23 <input checked="" type="checkbox"/> Spouse's 1988 deduction, from applicable worksheet on pages 14 or 15 24 <input checked="" type="checkbox"/> Self-employed health insurance deduction, if not mentioned on page 16 25 <input checked="" type="checkbox"/> Capital gains deduction and self-employed SEP deduction 26 <input checked="" type="checkbox"/> Premium paid for auto liability 27 <input checked="" type="checkbox"/> If no social security income 28 <input checked="" type="checkbox"/> Add lines 7 through 27 29 <input checked="" type="checkbox"/> Add lines 24 through 28. This is your adjusted gross income. If this is less than \$12,240, add a third worksheet, see "Reduced Income Credit" (See 8B) on page 16 of the instructions. If you used line 28, add lines 29 and 30 for your adjusted gross income 30 <input checked="" type="checkbox"/> Add lines 29 and 31 31 <input checked="" type="checkbox"/> Add lines 29 and 30 for your adjusted gross income 32 <input checked="" type="checkbox"/> Add lines 29 and 31 33 <input checked="" type="checkbox"/> Add lines 29 and 30 for your adjusted gross income 34 <input checked="" type="checkbox"/> Add lines 29 and 31 35 <input checked="" type="checkbox"/> Add lines 29 and 30 for your adjusted gross income 36 <input checked="" type="checkbox"/> Add lines 29 and 31 37 <input checked="" type="checkbox"/> Add lines 29 and 30 for your adjusted gross income 38 <input checked="" type="checkbox"/> Add lines 29 and 31 39 <input checked="" type="checkbox"/> Add lines 29 and 30 for your adjusted gross income 40 <input checked="" type="checkbox"/> Add lines 29 and 31 41 <input checked="" type="checkbox"/> Add lines 29 and 30 for your adjusted gross income 42 <input checked="" 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Applications for the IRS: Detecting Fraud

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

Bank Fraud

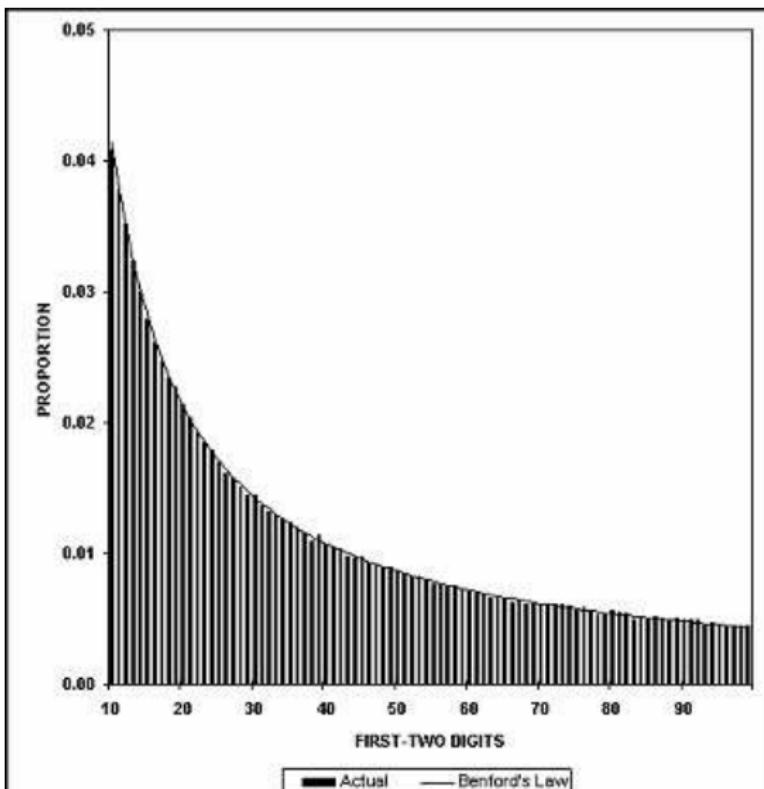
- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person..

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person..
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off..

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Election Fraud: Iran 2009

Numerous protests/complaints over Iran's 2009 elections..

Lot of analysis; data moderately suspicious:

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: enough tests, even if nothing wrong will find a suspicious result (but when all tests are on the boundary...).

The $3x + 1$ Problem and Benford's Law

3x + 1 Problem

- Kakutani (conspiracy), Erdős (not ready)...
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k \mid |3x + 1|$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.
- 7

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- $7 \rightarrow_1 11$

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- $7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \rightarrow_4 1 \rightarrow_2 1$,
2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
 m -path: (k_1, \dots, k_m) .

Heuristic Proof of 3x + 1 Conjecture

$$\begin{aligned}a_{n+1} &= T(a_n) \\ \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left(\frac{3a_n}{2^k} \right) \\ &= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\ &= \log a_n + \log \left(\frac{3}{4} \right).\end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

3x + 1 and Benford

Theorem (Kontorovich and M–, 2005)

As $m \rightarrow \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x + 1$ map are within $2N^{-1/36}$ of the Benford probabilities.

3x + 1 Data: random 10,000 digit number, $2^k \mid 3x + 1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319);
 $\chi^2 = 13.5$ (5% 15.5)...

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

3x + 1 Data: random 10,000 digit number, 2|3x + 1

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5)..

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

5x + 1 Data: random 10,000 digit number, $2^k \mid 5x + 1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

5x + 1 Data: random 10,000 digit number, 2|5x + 1

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Products of Random Variables

Preliminaries

- $X_1 \cdots X_n \Leftrightarrow Y_1 + \cdots + Y_n \bmod 1$, $Y_i = \log_B X_i$
- Density Y_i is g_i , density $Y_i + Y_j$ is

$$(g_i * g_j)(y) = \int_0^1 g_i(t)g_j(y - t)dt.$$

- $h_n = g_1 * \cdots * g_n$, $\widehat{g}(\xi) = \widehat{g}_1(\xi) \cdots \widehat{g}_n(\xi)$.

Modulo 1 Central Limit Theorem

Theorem (M– and Nigrini 2007)

$\{Y_m\}$ independent continuous random variables on $[0, 1]$ (not necc. i.i.d.), densities $\{g_m\}$.

$Y_1 + \cdots + Y_M \bmod 1$ converges to the uniform distribution as $M \rightarrow \infty$ in $L^1([0, 1])$ if and only if for all $n \neq 0$, $\lim_{M \rightarrow \infty} \widehat{g}_1(n) \cdots \widehat{g}_M(n) = 0$.

- ◊ Gives info on rate of convergence.

Proof under stronger conditions

- Use standard CLT to show $Y_1 + \cdots + Y_M$ tends to a Gaussian.
- Use Poisson Summation to show the Gaussian tends to the uniform modulo 1.

Proof under stronger conditions

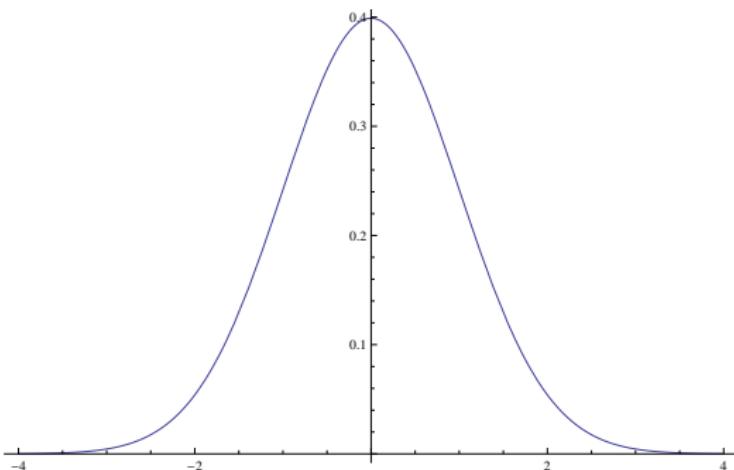


Figure: Plot of normal (mean 0, stdev 1).

Proof under stronger conditions

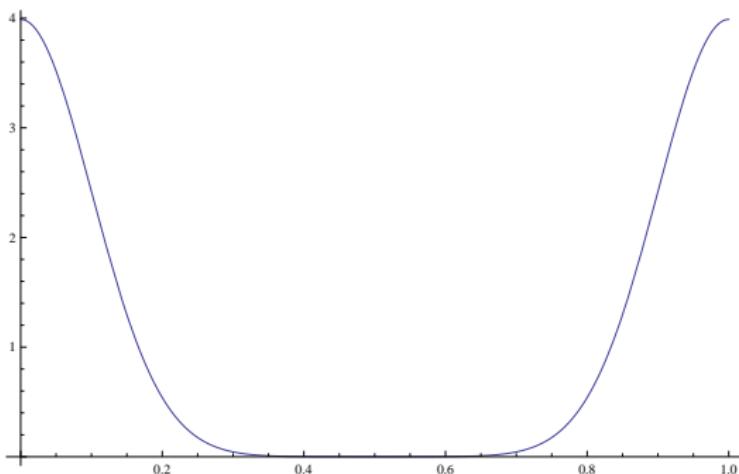


Figure: Plot of normal (mean 0, stdev .1) modulo 1.

Proof under stronger conditions

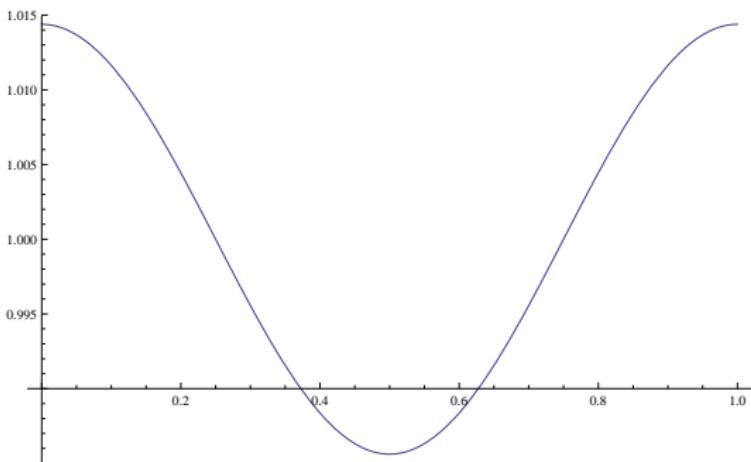


Figure: Plot of normal (mean 0, stdev .5) modulo 1.

Introduction
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General Theory
ooooooo

Applications
oooo

$3x + 1$
oooo

Products \mathcal{F}
ooo

Conclusions
oo

Refs

Conclusions

Current / Future Investigations

- Develop more sophisticated tests for fraud.
- Study digits of other systems.
 - ◊ Break rod of fixed length a variable number of times.
 - ◊ Break rods of variable length a variable number of times.
 - ◊ Break rods of variable length, each piece then breaks with given probability.
 - ◊ Break rods of variable integer length, each piece breaks until is a prime, or a square,

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

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