Benford's Law and Multinomial Distributions in Stick Fragmentation

Ella Lippelman
Colorado College
e_lippelman@coloradocollege.edu

Co-Author: Ava Irons
Advisor: Steven J. Miller
Williams College

YMC – Ohio State
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Summary

- Benford's Law

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- Fixed Proportion Decomposition Model

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- Fixed Proportion Decomposition Model
- Conjecture and Future Work

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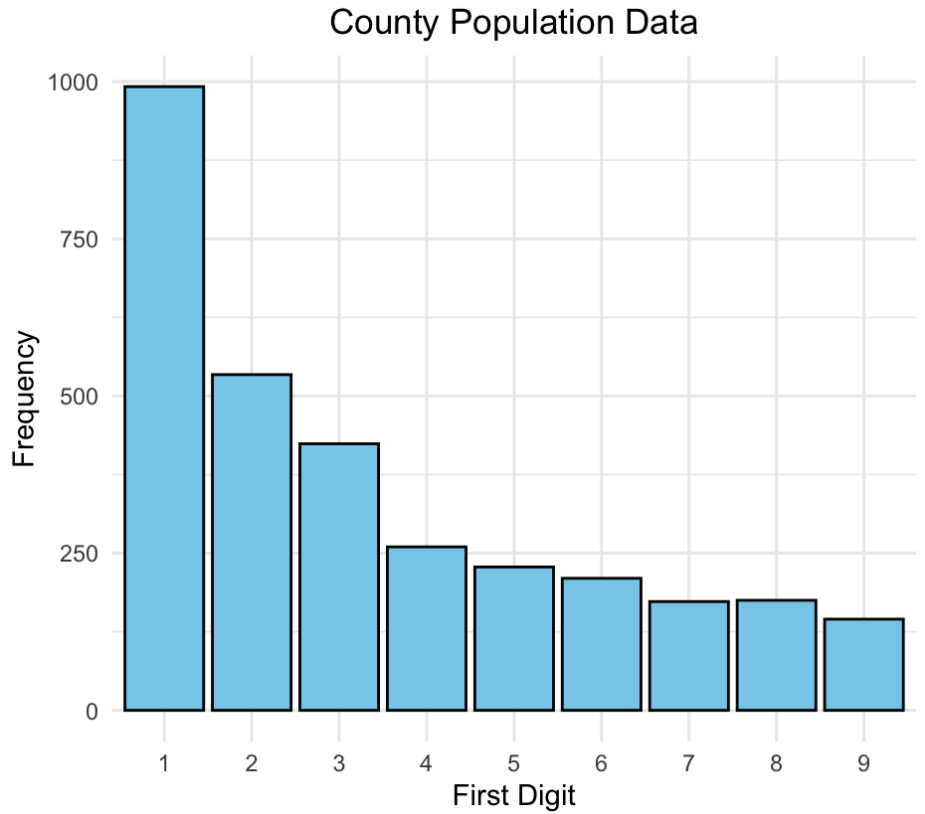

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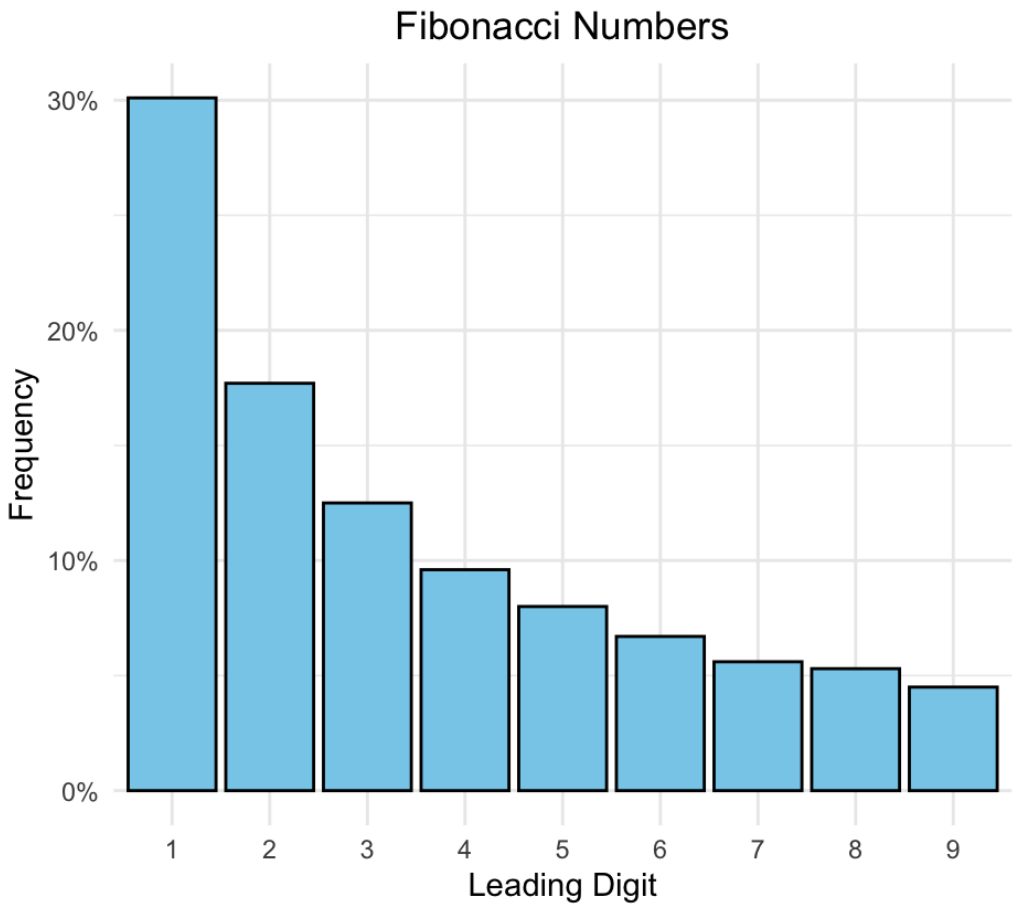
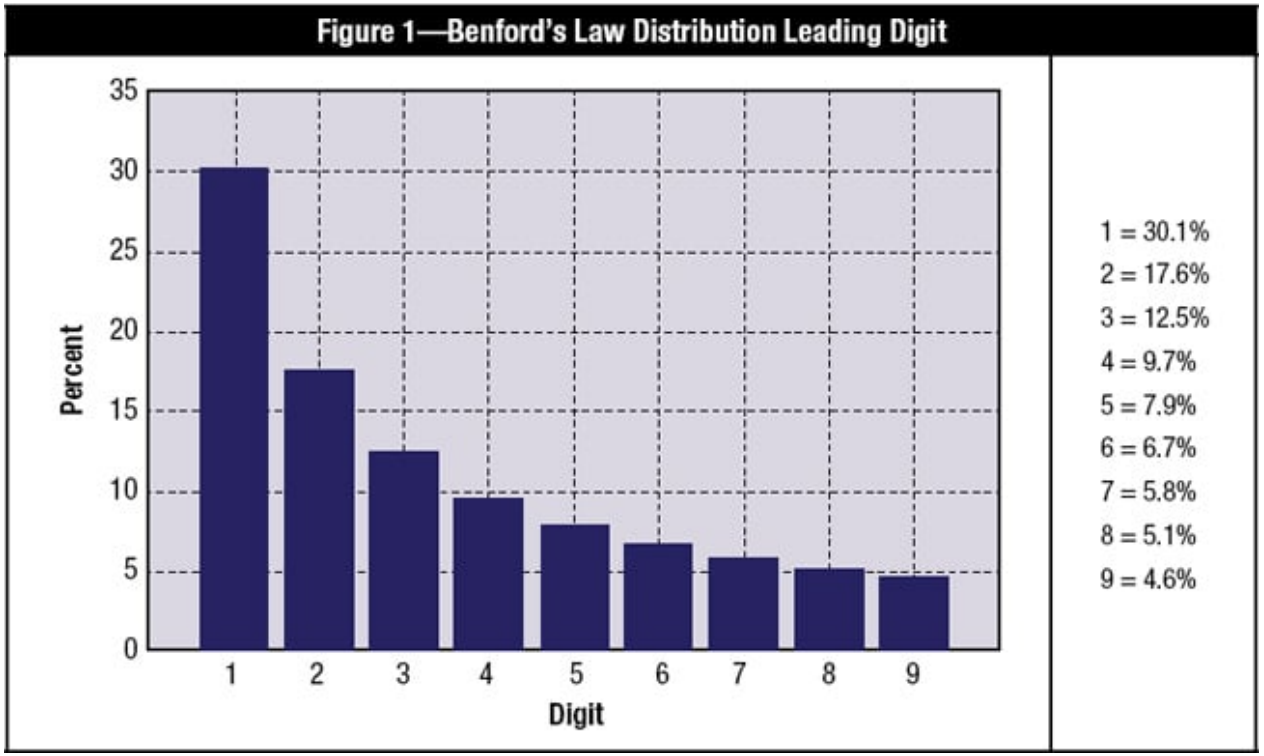


Benford's Law

Definition

A dataset is said to follow Benford's Law (base b) if the probability of observing a first digit of d is $\log_b \frac{1+d}{d}$.



Benford's Law and Logarithms Mod 1

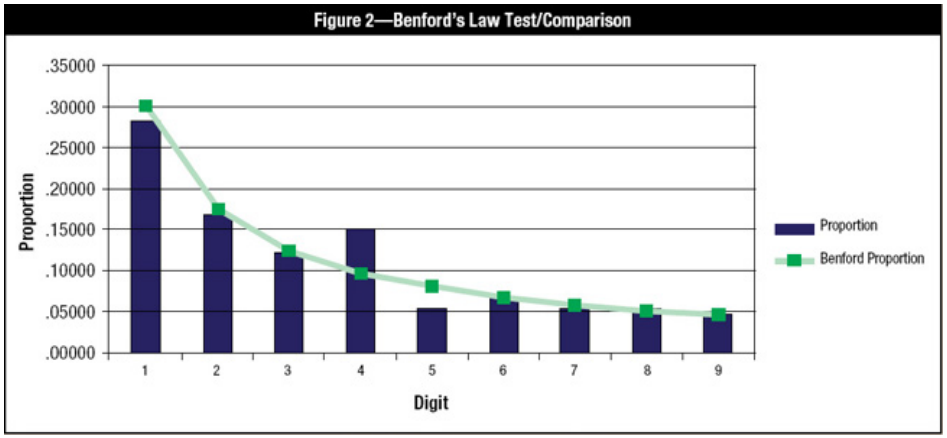
Uniform Distribution Modulo 1

A sequence (x_n) of real numbers is uniformly distributed modulo 1 if $\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \langle x_n \rangle \le s\}}{N} = s$ for all $s \in [0, 1)$.

Uniform Characterization Theorem

A sequence of real numbers is Benford base 10 if and only if the base 10 logarithms of its absolute values is uniformly distributed modulo 1.

Applications of Benford's Law



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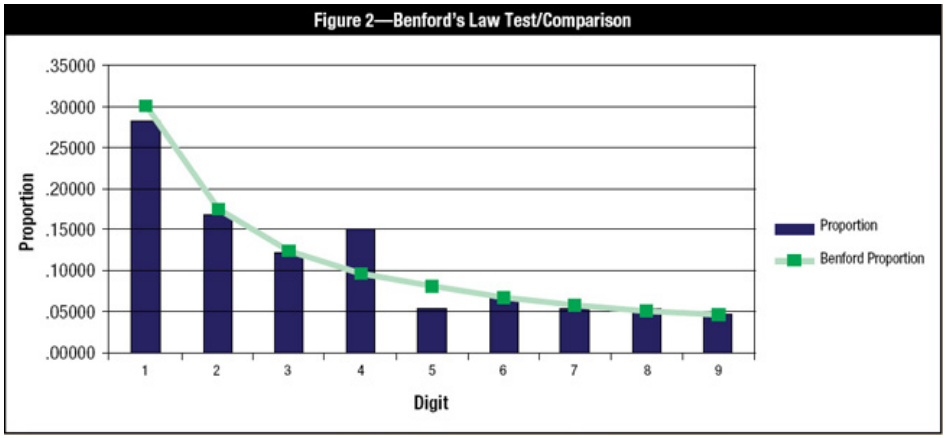

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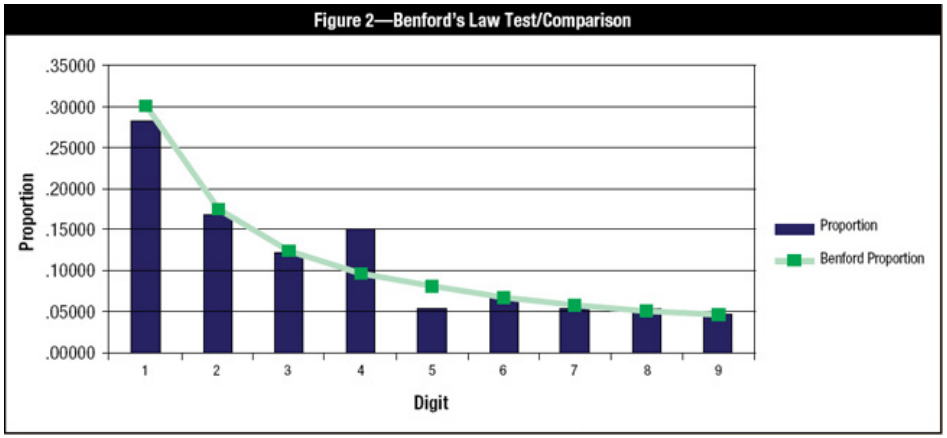

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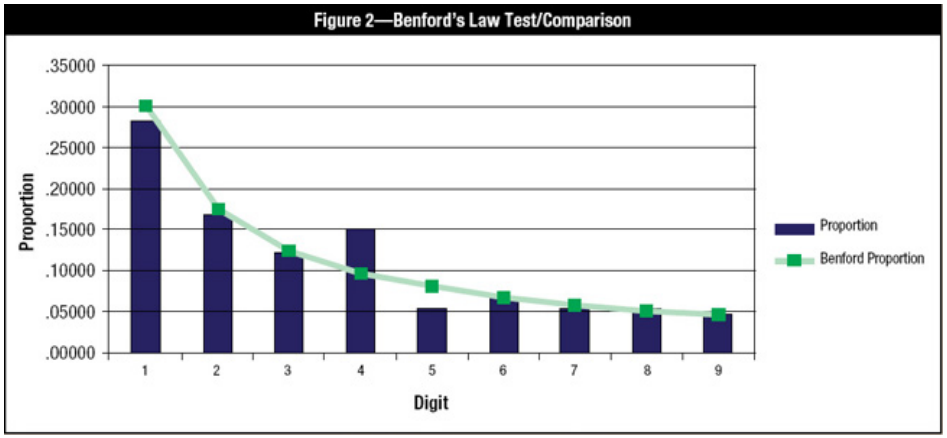

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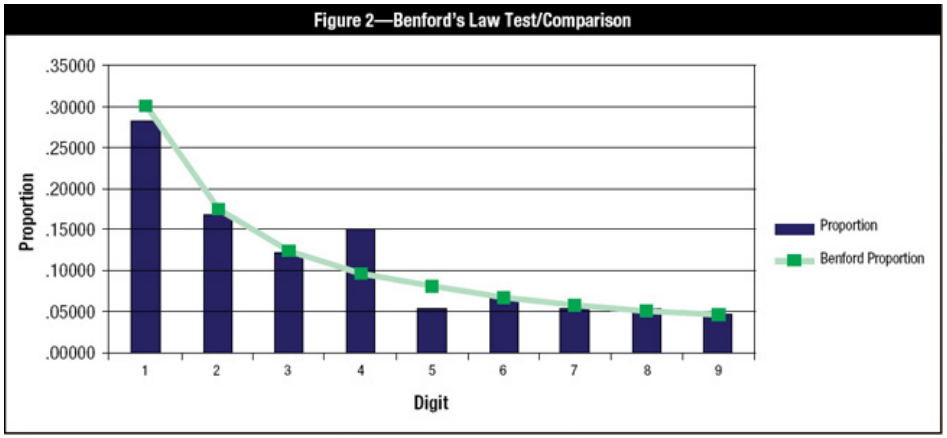

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- Lemons: fragmentation of conserved quantities.

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T. Becker, D. Burt, T. C. Corcoran, A. Greaves-Tunnell, J. R. Iafrate, J. Jing, S. J. Miller, J. D. Porfilio, R. Ronan, J. Samranvedhya, B. Talbut and F. W. Strauch (Becker et. al)

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- 2 sticks after first cut
- N iterations, results in 2^N sticks with $N + 1$ distinct lengths

Fixed Proportion Decomposition

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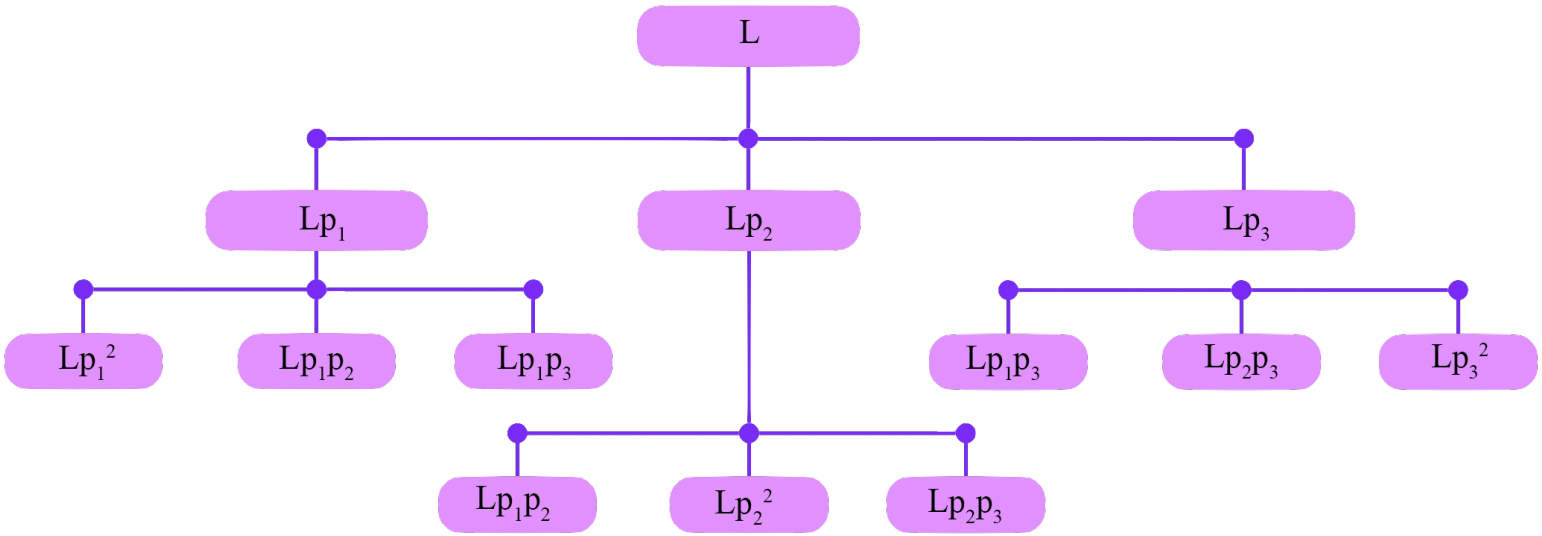
4. Repeat the process for N iterations, which results in
   - m^N sticks
   - $\binom{m+N-1}{N}$ distinct lengths for generic p_i

Fixed Proportion Decomposition Process Example



Example: Decomposition for $m = 3$ and $N = 2$.

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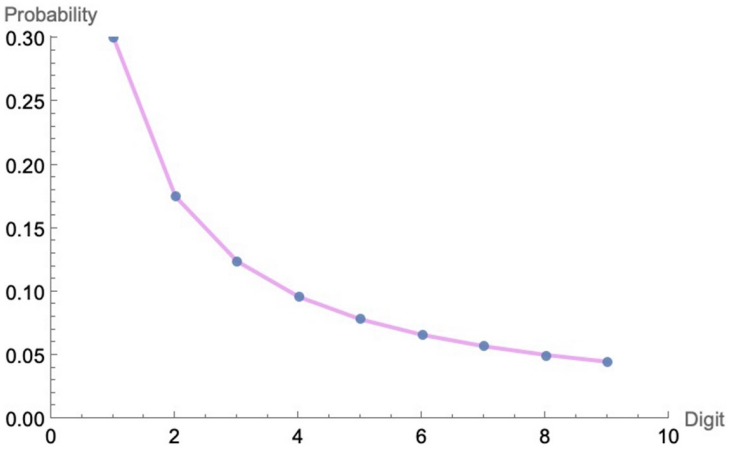
Multinomial Coefficient

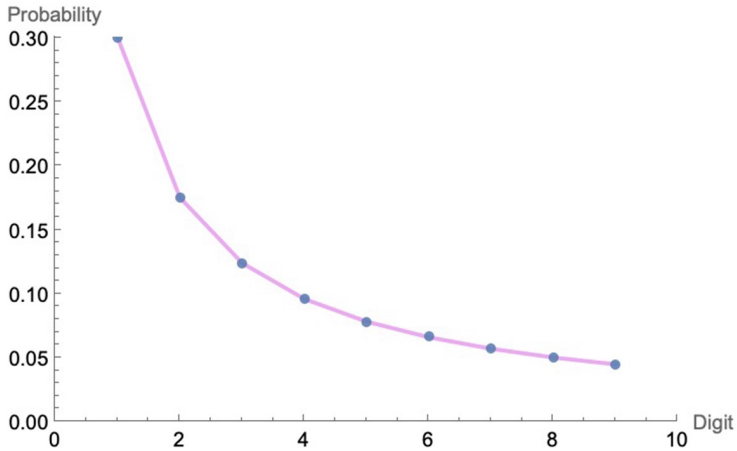
For any nonnegative integer N and positive integer m, we define the multinomial coefficient, for non-negative k summing to N, by $\binom{N}{k_1,k_2,\ldots,k_m} = \frac{N!}{k_1!k_2!\cdots k_m!}$.

Multinomial Theorem

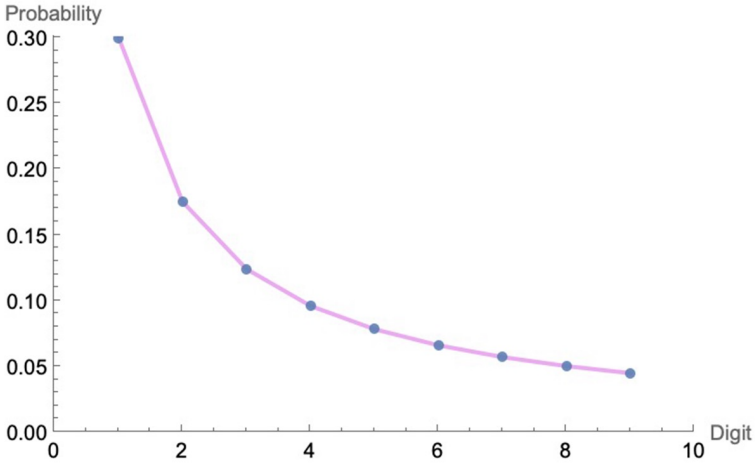
Let N be any non-negative integer and $p_1, p_2, \ldots, p_m$ real numbers. Then $(p_1 + p_2 + \cdots + p_m)^N = \sum_{k_1+k_2+\cdots+k_m\geq 0} \binom{N}{k_1,k_2,\ldots,k_m} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, where the k_j's are non-negative integers and 0^0 is interpreted as unity.



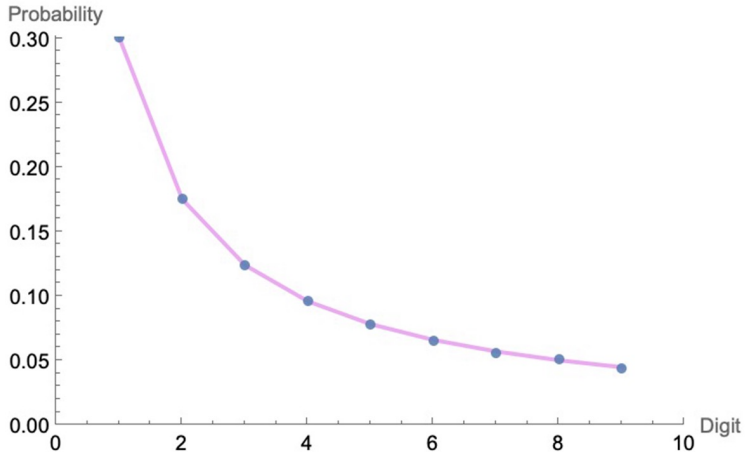
$p_1 = 0.05, p_2 = 0.16, N = 1000$



$p_1 = 0.07, p_2 = 0.34, N = 1000$



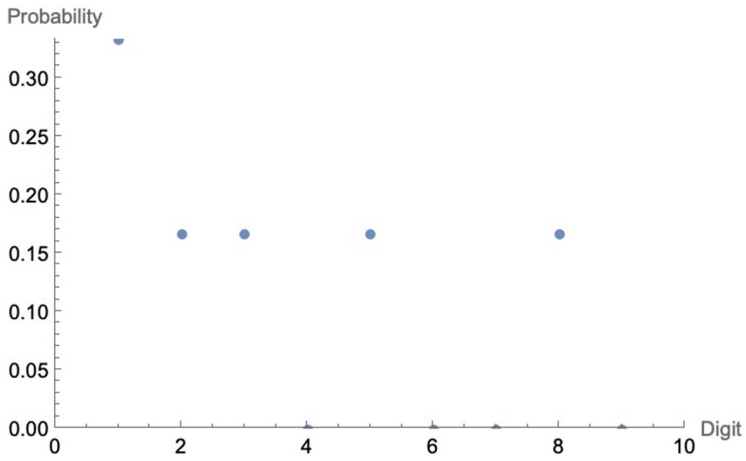
$p_1 = 0.11, p_2 = 0.23, N = 1000$



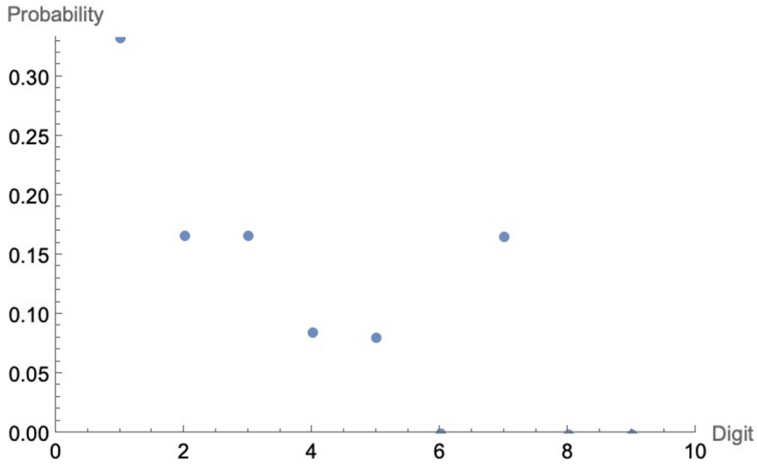
$p_1 = 0.13, p_2 = 0.3, N = 1000$

Theorem (Irons-Lippelman-Miller)

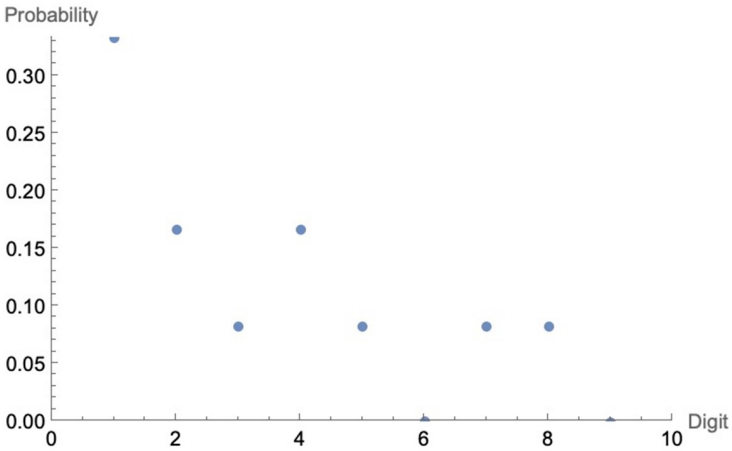
Consider the above described decomposition process. Choose $y_1, y_2, \ldots, y_{m-1}$ such that $10^{y_1} = p_1/p_2$, $10^{y_2} = p_2/p_3, \ldots, 10^{y_{m-1}} = p_{m-1}/p_m$, which are the ratios of adjacent lengths. The decomposition process results in a distribution of stick lengths that do not follow Benford's Law if all $y_1, y_2, \ldots, y_{m-1} \in \mathbb{Q}$.



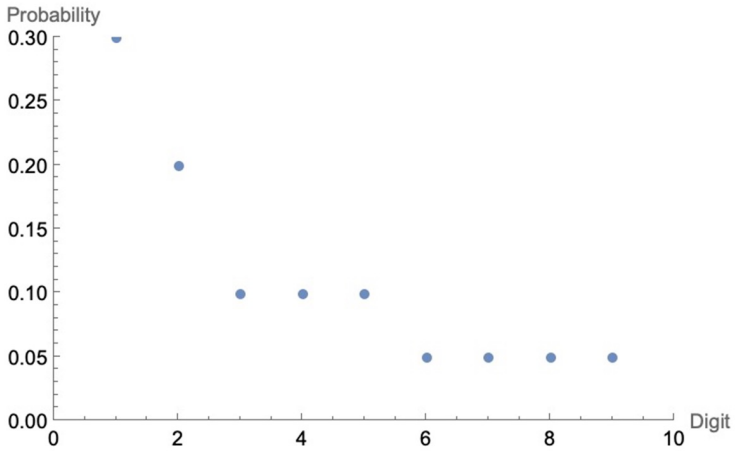
$p_1 \approx 0.1003, p_2 \approx 0.2161, y_1, y_2 \in \mathbb{Q}, N = 1000$



$p_1 \approx 0.0911, p_2 \approx 0.2881, y_1, y_2 \in \mathbb{Q}, N = 500$



$p_1 \approx 0.1856, p_2 \approx 0.33, y_1, y_2 \in \mathbb{Q}, N = 1000$



$p_1 \approx 0.1851, p_2 \approx 0.2933, y_1, y_2 \in \mathbb{Q}, N = 500$

$y_1, y_2, \ldots, y_{m-1} \in \mathbb{Q}$

Break a stick $m - 1$ times at distinct proportions $p_1, p_2, \ldots, p_{m-1}$.

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The frequency of lengths follows the m-multinomial distribution, as the lengths are given by

$$p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m},$$

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where k_j is the number of times the stick was cut at proportion p_i,

$$0 \leq k_1 \leq N, 0 \leq k_2 \leq N - k_1, \ldots, 0 \leq k_m \leq N - \sum_{j=1}^{m-1} k_j.$$

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Let $q_i = \sum_{j=1}^{i} k_j$. The above equation becomes

$$\left(\frac{p_1}{p_2}\right)^{q_1} \cdot \left(\frac{p_2}{p_3}\right)^{q_2} \cdot \left(\frac{p_3}{p_4}\right)^{q_3} \cdots \left(\frac{p_{m-1}}{p_m}\right)^{q_{m-1}} \cdot (p_m)^N .$$

$y_1, y_2, \ldots, y_{m-1} \in \mathbb{Q}$

Let $10^{y_i} = p_i/p_{i+1}$, where $y_i \in \mathbb{Q}$. Then there must exist $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{N}$ where $\gcd(a_i, b_i) = 1$ for all i where $y_i = a_i/b_i$.

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Substituting in we obtain

$$(p_m)^N \cdot \prod_{i=1}^{m-1} (10^{y_i})^{q_i} = (p_m)^N \cdot \prod_{i=1}^{m-1} \left(10^{\frac{a_i}{b_i}}\right)^{q_i}.$$

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Thus, for $1 \leq i \leq m-1$, the significands of $10^{a_i/b_i}$ repeat every b_i indices.

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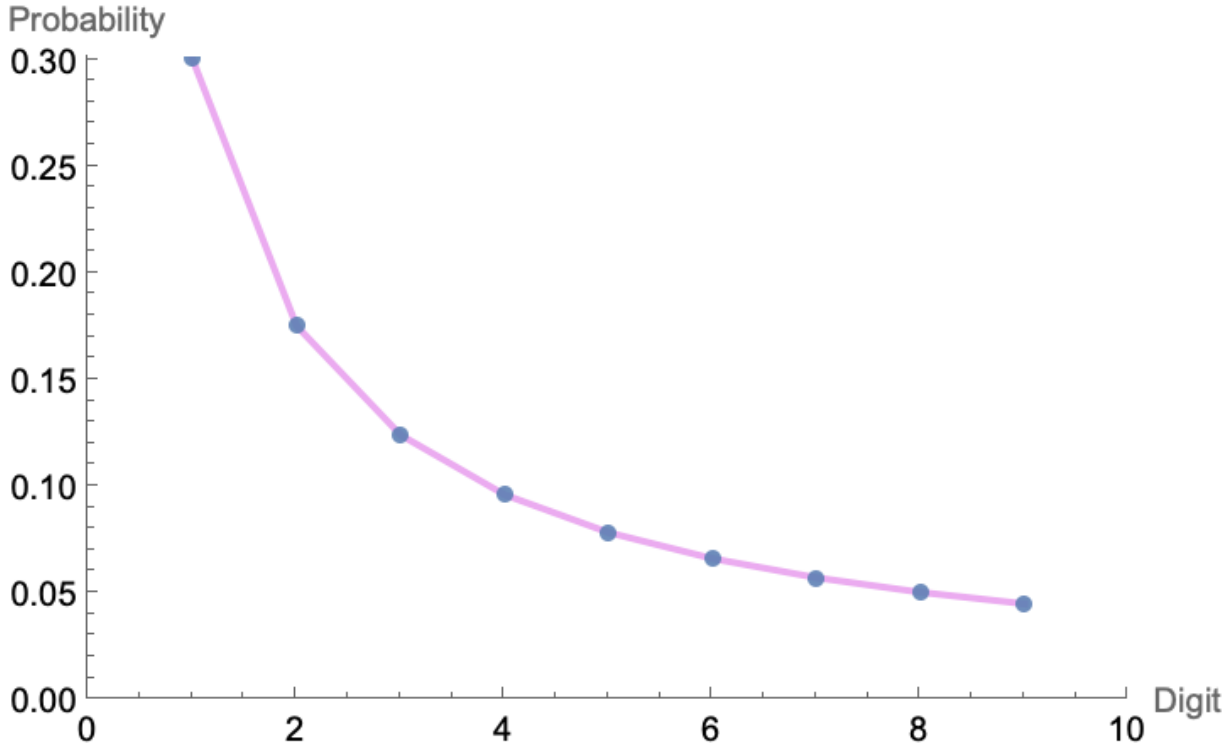
Thus, by the Uniform Characterization Theorem, the distribution of lengths cannot follow Benford's Law.

Conjecture, Future Work, and Conclusions

Conjecture

Choose $y_1, y_2, \ldots, y_{m-1}$ such that $10^{y_1} = p_1/p_2$, $10^{y_2} = p_2/p_3, \ldots, 10^{y_{m-1}} = p_{m-1}/p_m$, which is the ratio of adjacent lengths. If $y_1, y_2, \ldots, y_{m-1}$ are all irrational, then the distribution of stick lengths follow Benford's Law.



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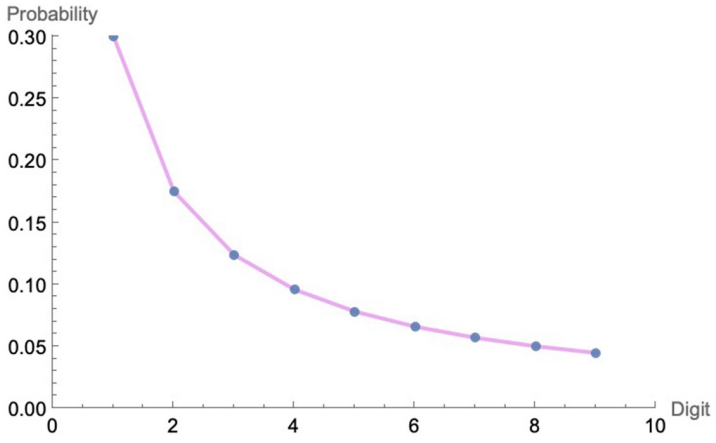
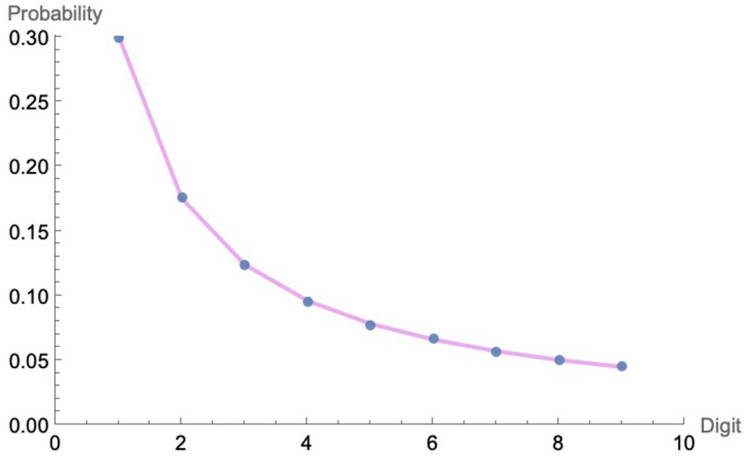
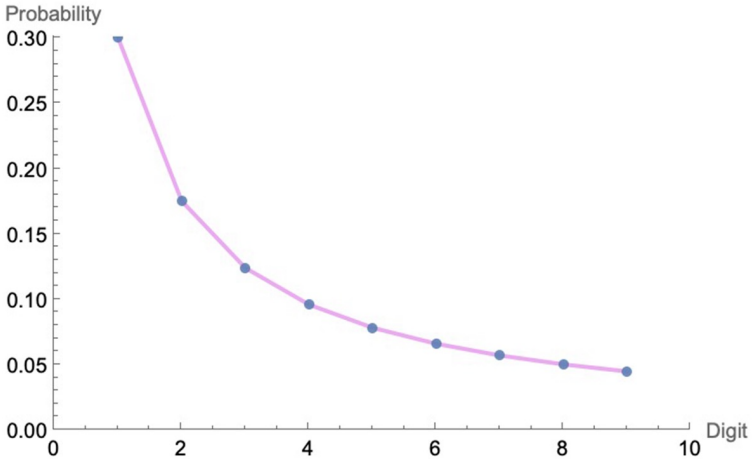


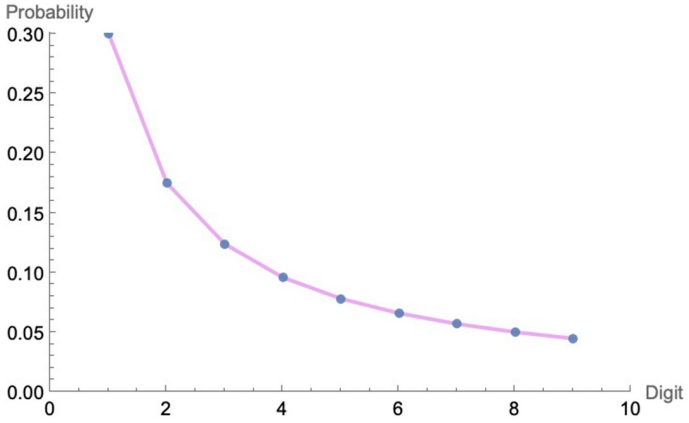
$p_1 = 0.1, p_2 = 0.3, y_1, y_2 \notin \mathbb{Q}, N = 1000$



$p_1 = 0.1, p_2 = 0.25, y_1, y_2 \notin \mathbb{Q}, N = 1000$

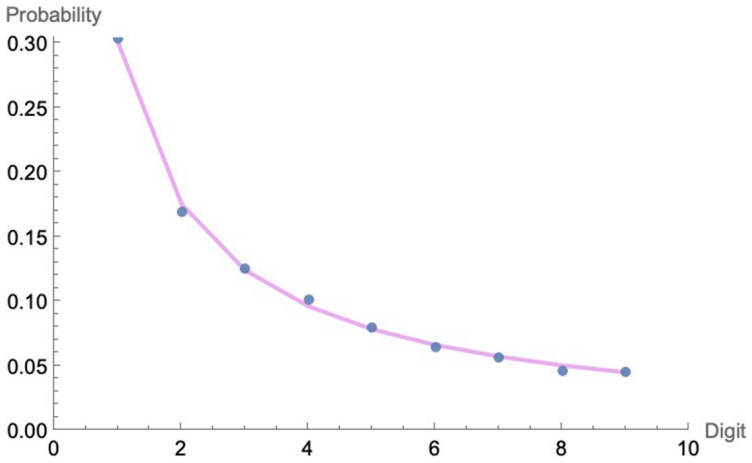


$p_1 = 0.09, p_2 = 0.39, y_1, y_2 \notin \mathbb{Q}, N = 1000$

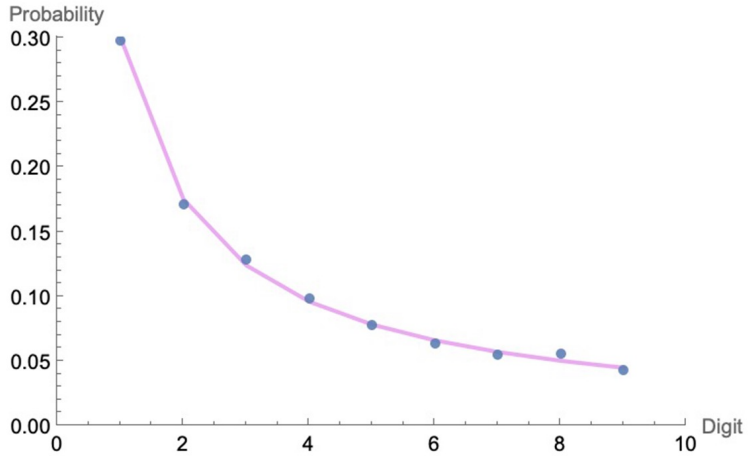


$p_1 = 0.05, p_2 = 0.25, y_1, y_2 \notin \mathbb{Q}, N = 1000$

Exploring the mixed case where some $y_i \in \mathbb{Q}$ and some y_i are irrational.



$p_1 \approx 0.0058, p_2 \approx 0.3152, y_1 \notin \mathbb{Q}, y_2 \in \mathbb{Q}, N = 1000$



$p_1 \approx 0.1160, p_2 \approx 0.3667, y_1 \in \mathbb{Q}, y_2 \notin \mathbb{Q}, N = 1000$

Future Work

- Prove Conjecture.

Future Work

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- Handling mixed cases where some y_i are irrational and some $y_i \in \mathbb{Q}$.

Conclusions

- Independent and Dependent Random Variables

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Conclusions

- Independent and Dependent Random Variables
- Fixed Proportion Decomposition
  - $y_1, y_2, \ldots, y_{m-1} \in \mathbb{Q}$, not Benford
- Future Work
  - $y_1, y_2, \ldots, y_{m-1}$ are all irrational
  - some y_i are irrational and some $y_i \in \mathbb{Q}$
  - other models

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