Poisson Summation and Benford's Law: From values of L-functions to the $3x + 1$ problem to products of random variables

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Santa Fe, NM, December 2007

Summary

- Review Benford's Law (statement, applications, proofs).

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    - (ii) Mellin transforms

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- Applications of Poisson Summation:
  - ◇ L-functions, RMT, $3x+1$.
  - ◇ Products Random Variables
    - (i) Fejér series
    - (ii) Mellin transforms
  - ◇ Sharp errors: $X_i \sim \mathrm{Exp}(1)$, $\Xi_n = X_1 \cdots X_n$, error $\le .057^n \log_{10} s$.

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$.

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
512	524288	536870912				

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x+1$ map
- differences of order statistics
- hydrology and financial data

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax fraud and data integrity

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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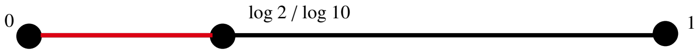

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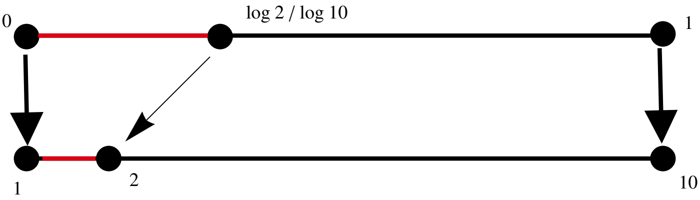

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Proof:

- $x = M_B(x) \cdot B^k$ for some $k \in \mathbb{Z}$.
- $\mathrm{FD}_B(x) = d$ iff $d \leq M_B(x) < d + 1$.
- $\log_B d \leq y < \log_B(d+1)$, $y = \log_B x$ mod 1.
- If $Y \sim \mathrm{Unif}(0, 1)$ then above probability is $\log_B\left(\frac{d+1}{d}\right)$.

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Kronecker-Weyl Theorem

If $\beta \notin \mathbb{Q}$ then $n\beta$ mod 1 is equidistributed. (Thus if $\log_B \alpha \notin \mathbb{Q}$, then α^n is Benford.)

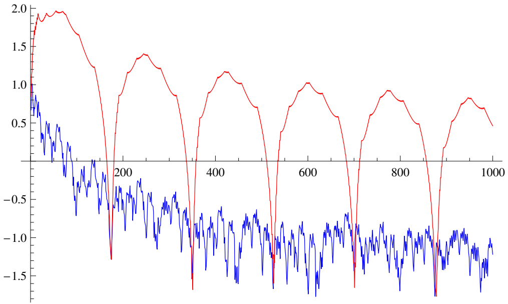
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

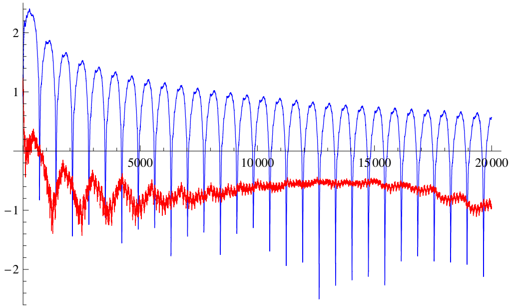
Logarithms and Benford's Law: Base 10

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1,\dots,N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5%, $\log(\chi^2) \approx 2.74$).



Logarithms and Benford's Law: Base 20

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \ldots, N\}$. Note $e^3 \approx 20.0855$, (5%, $\log(\chi^2) \approx 2.74$).



Benford Good Processes

Poisson Summation and Benford's Law: Definitions

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- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\overrightarrow{Y}_{T,B}}(y) = \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) := G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

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- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b+\ell) - E_T(a+\ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

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- Examples
  - ◇ L-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x + 1$ problem
  - ◇ geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
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  - ◇ apply Poisson summation

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  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right)$$

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$$\begin{aligned}
&\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right) \\
&= \sum_{|\ell|\leq Th(T)} \left[G_T(b+\ell) - G_T(a+\ell)\right] + o(1) \\
&= \int_a^b \sum_{|\ell|\leq Th(T)} \frac{1}{T} f\left(\frac{t}{T}\right) dt + \mathcal{E}(a,b,T) + o(1) \\
&= \widehat{f}(0)\cdot(b-a) + \sum_{\ell\neq 0} \widehat{f}(T\ell)\frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i\ell} + o(1).
\end{aligned}$$

L-functions and Random Matrix Theory

Good L-functions ($\zeta(s)$, full level cusp form)

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- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely many poles (all on the line $\mathrm{Re}(s) = 1$).
- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ prod Γ-fns:

$$e^{\mathrm{i}\omega} G(s) L(s,f) = e^{-\mathrm{i}\omega} \overline{G(1-\overline{s}) L(1-\overline{s})}.$$

Good L-functions

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$:

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} = \aleph \log\log x + c + O\left(\frac{1}{\log x}\right).$$

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- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- $N(\sigma, T) = \#\{\rho : L(\rho, f) = 0, \mathrm{Re}(\rho) \geq \sigma, \mathrm{Im}(\rho) \in [0, T]\}$. $\exists \beta > 0$

$$N(\sigma, T) = O\left(T^{1-\beta\left(\sigma - \frac{1}{2}\right)} \log T\right).$$

Log-Normal Law (Hejhal, Laurinčikas, Selberg

Log-Normal Law

$$\frac{\mu\left(\{t \in [T, 2T] : \log|L(\sigma + it, f)| \in [a, b]\}\right)}{T} =$$

$$\frac{1}{\sqrt{\psi(\sigma, T)}} \int_a^b e^{-\pi u^2/\psi(\sigma,T)} du + \text{Error}$$

$$\psi(\sigma, T) = \aleph \log\left[\min\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right)\right] + O(1)$$

$$\frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log^\delta T}, \quad \delta \in (0, 1).$$

Values of L-functions and Benford's Law

Theorem (Kontorovich and M–, 2005)

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Ingredients

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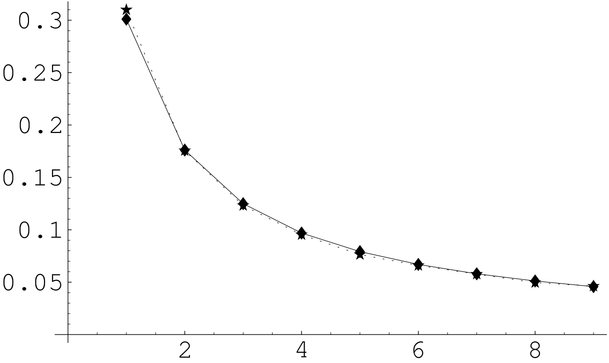
- Montgomery-Vaughan: $\int_T^{2T} \sum a_n n^{-it} \overline{\sum b_m m^{-it}} dt = H \sum a_n \overline{b}_n + O(1)\sqrt{\sum n|a_n|^2 \sum n|b_n|^2}$.

Riemann Zeta Function

$$\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$



Random Matrices: Preliminaries

- $N \times N$ unitary matrices U (Haar measure):

$$p_N(U) = \frac{1}{(2\pi)^N N!} \prod_{1 \le j < m \le N} \left| e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_m} \right|.$$

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- $\rho_N(x)$ the probability density for $\log|Z(U,\theta)|$:

$$\widetilde{\rho}_N(x) \;=\; \sqrt{Q_2(N)}\,\rho_N(\sqrt{Q_2(N)}\,x),$$

variance $Q_2(N) \sim (\log N)/2$.

Random Matrices and Benford's Law

Theorem (Kontorovich and M–, 2005)

As $N \to \infty$, the distribution of digits of the absolute values of the characteristic polynomials of $N \times N$ unitary matrices (with respect to Haar measure) converges to the Benford probabilities.

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- Key Ingredient: Keating-Snaith:

$$\widetilde{\rho}_N(x)dx \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx + O\left((\log N)^{-3/2}dx\right).$$

The $3x + 1$ Problem and Benford's Law

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x+1$ Conjecture

$$a_{n+1} = T(a_n)$$

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Geometric Brownian Motion, drift $\log(3/4) < 1$.

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n \le N : n \equiv 1,5 \bmod 6, n \in A\}}{\#\{n \le N : n \equiv 1,5 \bmod 6\}}.$$

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Theorem (Sinai, Kontorovich-Sinai)

k_i-values are i.i.d.r.v. (geometric, 1/2):

$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{\sqrt{2m}} \le a\right) = \mathbb{P}\left(\frac{S_m - 2m}{\sqrt{2m}} \le a\right)$$

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\le N: n\equiv 1,5 \bmod 6, n\in A\}}{\#\{n\le N: n\equiv 1,5 \bmod 6\}}.$$

$(k_1, \dots, k_m)$: two full arithm progressions:
$6 \cdot 2^{k_1+\cdots+k_m} p + q$.

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$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

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  $C = \log_B 2$ of irrationality type $\kappa < \infty$:
  $\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa})$.

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$,
$\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with
$d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$,
$\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| \, / q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} \left| N(b-a) - \#\{n \le N : x_n \in [a,b]\} \right|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Products
of
Random Variables

Preliminaries

- $X_1 \cdots X_n \Leftrightarrow Y_1 + \cdots + Y_n \bmod 1$, $Y_i = \log_B X_i$
- Density Y_i is g_i, density $Y_i + Y_j$ is

$$(g_i * g_j)(y) \ = \ \int_0^1 g_i(t) g_j(y-t) dt.$$

- $h_n = g_1 * \cdots * g_n$, $\widehat{g}(\xi) = \widehat{g}_1(\xi) \cdots \widehat{g}_n(\xi)$.
- Dirac delta functional: $\int \delta_\alpha(y) g(y) dy = g(\alpha)$.

Fourier input

- Fejér kernel:

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi inx}.$$

- Fejér series:

$$T_N f(x) = (f * F_N)(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \widehat{f}(n) e^{2\pi inx}.$$

- Lebesgue's Theorem: $f \in L^1([0,1])$. As $N \to \infty$, $T_N f$ converges to f in $L^1([0,1])$.
- $T_N(f * g) = (T_N f) * g$: convolution assoc.

Modulo 1 Central Limit Theorem

Theorem (M– and Nigrini 2007)

$\{Y_m\}$ *independent continuous random variables on* $[0,1)$ *(not necc. i.i.d.), densities* $\{g_m\}$*.* $Y_1 + \cdots + Y_M \bmod 1$ *converges to the uniform distribution as* $M \to \infty$ *in* $L^1([0,1])$ *iff* $\forall n \neq 0$, $\lim_{M\to\infty} \widehat{g_1}(n) \cdots \widehat{g_M}(n) = 0$.

Generalizations

- Levy proved for i.i.d.r.v. just one year after Benford's paper.
- Generalized to other compact groups, with estimates on the rate of convergence.
  ◇ Stromberg: n-fold convolution of a regular probability measure on a compact Hausdorff group G converges to normalized Haar measure in weak-star topology iff support of the distribution not contained in a coset of a proper normal closed subgroup of G.

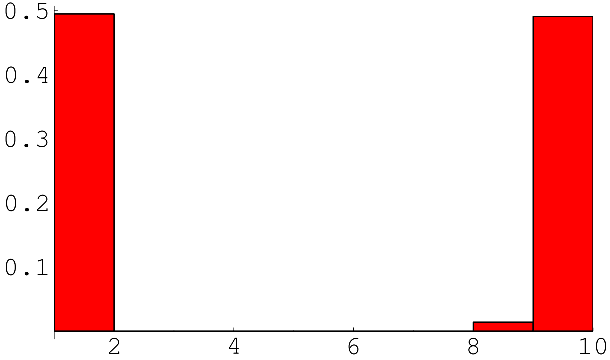
Theorem (M– and Nigrini 2007)

$\{Y_m\}$ indep. discrete random variables on $[0,1)$, not necc. identically distributed, densities

$$g_m(x) = \sum_{k=1}^{r_m} w_{k,m}\delta_{\alpha_{k,m}}(x),\ w_{k,m} > 0,\ \sum_{k=1}^{r_m} w_{k,m} = 1.$$

Assume that there is a finite set $A \subset [0,1)$ such that all $\alpha_{k,m} \in A$. $Y_1 + \cdots + Y_M \bmod 1$ converges weakly to the uniform distribution as $M \to \infty$ iff $\forall n \neq 0$, $\lim_{M\to\infty} \widehat{g_1}(n) \cdots \widehat{g_M}(n) = 0$.

Distribution of digits (base 10) of 1000 products $X_1 \cdots X_{1000}$, where $g_{10,m} = \phi_{11^m}$.
$\phi_m(x) = m$ if $|x - 1/8| \leq 1/2m$ (0 otherwise).



Proof of Modulo 1 CLT

- Density of sum is $h_\ell = g_1 * \cdots * g_\ell$.
- Suffices show $\forall \epsilon$: $\lim_{M\to\infty} \int_0^1 |h_M(x) - 1|dx < \epsilon$.
- Lebesgue's Theorem: N large,

$$||h_1 - T_N h_1||_1 = \int_0^1 |h_1(x) - T_N h_1(x)|dx < \frac{\epsilon}{2}.$$

- Claim: above holds for h_M for all M.

Proof of claim

$$T_N h_{M+1} = T_N(h_M * g_{M+1}) = (T_N h_M) * g_{M+1}$$

$$\begin{aligned}
||h_{M+1} - T_N h_{M+1}||_1 &= \int_0^1 |h_{M+1}(x) - T_N h_{M+1}(x)| dx \\
&= \int_0^1 |(h_M * g_{M+1})(x) - (T_N h_M) * g_{M+1}(x)| dx \\
&= \int_0^1 \left| \int_0^1 (h_M(y) - T_N h_M(y))\, g_{M+1}(x-y) \right| dydx \\
&\le \int_0^1 \int_0^1 |h_M(y) - T_N h_M(y)| g_{M+1}(x-y) dxdy \\
&= \int_0^1 |h_M(y) - T_N h_M(y)| dy \cdot 1 \; < \; \frac{\epsilon}{2}.
\end{aligned}$$

Proof of Modulo 1 CLT (continued)

Show $\lim_{M\to\infty} ||h_M - 1||_1 = 0$.
Triangle inequality:

$$||h_M - 1||_1 \le ||h_M - T_N h_M||_1 + ||T_N h_M - 1||_1.$$

Choices of N and ϵ:

$$||h_M - T_N h_M||_1 < \epsilon/2.$$

Show $||T_N h_M - 1||_1 < \epsilon/2$.

$$||T_N h_M - 1||_1 = \int_0^1 \left| \sum_{\substack{n=-N \\ n\neq 0}}^{N} \left(1 - \frac{|n|}{N}\right) \widehat{h_M}(n) e^{2\pi i n x} \right| dx$$

$$\leq \sum_{\substack{n=-N \\ n\neq 0}}^{N} \left(1 - \frac{|n|}{N}\right) |\widehat{h_M}(n)|$$

$\widehat{h_M}(n) = \widehat{g}_1(n) \cdots \widehat{g_M}(n) \longrightarrow_{M\to\infty} 0$.
For fixed N and ϵ, choose M large so that $|\widehat{h_M}(n)| < \epsilon/4N$ whenever $n \neq 0$ and $|n| \leq N$.

Where's the Poisson Summation?

- Mellin transform and Fourier transform related by logarithmic change of variable.

- Poisson summation from collapsing to modulo 1 random variables.

Preliminaries

- $\Xi_1, \dots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.

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◇ Proof: $\mathrm{Prob}(\Xi_1 \cdot \Xi_2 \in [0, x])$:

$$\begin{aligned} & \int_{t=0}^\infty \mathrm{Prob}\left(\Xi_2 \in \left[0, \frac{x}{t}\right]\right) f_1(t) dt \\ = & \int_{t=0}^\infty F_2\left(\frac{x}{t}\right) f_1(t) dt, \end{aligned}$$

differentiate.

Mellin Transform

$$\begin{aligned}(\mathcal{M}f)(s) &= \int_0^\infty f(x)x^s\frac{dx}{x}\\ (\mathcal{M}^{-1}g)(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)x^{-s}ds\end{aligned}$$

$$g(s) = (\mathcal{M}f)(s), f(x) = (\mathcal{M}^{-1}g)(x).$$

$$\begin{aligned}(f_1 \star f_2)(x) &= \int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t)\frac{dt}{t}\\ (\mathcal{M}(f_1 \star f_2))(s) &= (\mathcal{M}f_1)(s)\cdot(\mathcal{M}f_2)(s).\end{aligned}$$

Mellin Transform Formulation: Products Random Variables

Theorem

X_i's independent, densities f_i. $\Xi_n = X_1 \cdots X_n$,

$$\begin{aligned} h_n(x_n) &= (f_1 \star \cdots \star f_n)(x_n) \\ (\mathcal{M}h_n)(s) &= \prod_{m=1}^{n} (\mathcal{M}f_m)(s). \end{aligned}$$

As $n \to \infty$, Ξ_n becomes Benford: $Y_n = \log_B \Xi_n$,
$|\mathrm{Prob}(Y_n \bmod 1 \in [a,b]) - (b-a)| \leq$

$$(b-a) \cdot \sum_{\ell \neq 0, \ell=-\infty}^{\infty} \prod_{m=1}^{n} (\mathcal{M}f_i)\left(1 - \frac{2\pi i \ell}{\log B}\right).$$

Proof of Mellin formulation of product

$Y_n = \log_B \Xi_n$:

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Differentiate: $g_n(y) = h_n(B^y)B^y \log B$.

Poisson summation:

$$\sum_{\ell=-\infty}^{\infty} g_n(y+\ell) = \sum_{\ell=-\infty}^{\infty} e^{2\pi i y \ell} \widehat{g}_n(\ell)$$

where $\widehat{f}$ denotes the Fourier transform of f:

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Proof of the Mellin formulation for the product (continued)

Prob(Y_n mod 1 $\in [a, b]$) is

$$\sum_{\ell=-\infty}^{\infty} \int_{a+\ell}^{b+\ell} g_n(y)dy$$

$$= \int_a^b \sum_{\ell=-\infty}^{\infty} g_n(y+\ell)dy$$

Proof of the Mellin formulation for the product (continued)

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$$= \int_a^b \sum_{\ell=-\infty}^{\infty} e^{2\pi i y \ell}\widehat{g}_n(\ell)dy$$

$$= b-a+(b-a)\sum_{\ell\neq 0}|\widehat{g}_n(\ell)|.$$

Proof of the Mellin formulation for the product (continued)

Must show sum over ℓ tends to zero as $n \to \infty$.

$$\widehat{g}_n(\xi) = \int_{-\infty}^{\infty} g_n(y) e^{-2\pi i y \xi} dy$$

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$$\widehat{g}_n(\xi) = \int_{-\infty}^{\infty} g_n(y) e^{-2\pi i y \xi} dy$$

$$= \int_{-\infty}^{\infty} h_n(B^y) B^y \log B \cdot e^{-2\pi i y \xi} dy$$

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$$= \int_0^{\infty} h_n(t) t^{-2\pi i \xi / \log B} dt$$

$$= (\mathcal{M} h_n)\left(1 - \frac{2\pi i \xi}{\log B}\right) = \prod_{m=1}^{n} (\mathcal{M} f_m)\left(1 - \frac{2\pi i \xi}{\log B}\right).$$

Proof of Kossovsky's Chain Conjecture for certain densities

Conditions

- $\{\mathcal{D}_i(\theta)\}_{i\in I}$: one-parameter distributions, densities $f_{\mathcal{D}_i(\theta)}$ on $[0,\infty)$.

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- $m \geq 2$,

$$f_m(x_m) \;=\; \int_0^\infty f_{\mathcal{D}_{p(m)}(1)}\left(\frac{x_m}{x_{m-1}}\right) f_{m-1}(x_{m-1})\frac{dx_{m-1}}{x_{m-1}}$$

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-

$$\lim_{n\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} (\mathcal{M} f_{\mathcal{D}_{p(m)}(1)})\left(1 - \frac{2\pi i \ell}{\log B}\right) \;=\; 0$$

Proof of Kossovsky's Chain Conjecture for certain densities

Theorem (JKKKM)

- *If conditions hold, as $n \to \infty$ the distribution of leading digits of X_n tends to Benford's law.*
- *The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then*

$$\left|\mathrm{Prob}(Y_n \bmod 1 \in [a,b]) - (b+a)\right| \le$$

$$\left|(b-a)\cdot \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} (\mathcal{M} f_{\mathcal{D}_{p(m)}}(1)) \left(1 - \frac{2\pi i \ell}{\log B}\right)\right|$$

Example: All $X_i \sim \mathrm{Unif}(0, k)$

- $X_i \sim \mathrm{Unif}(0, k)$: without loss of generality $k \in [1, 10)$.
- $P_n(s) = \mathrm{Prob}(M_{10}(\Xi_n) \leq s)$.
- $|P_n(s) - \log_{10}(s)| \leq$

$$\frac{k}{s}\frac{(\log k)^{n-1}}{\Gamma(n)} + \left(\frac{1}{2.9^n} + \frac{\zeta(n) - 1}{2.7^n}\right) 2\log_{10} s.$$

Example: All $X_i \sim \mathrm{Exp}(1)$

- $X_i \sim \mathrm{Exp}(1)$, $Y_n = \log_B \Xi_n$.
- Needed ingredients:
  - ◇ $\int_0^\infty \exp(-x)x^{s-1}dx = \Gamma(s)$.
  - ◇ $|\Gamma(1+ix)| = \sqrt{\pi x / \sinh(\pi x)}, x \in \mathbb{R}$.
- $|P_n(s) - \log_{10}(s)| \le$

$$\log_B s \sum_{\ell=1}^{\infty} \left(\frac{2\pi^2\ell / \log B}{\sinh(2\pi^2\ell/\log B)} \right)^{n/2}.$$

Example: All $X_i \sim \mathrm{Exp}(1)$ (continued)

Bounds on the error

- $|P_n(s) - \log_{10} s| \le$
  - $\diamond$ $3.3 \cdot 10^{-3} \log_B s$ if $n = 2$,
  - $\diamond$ $1.9 \cdot 10^{-4} \log_B s$ if $n = 3$,
  - $\diamond$ $1.1 \cdot 10^{-5} \log_B s$ if $n = 5$, and
  - $\diamond$ $3.6 \cdot 10^{-13} \log_B s$ if $n = 10$.
- Error at most

$$\log_{10} s \sum_{\ell=1}^{\infty} \left(\frac{17.148\ell}{\exp(8.5726\ell)} \right)^{n/2} \le .057^n \log_{10} s$$

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