Benford’s Law: Why the IRS might care about the $3x + 1$ problem and $\zeta(s)$.

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Summary

- Review Benford’s Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.
Caveats!

- Not all fraud can be detected by Benford’s Law.
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- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.
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Benford’s Law: Newcomb (1881), Benford (1938)

Statement
For many data sets, probability of observing a first digit of $d$ base $B$ is $\log_B \left( \frac{d+1}{d} \right)$; base 10 about 30% are 1s.
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  - Long street \([1, L]\): \(L = 199\) versus \(L = 999\).
  - Oscillates between 1/9 and 5/9 with first digit 1.
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- Not all data sets satisfy Benford’s Law.
  - Long street \([1, L]\): \(L = 199\) versus \(L = 999\).
  - Oscillates between 1/9 and 5/9 with first digit 1.
  - Many streets of different sizes: close to Benford.
Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- $L$-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models
Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity
General Theory
Mantissas

\[ x = M_{10}(x) \cdot 10^k, \text{ } k \text{ integer.} \]
Mantissas

Mantissa: $x = M_{10}(x) \cdot 10^k$, $k$ integer.

$M_{10}(x) = M_{10}(\tilde{x})$ if and only if $x$ and $\tilde{x}$ have the same leading digits.
Mantissas

Mantissa: \( x = M_{10}(x) \cdot 10^k, \quad k \text{ integer.} \)

\( M_{10}(x) = M_{10}(\tilde{x}) \) if and only if \( x \) and \( \tilde{x} \) have the same leading digits.

**Key observation:** \( \log_{10}(x) = \log_{10}(\tilde{x}) \mod 1 \) if and only if \( x \) and \( \tilde{x} \) have the same leading digits. Thus often study \( y = \log_{10} x \).
Equidistribution and Benford’s Law

Equidistribution

\( \{y_n\}_{n=1}^\infty \) is equidistributed modulo 1 if probability 
\( y_n \mod 1 \in [a, b] \) tends to \( b - a \):

\[
\frac{\# \{ n \leq N : y_n \mod 1 \in [a, b] \}}{N} \rightarrow b - a.
\]
Equidistribution and Benford’s Law

**Equidistribution**

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- Thm: \( \beta \notin \mathbb{Q} \), \( n\beta \) is equidistributed mod 1.
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- Examples: \( \log_{10} 2, \log_{10} \left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q} \).
Equidistribution and Benford’s Law

**Equidistribution**

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**Proof:** if rational: \( 2 = 10^{p/q} \).
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  \textit{Proof}: if rational: \( 2 = 10^{p/q} \).
  
  Thus \( 2^q = 10^p \) or \( 2^{q-p} = 5^p \), impossible.
Example of Equidistribution: \( n\sqrt{\pi} \mod 1 \)
Example of Equidistribution: \( n\sqrt{\pi} \mod 1 \) for \( n \leq 100 \)
Example of Equidistribution: $n\sqrt{\pi} \mod 1$

$n\sqrt{\pi} \mod 1$ for $n \leq 1000$
Example of Equidistribution: $n\sqrt{\pi} \mod 1$ for $n \leq 10,000$
Denseness

Dense

A sequence \( \{z_n\}_{n=1}^{\infty} \) of numbers in \([0, 1]\) is dense if for any interval \([a, b]\) there are infinitely many \(z_n\) in \([a, b]\).

- **Dirichlet’s Box (or Pigeonhole) Principle:**
  If \(n + 1\) objects are placed in \(n\) boxes, at least one box has two objects.

- **Denseness of \(n\alpha\):**
  Thm: If \(\alpha \notin \mathbb{Q}\) then \(z_n = n\alpha \mod 1\) is dense.
**Proof** $n\alpha \mod 1$ dense if $\alpha \notin \mathbb{Q}$

- Enough to show in $[0, b]$ infinitely often for any $b$.
- Choose any integer $Q > 1/b$.
- $Q$ bins: $\left[0, \frac{1}{Q}\right], \left[\frac{1}{Q}, \frac{2}{Q}\right], \ldots, \left[\frac{Q-1}{Q}, Q\right]$.
- $Q + 1$ objects: $\{\alpha \mod 1, 2\alpha \mod 1, \ldots, (Q + 1)\alpha \mod 1\}$.
- Two in same bin, say $q_1\alpha \mod 1$ and $q_2\alpha \mod 1$.
- Exists integer $p$ with $0 < q_2\alpha - q_1\alpha - p < \frac{1}{Q}$.
- Get $(q_2 - q_1)\alpha \mod 1 \in [0, b]$. 
Logarithms and Benford’s Law

**Fundamental Equivalence**

Data set \( \{x_i\} \) is Benford base \( B \) if \( \{y_i\} \) is equidistributed mod 1, where \( y_i = \log_B x_i \).
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Proof:

- \( x = M_B(x) \cdot B^k \) for some \( k \in \mathbb{Z} \).
- \( \text{FD}_B(x) = d \) iff \( d \leq M_B(x) < d + 1 \).
- \( \log_B d \leq y < \log_B(d + 1) \), \( y = \log_B x \mod 1 \).
- If \( Y \sim \text{Unif}(0, 1) \) then above probability is \( \log_B \left( \frac{d+1}{d} \right) \).
Examples

- $2^n$ is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$. 
Examples

- Fibonacci numbers are Benford base 10.
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\[ a_{n+1} = a_n + a_{n-1}. \]
Fibonacci numbers are Benford base 10.

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Roots \( r = \left(1 \pm \sqrt{5}\right)/2 \).
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Binet: \( a_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n \).
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- Most linear recurrence relations Benford:
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- Most linear recurrence relations Benford:
  
  \( \Diamond \) \( a_{n+1} = 2a_n \)
Fibonacci numbers are Benford base 10.

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Most linear recurrence relations Benford:

\[ \diamond a_{n+1} = 2a_n - a_{n-1} \]
Examples

- **Fibonacci numbers** are Benford base 10.
  \[ a_{n+1} = a_n + a_{n-1}. \]
  
  Guess \( a_n = n^r: \) \( r^{n+1} = r^n + r^{n-1} \) or \( r^2 = r + 1. \)
  
  Roots \( r = (1 \pm \sqrt{5})/2. \)
  
  General solution: \( a_n = c_1 r_1^n + c_2 r_2^n. \)
  
  Binet: \( a_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n. \)

- **Most linear recurrence relations** Benford:
  - \( a_{n+1} = 2a_n - a_{n-1} \)
  - Take \( a_0 = a_1 = 1 \) or \( a_0 = 0, \) \( a_1 = 1. \)
## Digits of $2^n$

First 60 values of $2^n$ (only displaying 30)

<table>
<thead>
<tr>
<th>1</th>
<th>1024</th>
<th>1048576</th>
<th>digit</th>
<th>#</th>
<th>Obs Prob</th>
<th>Benf Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2048</td>
<td>2097152</td>
<td>1</td>
<td>18</td>
<td>.300</td>
<td>.301</td>
</tr>
<tr>
<td>4</td>
<td>4096</td>
<td>4194304</td>
<td>2</td>
<td>12</td>
<td>.200</td>
<td>.176</td>
</tr>
<tr>
<td>8</td>
<td>8192</td>
<td>8388608</td>
<td>3</td>
<td>6</td>
<td>.100</td>
<td>.125</td>
</tr>
<tr>
<td>16</td>
<td>16384</td>
<td>16777216</td>
<td>4</td>
<td>6</td>
<td>.100</td>
<td>.097</td>
</tr>
<tr>
<td>32</td>
<td>32768</td>
<td>33554432</td>
<td>5</td>
<td>6</td>
<td>.100</td>
<td>.079</td>
</tr>
<tr>
<td>64</td>
<td>65536</td>
<td>67108864</td>
<td>6</td>
<td>4</td>
<td>.067</td>
<td>.067</td>
</tr>
<tr>
<td>128</td>
<td>131072</td>
<td>134217728</td>
<td>7</td>
<td>2</td>
<td>.033</td>
<td>.058</td>
</tr>
<tr>
<td>256</td>
<td>262144</td>
<td>268435456</td>
<td>8</td>
<td>5</td>
<td>.083</td>
<td>.051</td>
</tr>
<tr>
<td>512</td>
<td>524288</td>
<td>536870912</td>
<td>9</td>
<td>1</td>
<td>.017</td>
<td>.046</td>
</tr>
</tbody>
</table>
Data Analysis

- $\chi^2$-Tests: Test if theory describes data
  - Expected probability: $p_d = \log_{10} \left( \frac{d+1}{d} \right)$.
  - Expect about $Np_d$ will have first digit $d$.
  - Observe $\text{Obs}(d)$ with first digit $d$.
  - $\chi^2 = \sum_{d=1}^{9} \frac{\left( \text{Obs}(d) - Np_d \right)^2}{Np_d}$.
  - Smaller $\chi^2$, more likely correct model.

- Will study $\gamma^n$, $e^n$, $\pi^n$. 
Logarithms and Benford’s Law

$\chi^2$ values for $\alpha^n$, $1 \leq n \leq N$ (5% 15.5).

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\chi^2(\gamma)$</th>
<th>$\chi^2(e)$</th>
<th>$\chi^2(\pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.72</td>
<td>0.30</td>
<td>46.65</td>
</tr>
<tr>
<td>200</td>
<td>0.24</td>
<td>0.30</td>
<td>8.58</td>
</tr>
<tr>
<td>400</td>
<td>0.14</td>
<td>0.10</td>
<td>10.55</td>
</tr>
<tr>
<td>500</td>
<td>0.08</td>
<td>0.07</td>
<td>2.69</td>
</tr>
<tr>
<td>700</td>
<td>0.19</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>800</td>
<td>0.04</td>
<td>0.03</td>
<td>6.19</td>
</tr>
<tr>
<td>900</td>
<td>0.09</td>
<td>0.09</td>
<td>1.71</td>
</tr>
<tr>
<td>1000</td>
<td>0.02</td>
<td>0.06</td>
<td>2.90</td>
</tr>
</tbody>
</table>
\[ \log(\chi^2) \text{ vs } N \text{ for } \pi^n \text{ (red) and } e^n \text{ (blue), } \\
\pi \in \{1, \ldots, N\}. \text{ Note } \pi^{175} \approx 1.0028 \cdot 10^{87}, \text{ (5\%, } \\
\log(\chi^2) \approx 2.74). \]
Logarithms and Benford’s Law: Base 20

\[ \log(x^2) \text{ vs } N \text{ for } \pi^n \text{ (red)} \text{ and } e^n \text{ (blue)}, \]
\[ n \in \{1, \ldots, N\}. \text{ Note } e^3 \approx 20.0855, \text{ (5\%),} \]
\[ \log(x^2) \approx 2.74. \]
Applications
Applications for the IRS: Detecting Fraud
Applications for the IRS: Detecting Fraud
Applications for the IRS: Detecting Fraud

Exhibit 3: Check Fraud in Arizona

The table lists the checks that a manager in the office of the Arizona State Treasurer wrote to divert funds for his own use. The vendors to whom the checks were issued were fictitious.

<table>
<thead>
<tr>
<th>Date of Check</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 9, 1992</td>
<td>$1,927.48</td>
</tr>
<tr>
<td></td>
<td>27,902.31</td>
</tr>
<tr>
<td>October 14, 1992</td>
<td>86,241.90</td>
</tr>
<tr>
<td></td>
<td>72,117.46</td>
</tr>
<tr>
<td></td>
<td>81,321.75</td>
</tr>
<tr>
<td></td>
<td>97,473.96</td>
</tr>
<tr>
<td>October 19, 1992</td>
<td>93,249.11</td>
</tr>
<tr>
<td></td>
<td>89,658.17</td>
</tr>
<tr>
<td></td>
<td>87,776.89</td>
</tr>
<tr>
<td></td>
<td>92,105.83</td>
</tr>
<tr>
<td></td>
<td>79,949.16</td>
</tr>
<tr>
<td></td>
<td>87,602.93</td>
</tr>
<tr>
<td></td>
<td>96,879.27</td>
</tr>
<tr>
<td></td>
<td>91,806.47</td>
</tr>
<tr>
<td></td>
<td>84,991.67</td>
</tr>
<tr>
<td></td>
<td>90,831.83</td>
</tr>
<tr>
<td></td>
<td>93,766.67</td>
</tr>
<tr>
<td></td>
<td>88,338.72</td>
</tr>
<tr>
<td></td>
<td>94,639.49</td>
</tr>
<tr>
<td></td>
<td>83,709.28</td>
</tr>
<tr>
<td></td>
<td>96,412.21</td>
</tr>
<tr>
<td></td>
<td>88,432.86</td>
</tr>
<tr>
<td></td>
<td>71,552.16</td>
</tr>
</tbody>
</table>

TOTAL $1,878,687.58
Applications for the IRS: Detecting Fraud (cont)

- Embezzler started small and then increased dollar amounts.

- Most amounts below $100,000 (critical threshold for data requiring additional scrutiny).

- Over 90% had first digit of 7, 8 or 9.
Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.

- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.
Detecting Fraud

**Enron**

- Benford’s Law detected manipulation of revenue numbers.

- Results showed a tendency towards round Earnings Per Share (0.10, 0.20, etc.). Consistent with a small but noticeable increase in earnings management in 2002.
Data Integrity: Stream Flow Statistics: 130 years, 457,440 records
Analysis of Williams College Transactions (thanks to Richard McDowell): September 6, 2006 to June 29, 2007: 64,000+ transactions
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Benford Good Processes
Poisson Summation and Benford’s Law: Definitions

- Feller, Pinkham (often exact processes)
- data \( Y_{T,B} = \log_B \overrightarrow{X}_T \) (discrete/continuous):
  \[
  \mathbb{P}(A) = \lim_{T \to \infty} \frac{\#\{n \in A : n \leq T\}}{T}
  \]
- Poisson Summation Formula: \( f \) nice:
  \[
  \sum_{\ell=-\infty}^{\infty} f(\ell) = \sum_{\ell=-\infty}^{\infty} \hat{f}(\ell),
  \]
  Fourier transform \( \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \).
Benford Good Process

$X_T$ is Benford Good if there is a nice $f$ st

$$\text{CDF}_{Y_{T,B}}(y) = \int_{-\infty}^{y} \frac{1}{T} f \left( \frac{t}{T} \right) dt + E_T(y) := G_T(y)$$

and monotonically increasing $h (h(|T|) \to \infty)$:

- **Small tails:** $G_T(\infty) - G_T(Th(T)) = o(1)$,
  $G_T(- Th(T)) - G_T(-\infty) = O(1)$.

- **Decay of the Fourier Transform:**
  $$\sum_{\ell \neq 0} \left| \frac{\hat{f}(T\ell)}{\ell} \right| = o(1).$$

- **Small translated error:** $\mathcal{E}(a, b, T)) =
  \sum_{|\ell| \leq Th(T)} [E_T(b + \ell) - E_T(a + \ell)] = o(1).$
Main Theorem

Theorem (Kontorovich and M–, 2005)

\[ X_T \text{ converging to } X \text{ as } T \to \infty \text{ (think spreading Gaussian)}. \text{ If } X_T \text{ is Benford good, then } X \text{ is Benford.} \]

- **Examples**
  - \( L \)-functions
  - characteristic polynomials (RMT)
  - \( 3x + 1 \) problem
  - geometric Brownian motion.
Sketch of the proof

- **Structure Theorem:**
  - main term is something nice spreading out
  - apply Poisson summation

- **Control translated errors:**
  - hardest step
  - techniques problem specific
Sketch of the proof (continued)

\[
\sum_{\ell = -\infty}^{\infty} \mathbb{P} \left( a + \ell \leq Y_{T,B} \leq b + \ell \right) \\
= \sum_{|\ell| \leq Th(T)} \left[ G_T(b + \ell) - G_T(a + \ell) \right] + o(1) \\
= \int_a^b \sum_{|\ell| \leq Th(T)} \frac{1}{T} f \left( \frac{t}{T} \right) dt + E(a, b, T) + o(1) \\
= \hat{f}(0) \cdot (b - a) + \sum_{\ell \neq 0} \hat{f}(T \ell) \frac{e^{2\pi ib\ell} - e^{2\pi i a\ell}}{2\pi i \ell} + o(1).
\]
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime} \ p} \left(1 - \frac{1}{p^s}\right)^{-1}. \]
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\[ \prod_{\text{prime } p} \left( 1 - \frac{1}{p^s} \right)^{-1} = \prod_{\text{prime } p} \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) \]

\[ = \left( 1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots \right) \left( 1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots \right) \]

\[ = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{(2 \cdot 3)^s} + \cdots \]
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\rho \text{ prime}} \left(1 - \frac{1}{\rho^s}\right)^{-1}. \]

\[ \lim_{s \to 1^+} \zeta(s) = \infty \text{ implies infinitely many primes.} \]
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\rho \text{ prime}} \left(1 - \frac{1}{\rho^s}\right)^{-1}. \]

\[ \lim_{s \to 1^+} \zeta(s) = \infty \text{ implies infinitely many primes.} \]

\[ \zeta(2) = \frac{\pi^2}{6} \text{ implies infinitely many primes.} \]
Riemann Zeta Function

\[ |\zeta \left( \frac{1}{2} + i\frac{k}{4} \right) |, \ k \in \{0, 1, \ldots, 65535\}. \]
The $3x + 1$ Problem and Benford's Law
3x + 1 Problem

- Kakutani (conspiracy), Erdös (not ready).

- x odd, $T(x) = \frac{3x+1}{2^k}, 2^k \parallel 3x + 1$. 
3x + 1 Problem

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- $x$ odd, $T(x) = \frac{3x+1}{2^k}$, $2^k \parallel 3x + 1$.

- Conjecture: for some $n = n(x)$, $T^n(x) = 1$. 
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- 7
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- $7 \rightarrow_1 11$
Kakutani (conspiracy), Erdős (not ready).

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- $7 \rightarrow_1 11 \rightarrow_1 17$
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$7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13$
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- \( 7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \)
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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1$
Kakutani (conspiracy), Erdös (not ready).

\( x \) odd, \( T(x) = \frac{3x+1}{2^k} \), \( 2^k \mid 3x + 1 \).

Conjecture: for some \( n = n(x) \), \( T^n(x) = 1 \).

\( 7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \rightarrow_4 1 \rightarrow_2 1 \),
3x + 1 Problem

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- 7 \( \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \rightarrow_4 1 \rightarrow_2 1 \),
  2-path (1, 1), 5-path (1, 1, 2, 3, 4).
  \( m \)-path: (\( k_1, \ldots, k_m \)).
Heuristic Proof of $3x + 1$ Conjecture

\[ a_{n+1} = T(a_n) \]
Heuristic Proof of $3x + 1$ Conjecture

\[
a_{n+1} = T(a_n)
\]

\[
\mathbb{E}[\log a_{n+1}] \approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left( \frac{3a_n}{2^k} \right)
\]
Heuristic Proof of $3x + 1$ Conjecture

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\[ = \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \]
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\[ = \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \]

\[ = \log a_n + \log \left( \frac{3}{4} \right). \]

Geometric Brownian Motion, drift $\log(3/4) < 1$. 
Structure Theorem: Sinai, Kontorovich-Sinai

\[ \mathbb{P}(A) = \lim_{N \to \infty} \frac{\# \{ n \leq N : n \equiv 1,5 \text{ mod } 6, n \in A \}}{\# \{ n \leq N : n \equiv 1,5 \text{ mod } 6 \}}. \]
Structure Theorem: Sinai, Kontorovich-Sinai

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\((k_1, \ldots, k_m)\): two full arithm progressions:

\[ 6 \cdot 2^{k_1 + \cdots + k_m} p + q. \]
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Theorem (Sinai, Kontorovich-Sinai)

\(k_i\)-values are i.i.d.r.v. (geometric, 1/2):

\[
P \left( \log_2 \left[ \frac{x_m}{\left( \frac{3}{4} \right)^m x_0} \right] \leq a \right) = P \left( \frac{S_m - 2m}{\sqrt{2m}} \leq a \right)
\]
Structure Theorem: Sinai, Kontorovich-Sinai

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\mathbb{P} \left( \frac{\log_2 \left[ \frac{x_m \left( \frac{3}{4} \right)^m x_0}{(\log_2 B)^{\sqrt{2m}}} \right]}{\log_2 B \sqrt{2m}} \leq a \right) = \mathbb{P} \left( \frac{S_m - 2m}{(\log_2 B)^{\sqrt{2m}}} \leq a \right)
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Structure Theorem: Sinai, Kontorovich-Sinai

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\( k_i \)-values are i.i.d.r.v. (geometric, 1/2):

\[ P \left( \log_B \left[ \frac{x_m}{\left(\frac{3}{4}\right)^m x_0} \right] \leq a \right) = P \left( \frac{(S_m-2m)}{\log_2 B} \leq a \right) \]
3x + 1 and Benford

Theorem (Kontorovich and M–, 2005)

As \( m \to \infty \), \( x_m/(3/4)^m x_0 \) is Benford.

Theorem (Lagarias-Soundararajan 2006)

\( X \geq 2^N \), for all but at most \( c(B)N^{-1/36}X \) initial seeds the distribution of the first \( N \) iterates of the 3x + 1 map are within \( 2N^{-1/36} \) of the Benford probabilities.
Failed Proof: lattices, bad errors.

CLT: \((S_m - 2m)/\sqrt{2m} \to N(0, 1)\):

\[
\mathbb{P}(S_m - 2m = k) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).
\]

Quantified Equidistribution:

- \(I_\ell = \{\ell M, \ldots, (\ell + 1)M - 1\}, M = m^c, c < 1/2\)
- \(k_1, k_2 \in I_\ell: \left| \eta \left(\frac{k_1}{\sqrt{m}}\right) - \eta \left(\frac{k_2}{\sqrt{m}}\right) \right| \text{ small}\)
- \(C = \log_B 2\) of irrationality type \(\kappa < \infty\):

\[
\#\{k \in I_\ell : kC \in [a, b]\} = M(b - a) + O(M^{1+\epsilon-1/\kappa}).
\]
### Irrationality Type

**Irrationality type**

\( \alpha \) has irrationality type \( \kappa \) if \( \kappa \) is the supremum of all \( \gamma \) with

\[
\lim_{q \to \infty} q^{\gamma+1} \min_p \left| \alpha - \frac{p}{q} \right| = 0.
\]

- Algebraic irrationals: type 1 (Roth’s Thm).
- Theory of Linear Forms: \( \log_B 2 \) of finite type.
Linear Forms

**Theorem (Baker)**

\[ \alpha_1, \ldots, \alpha_n \text{ algebraic numbers height } A_j \geq 4, \]
\[ \beta_1, \ldots, \beta_n \in \mathbb{Q} \text{ with height at most } B \geq 4, \]
\[ \Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n. \]

*If* \( \Lambda \neq 0 \) *then* \( |\Lambda| > B^{-C\Omega \log \Omega'} \), *with*
\[ d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}], \ C = (16nd)^{200n}, \]
\[ \Omega = \prod_j \log A_j, \ \Omega' = \Omega / \log A_n. \]

Gives \( \log_{10} 2 \) of finite type, with \( \kappa < 1.2 \cdot 10^{602} : \)
\[ |\log_{10} 2 - p/q| = |q \log 2 - p \log 10| / q \log 10. \]
Quantified Equidistribution

**Theorem (Erdös-Turan)**

\[
D_N = \sup_{[a,b]} \left| N(b - a) - \#\{n \leq N : x_n \in [a, b]\} \right| / N
\]

There is a \( C \) such that for all \( m \):

\[
D_N \leq C \cdot \left( \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ihx_n} \right| \right)
\]
Proof of Erdös-Turan

Consider special case \( x_n = n\alpha, \alpha \notin \mathbb{Q} \).

- Exponential sum \( \leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||} \).

- Must control \( \sum_{h=1}^{m} \frac{1}{h||h\alpha||} \), see irrationality type enter.

- type \( \kappa \), \( \sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right) \), take \( m = \lfloor N^{1/\kappa} \rfloor \).
3x + 1 Data: random 10,000 digit number, $2^k \| 3x + 1$

80,514 iterations ($((4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

<table>
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<tr>
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<th>Number</th>
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<th>Benford</th>
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<td>0.051</td>
</tr>
<tr>
<td>9</td>
<td>3650</td>
<td>0.045</td>
<td>0.046</td>
</tr>
</tbody>
</table>
3\(x + 1\) Data: random 10,000 digit number, \(2 | 3x + 1\)

241,344 iterations, \(\chi^2 = 11.4\) (5% 15.5).

<table>
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<tr>
<td>9</td>
<td>11073</td>
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</table>
5\times 1 \text{ Data: random 10,000 digit number, } 2^k \parallel 5x + 1

27,004 iterations, \chi^2 = 1.8 (5\% 15.5).

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<tr>
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</table>
**Data: random 10,000 digit number, $2|5x + 1$**

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

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Conclusions
Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
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- Ingredients of proofs (logarithms, equidistribution).
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- Applications to fraud detection / data integrity.
Conclusions and Future Investigations

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- Ingredients of proofs (logarithms, equidistribution).

- Applications to fraud detection / data integrity.

- Future work:
  - Study digits of other systems.
  - Develop more sophisticated tests for fraud.
References


S. J. Miller, *When the Cramér-Rao Inequality provides no information*, to appear in Communications in Information and Systems.


http://arxiv.org/abs/math/0601344


Products and Chains of Random Variables
Key Ingredients

- Mellin transform and Fourier transform related by logarithmic change of variable.

- Poisson summation from collapsing to modulo 1 random variables.
Preliminaries

- $\Xi_1, \ldots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.
- Density $\Xi_1 \cdot \Xi_2$:

$$\int_0^\infty f_2 \left( \frac{x}{t} \right) f_1(t) \frac{dt}{t}$$

◊ Proof: $\text{Prob}(\Xi_1 \cdot \Xi_2 \in [0, x])$:

$$\int_{t=0}^\infty \text{Prob} \left( \Xi_2 \in \left[ 0, \frac{x}{t} \right] \right) f_1(t) dt$$

$$= \int_{t=0}^\infty F_2 \left( \frac{x}{t} \right) f_1(t) dt,$$

differentiate.
Mellin Transform

\[ (Mf)(s) = \int_0^\infty f(x)x^s \frac{dx}{x} \]

\[ (M^{-1}g)(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)x^{-s} ds \]

\[ g(s) = (Mf)(s), \quad f(x) = (M^{-1}g)(x). \]

\[ (f_1 \ast f_2)(x) = \int_0^\infty f_2 \left( \frac{x}{t} \right) f_1(t) \frac{dt}{t} \]

\[ (M(f_1 \ast f_2))(s) = (Mf_1)(s) \cdot (Mf_2)(s). \]
Theorem

Let $X_i$ be independent, densities $f_i$. Then $\Xi_n = X_1 \cdots X_n$,

\[
h_n(x_n) = (f_1 \ast \cdots \ast f_n)(x_n)
\]

\[
(Mh_n)(s) = \prod_{m=1}^{n} (Mf_m)(s).
\]

As $n \to \infty$, $\Xi_n$ becomes Benford: $Y_n = \log_B \Xi_n$,

\[
|\text{Prob}(Y_n \mod 1 \in [a, b]) - (b - a)| \leq
\]

\[
(b - a) \cdot \sum_{\ell \neq 0, \ell = -\infty}^{\infty} \prod_{m=1}^{n} (Mf_i) \left( 1 - \frac{2\pi i \ell}{\log B} \right).
\]
Proof of Kossovsky’s Chain Conjecture for certain densities

Conditions

- \{D_i(\theta)\}_{i \in I}: one-parameter distributions, densities \( f_{D_i(\theta)} \) on \( [0, \infty) \).
- \( p: \mathbb{N} \to I, X_1 \sim D_{p(1)}(1), X_m \sim D_{p(m)}(X_{m-1}) \).
- \( m \geq 2 \),

\[
 f_m(x_m) = \int_0^\infty f_{D_{p(m)}(1)} \left( \frac{x_m}{x_{m-1}} \right) f_{m-1}(x_{m-1}) \frac{dx_{m-1}}{x_{m-1}}
\]

\[
 \lim_{n \to \infty} \sum_{\ell=-\infty}^{\infty} \prod_{m=1}^{n} (M f_{D_{p(m)}(1)}) \left( 1 - \frac{2\pi i \ell}{\log B} \right) = 0
\]
Proof of Kossovsky’s Chain Conjecture for certain densities

Theorem (JKKKM)

- If conditions hold, as $n \to \infty$ the distribution of leading digits of $X_n$ tends to Benford’s law.
- The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then

\[
|\text{Prob}(Y_n \mod 1 \in [a, b]) - (b + a)| \leq (b - a) \cdot \sum_{\ell = -\infty}^{\infty} \prod_{m=1}^{n} (\mathcal{M}f_{D_p(m)}(1)) \left(1 - \frac{2\pi i \ell}{\log B}\right)
\]
Example: All $X_i \sim \text{Exp}(1)$

- $X_i \sim \text{Exp}(1)$, $Y_n = \log_B \Xi_n$.
- Needed ingredients:
  - $\int_0^\infty \exp(-x)x^{s-1}dx = \Gamma(s)$.
  - $|\Gamma(1 + ix)| = \sqrt{\pi x / \sinh(\pi x)}$, $x \in \mathbb{R}$.
- $|P_n(s) - \log_{10}(s)| \leq$

\[
\log_B s \sum_{\ell=1}^\infty \left( \frac{2\pi^2 \ell / \log B}{\sinh(2\pi^2 \ell / \log B)} \right)^{n/2}.
\]
Example: All $X_i \sim \text{Exp}(1)$

**Bounds on the error**

- $|P_n(s) - \log_{10} s| \leq$
  - $3.3 \cdot 10^{-3} \log_B s$ if $n = 2$,
  - $1.9 \cdot 10^{-4} \log_B s$ if $n = 3$,
  - $1.1 \cdot 10^{-5} \log_B s$ if $n = 5$, and
  - $3.6 \cdot 10^{-13} \log_B s$ if $n = 10$.

- Error at most

$$\log_{10} s \sum_{\ell=1}^{\infty} \left( \frac{17.148 \ell}{\exp(8.5726 \ell)} \right)^{n/2} \leq .057^n \log_{10} s$$