Introduction
 Theory of Benford's Law
 Applications
 Conclusions
 Refs

 Benford's Law and Fraud Detection, or:

Benford's Law and Fraud Detection, or: Why the IRS Should Care About Number Theory!

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Bronfman Science Lunch Williams College, October 21, 2008

Introduction ●○○○○	Theory of Benford's Law	Applications	Conclusions	Refs
Summary				

- Review Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.



Introduction •••••	Theory of Benford's Law ০০০০০	Applications 00000	Conclusions	Refs
Caveats!				

• Not all fraud can be detected by Benford's Law.

Introduction ○●○○○	Theory of Benford's Law	Applications 00000	Conclusions	Refs
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Introduction •••••	Theory of Benford's Law	Applications	Conclusions	Refs
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Introduction	
00000	

Applications

Conclusions

Refs

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of *d* base *B* is $\log_B(\frac{d+1}{d})$; base 10 about 30% are 1s.

Introduction	The
00000	000

Applications

Conclusions

Refs

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Introduction	
00000	

Applications

Conclusions

Refs

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Introduction	
00000	

Applications

Conclusions

Refs

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Introduction	
00000	

Applications

Conclusions

Refs

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 - Many streets of different sizes: close to Benford.

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs
Examples				

- recurrence relations
- special functions (such as n!)
- iterates of power, exponential, rational maps
- products of random variables
- *L*-functions, characteristic polynomials
- iterates of the 3x + 1 map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Introduction ○○○○●	Theory of Benford's Law	Applications	Conclusions	Refs
Applications				

analyzing round-off errors

- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity



Introduction	Theory of Benford's Law	Applications	Conclusions	Refs

General Theory

Introduction 00000	Theory of Benford's Law ●○○○○	Applications	Conclusions	Refs
Mantissas				

Mantissa: $x = M_{10}(x) \cdot 10^k$, k integer.

Introduction 00000	Theory of Benford's Law ●○○○○	Applications 00000	Conclusions	Refs
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Key observation: $\log_{10}(x) = \log_{10}(\tilde{x}) \mod 1$ if and only if x and \tilde{x} have the same leading digits. Thus often study $y = \log_{10} x$.

Introduction	Theory of Benford's Law ○●○○○	Applications	Conclusions	Refs

Equidistribution

 $\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \mod 1 \in [a, b]$ tends to b - a:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a, b]\}}{N} \to b - a.$$

Introduction 00000	Theory of Benford's Law	Applications	Conclusions	Refs

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• Examples:
$$\log_{10} 2$$
, $\log_{10} \left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$.

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs

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Introduction	Theory of Benford's Law	Applications	Conclusions	Refs

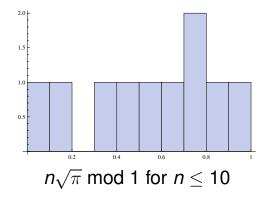
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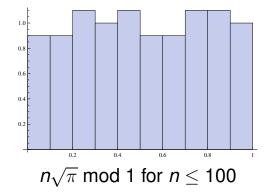
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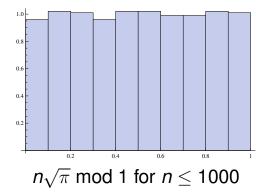
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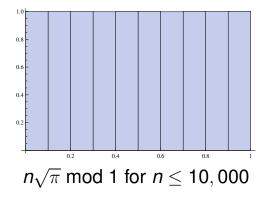
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Introduction 00000	Theory of Benford's Law ○○●○○	Applications 00000	Conclusions	Refs
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Introduction	Theory of Benford's Law ○○○●○	Applications	Conclusions	Refs
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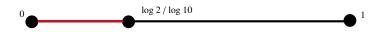
Fundamental Equivalence

Data set $\{x_i\}$ is Benford base *B* if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

Introduction	Theory of Benford's Law ○○○●○	Applications	Conclusions	Refs
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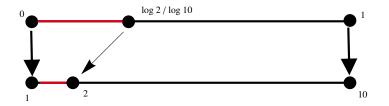
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Introduction	Theory of Benford's Law ○○●○	Applications	Conclusions	Refs
Logarithm	s and Benford's Law			

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- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.
- Fibonacci numbers are Benford base 10.

 $a_{n+1} = a_n + a_{n-1}.$ Guess $a_n = n^r$: $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$. Roots $r = (1 \pm \sqrt{5})/2$. General solution: $a_n = c_1 r_1^n + c_2 r_2^n$. Binet: $a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$.

Introduction 00000	Theory of Benford's Law	Applications	Conclusions	Refs

Applications

Introduction	

Applications

Conclusions

Refs

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Introduction 00000	Theory of Benford's Law	Applications ●0000	Conclusions	Refs

Applications for the IRS: Detecting Fraud

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Introduction	

Applications

Conclusions

Refs

Applications for the IRS: Detecting Fraud

e table lists the checks that a ma izona State Treasurer wrote to div e vendors to whom the checks w	ert funds for his own use. are issued were fictitious.
Date of Check	Amount
October 9, 1992	\$ 1,927.48
*	27,902.31
October 14, 1992	86,241.90
	72,117.46
	81,321.75
*	97,473.96
October 19, 1992	93,249.11
	89,658.17
	87,776.89
Real sector (all the sector sector)	92,105.83
	79,949.16
	87,602.93
	96,879.27
	91,806.47
	84,991.67
Contraction of the second second	90,831.83
	93,766.67
	88,338.72
	94,639.49
	83,709.28
	96,412.21
	88,432.86 71.552.16

Introduction	Theory of Benford's Law	Applications ○●○○○	Conclusions	Refs
Applicatio	ons for the IRS: Detec	ting Fraud (coi	nt)	

- Embezzler started small and then increased dollar amounts.
- Most amounts below \$100,000 (critical threshold for data requiring additional scrutiny).
- Over 90% had first digit of 7, 8 or 9.

Introduction 00000	Theory of Benford's Law 00000	Applications ○○●○○	Conclusions	Refs
Detecting	Fraud			

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off.

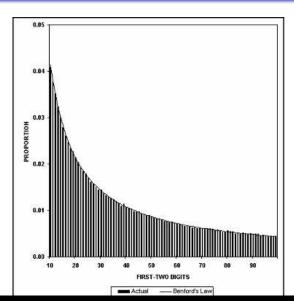
Introduction 00000	Theory of Benford's Law	Applications	Conclusions	Refs
Detecting	Fraud			

Enron

- Benford's Law detected manipulation of revenue numbers.
- Results showed a tendency towards round Earnings Per Share (0.10, 0.20, etc.). Consistent with a small but noticeable increase in earnings management in 2002.

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs
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Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



37

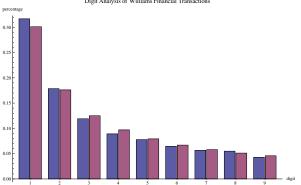
Introduction	

Theory of Benford's Law

Applications 00000

Refs

Analysis of Williams College Transactions (thanks to Richard McDowell): September 6, 2006 to June 29, 2007: 64,000+ transactions



Digit Analysis of Williams Financial Transactions

38

Introduction	

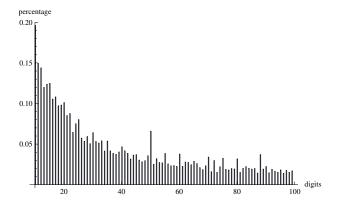
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Applications

Conclusions

Refs

Analysis of Williams College Transactions (thanks to Richard McDowell): September 6, 2006 to June 29, 2007: 64,000+ transactions



Introduction	

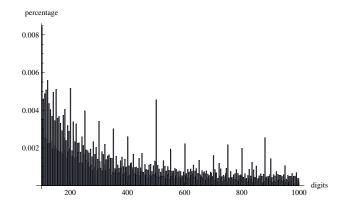
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Applications

Conclusions

Refs

Analysis of Williams College Transactions (thanks to Richard McDowell): September 6, 2006 to June 29, 2007: 64,000+ transactions



Introduction 00000	Theory of Benford's Law	Applications	Conclusions	Refs

Conclusions

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs		
Conclusions and Future Investigations						

• Diverse systems exhibit Benford behavior.

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs
Conclusio	ns and Future Investi	igations		

- Diverse systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs		
Conclusions and Future Investigations						

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- Applications to fraud detection / data integrity.

Introduction	Theory of Benford's Law	Applications	Conclusions	Refs
Conclusio	ns and Future Invest	igations		

- Diverse systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

• Future work:

- ◊ Study digits of other systems.
- Develop more sophisticated tests for fraud.

Introduction 00000	Theory of Benford's Law	Applications	Conclusions	Refs

References

Introduction 00000	Theory of Benford's Law	Applications	Conclusions	Refs

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