Benfordness of lower-dimensional measurements resulting from box fragmentation

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1. Background
   - Benford's Law
   - Fragmentation Processes
2. Benford Behavior from Fragmentation Processes
   - Probability Background
   - Benfordness of Volumes resulting from Box Fragmentation
3. Lower-dimensional Volumes in Box Fragmentation
   - Uniform Distributions: Experimental Data
   - Proof for Perimeter
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Benford's Law

Definition (Benford's Law)

The frequencies of first digits d (in base B) follow closely to the logarithmic relation:

$$\log_B\left(\frac{d+1}{d}\right)$$

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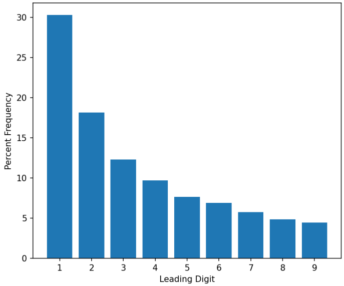
Base 10:



How to show Benford Behavior

Definition (Significand)

Given a base $B \geq 2$, we write $x = aB^n$ (scientific notation) with $|a| \in [1, B)$, $n \in \mathbb{Z}$. The *significand* of x is $|a|$.

- Example ($B = 10$): $1022 = 1.022 \times 10^3$ has significand 1.022.

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- Example ($B = 10$): $1022 = 1.022 \times 10^3$ has significand 1.022.

Definition (Strong Benford's Law)

A data set satisfies *Strong Benford's Law base B* if the probability of observing a significand of at most s in base B is $\log_B s$

How to show Benford Behavior

Definition (Equidistributed)

A sequence $\{x_n\} \subset [0, 1]$ is equidistributed in $[0, 1]$ if for all $[a, b] \subset [0, 1]$,

$$\lim_{N\to\infty} \frac{\#\{n : |n| \leq N, x_n \in [a, b]\}}{2N + 1} = b - a$$

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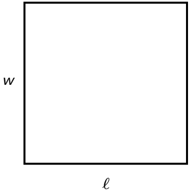
Theorem

If $y_n = \log_B x_n$ is equidistributed $\pmod 1$ then x_n is Benford base B.

Box Fragmentation

Fragmenting a rectangle in 2-dimensions:

- Begin with an 2-dimensional box and density functions f_1, f_2 on $(0, 1)$.



Box Fragmentation

Fragmenting a rectangle in 2-dimensions:

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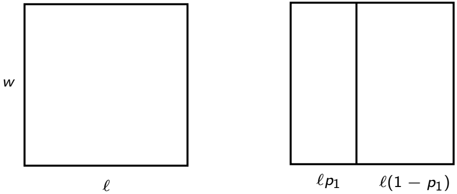

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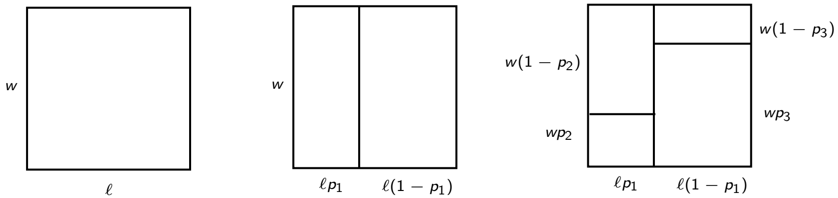
- Draw p_2 and p_3 from f_2. Cut one subrectangle at proportion p_2 along the vertical axis and the other subrectangle at proportion p_3 along the vertical axis.



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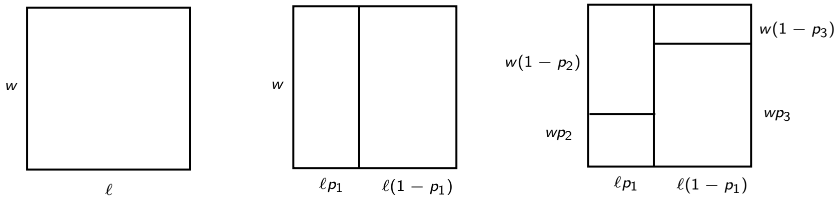
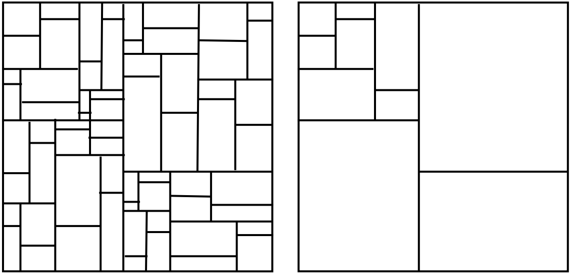
- Continue this way for n iterations.



Decomposition Models

We can vary the way we apply the fragmentation to the box:

- Unrestricted Decomposition Model: Each piece decomposes.

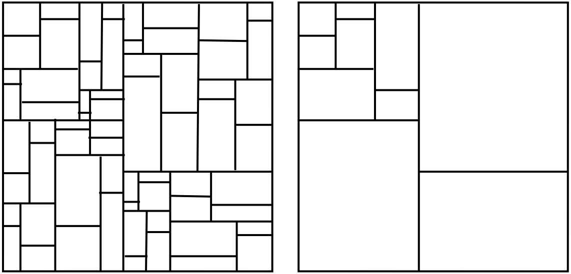


Unrestricted vs. Restricted Fragmentation

Decomposition Models

We can vary the way we apply the fragmentation to the box:

- Unrestricted Decomposition Model: Each piece decomposes.
- Restricted Decomposition Model: Only one resulting piece decomposes further.



Unrestricted vs. Restricted Fragmentation

Questions we can ask

What has been studied: Benfordness of m-dimensional volumes resulting from fragmentation of an m-dimensional box. For example:

- Area of 2-dimensional rectangle
- Volume of 3-dimensional box

These processes are known to exhibit Benford behavior as the number of iterations, n, goes to infinity!

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What we are studying: Benfordness of lower-dimensional volumes resulting from hyper-box fragmentation. For example:

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- Surface Area of 3-dimensional box
- Frame of 3-dimensional box

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Do these processes exhibit Benford behavior? Yes!

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Probability

Definition (Probability Density Function (pdf))

Let X be a continuous random variable. If f satisfies

$$\mathbb{P}(X \leq b) = \int_{-\infty}^{b} f(t)dt$$

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Definition (Convolution)

If X, Y are independent continuous random variables with pdfs f_X, f_Y, then the pdf of $X + Y$ is the called the *convolution* of f_X and f_Y, and

$$f_{X+Y}(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z-t)dt$$

Mellin Convolution and Mellin Transform

Definition (Mellin Convolution)

If X, Y are independent continuous random variables with pdfs f_X, f_Y, then the pdf of $X \cdot Y$ is the called the *Mellin convolution* of f_X and f_Y, and

$$f_{X \cdot Y}(z) = (f_X *_{\mathcal{M}} f_Y)(z) = \int_{-\infty}^{\infty} f_X(t) f_Y\left(\frac{z}{t}\right) \frac{dt}{t}$$

Mellin Transform

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Let f be a continuous real-valued function on $[0, \infty)$. The *Mellin Transform* of f, denoted $\mathcal{M}_f$, is

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Theorem

Let X, Y be independent continuous random variables with pdfs f_X, f_Y defined on $[0, \infty)$. Then,

$$\mathcal{M}_{(f_X *_{\mathcal{M}} f_Y)}(s) = \mathcal{M}_{f_X}(s) \cdot \mathcal{M}_{f_Y}(s)$$

Benfordness of Volumes resulting from Box Fragmentation

From SMALL 2018 and Irfan Durmić:

Consider an m-dimensional box with volume V. For random variables $X_1, \dots, X_N$, fix a continuous probability density f_k on $(0,1)$ such that

$$\lim_{N\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \left| \prod_{k=1}^{Nm} \mathcal{M}_{f_k}\left(1 - \frac{2\pi i \ell}{\log 10}\right) \right| = 0.$$

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Then, if unrestricted fragmentation is performed on the m-dimensional box, the resulting m-dimensional volumes follow Benford behavior, and

$$\begin{aligned} &|\mathbb{P}(\log(X_1 \cdots X_N) \pmod 1) \in [a,b] - (b-a))| \\ &\leq (b-a) \cdot \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \left| \prod_{k=1}^{Nm} \mathcal{M}_{f_k}\left(1 - \frac{2\pi i \ell}{\log 10}\right) \right| \end{aligned}$$

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From before, we have

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Where does this condition break when we look at lower-dimensional volumes?

Mellin Transforms do not behave nicely under addition! For $X \cdot Y$,

$$\mathcal{M}_{(f_X *_{\mathcal{M}} f_Y)}(s) = \mathcal{M}_{f_X}(s) \cdot \mathcal{M}_{f_Y}(s)$$

But, for $X + Y$

$$\mathcal{M}_{(f_X * f_Y)}(s) \neq \mathcal{M}_{f_X}(s) \cdot \mathcal{M}_{f_Y}(s)$$

Perimeter

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- Draw proportions p_i and $q_{2i+1}, q_{2(i+1)}$ from distributions with a piecewise continuous pdf.
- The perimeter of an arbitrary box after N iterations is

$$Per = 2\left(\underbrace{\ell\prod_{i=1}^{k} p_i \prod_{j=k+1}^{N}(1-p_j)} + \underbrace{w\prod_{i=1}^{k} q_i \prod_{j=k+1}^{N}(1-q_j)}\right)$$

Uniform Distributions

First, we looked at the fragmentation process resulting from p_i and q_{2i}, q_{2i+1} being drawn from uniform distributions.

Lemma

The product of n uniform distributions has pdf

$$f_n(x) = \begin{cases} \frac{(-\ln(x))^{n-1}}{(n-1)!} & \textit{if } x \in (0,1] \\ 0 & \textit{otherwise} \end{cases}$$

Integration Attempt

Proportion of perimeters with leading digit d is given by:

$$\sum_m \mathbb{P}(d \cdot 10^{-m} \leq Per < (d+1)10^{-m})$$

$$= \sum_m \int_0^{d \cdot 10^{-m}} f_n(y) \int_{d \cdot 10^{-m}}^{(d+1) \cdot 10^{-m}} f_n(x) dx dy + \int_{d \cdot 10^{-m}}^{(d+1) \cdot 10^{-m}} f_n(y) \int_0^{(d+1) \cdot 10^{-m}} f_n(x) dx dy$$

Approximation

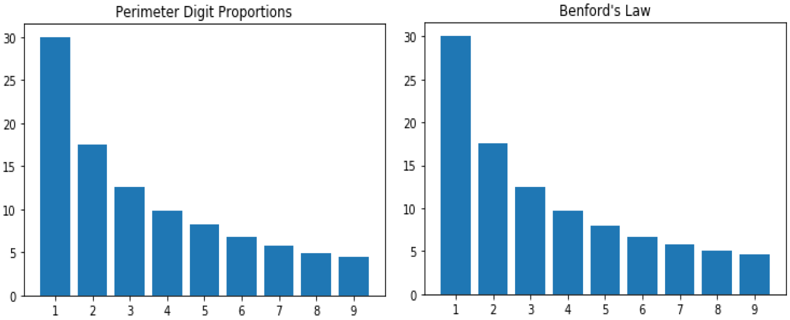
For 8 iterations of fragmentation, we can approximate the proportion of perimeters with leading digit 1:

$$\sum_{m=0}^{100}\left(\int_0^{10^{-m}} f_8(y)\int_{10^{-m}-y}^{2\cdot 10^{-m}-y} f_8(x)dxdy + \int_{10^{-m}}^{2\cdot 10^{-m}} f_8(y)\int_0^{2\cdot 10^{-m}-y} f_8(x)dxdy\right)$$

$$\approx \quad 0.301032\dots$$

$$\approx \quad \log_{10}\left(\frac{1+1}{1}\right) = 0.3010299\dots$$

Approximation



Modeling 8 iterations of the Unrestricted Model.

Using the Maximum of two Benford RVs

Let $L(\cdot, B) : \mathbb{R} \to \mathbb{N}$ denote the function that returns the leading digit of a real number base B. (Note this is not the same as the significand.)

When is $L(X_n + Y_n, B) = L(\max\{X_n, Y_n\}, B)$?

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Theorem (SMALL 2022)

Let f_{X_n} and f_{Y_n} be pdfs of X_n, Y_n respectively. Suppose that f_{X_n}, f_{Y_n} are piecewise continuous pdfs. Then, for any ε, the probability that $L(X_n + Y_n, B) = L(\max\{X_n, Y_n\}, B)$ is at least $(1-\varepsilon)\left(1 - O\left(\frac{1}{\sqrt{n}}\right)\right)$.

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Theorem (SMALL 2022)

$\max\{X_n, Y_n\}$ *converges to Benford behavior base B as $n \to \infty$.*

Using the Maximum of two Benford RV's

We have

$$\begin{aligned}\mathbb{P}(L(X_n+Y_n)=d) &= \left(1-O\left(\frac{1}{\sqrt{n}}\right)\right)(1-\varepsilon)\mathbb{P}(L(\max\{X_n, Y_n\})=d) \\ &+ \left(1-\left(1-O\left(\frac{1}{\sqrt{n}}\right)\right)(1-\varepsilon)\right)\mathbb{P}^*(L(X_n+Y_n)=d)\end{aligned}$$

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Substitute in

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Cancelling out terms and using that

$$\mathbb{P}(L(X_n) = d) = \log_B\left(\frac{d+1}{d}\right) \quad \mathbb{P}(L(Y_n) = d) = \log_B\left(\frac{d+1}{d}\right),$$

we get

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Theorem (SMALL 2022)

Let $k \leq m$. The k-dimensional volumes of boxes resulting from unrestricted fragmentation of an m-dimensional box with proportions drawn from piecewise continuous pdfs follow Benford behavior.

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Note: This is not Strong Benford Behavior!

Future Work

- Prove Strong Benford Behavior for the lower-dimensional volumes resulting from fragmentation processes
- Determine classes of functions which satisfy the conditions necessary for Benford behavior and generalize our proof to other distributions
- Expand the library of distributions that satisfy the conditions of the Mellin Transform theorem used to prove the original volume problem

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Acknowledgments

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Reference

- Thealexa Becker, David Burt, Taylor C. Corcoran, Alec Greaves-Tunnell, Joseph R. Iafrate, Joy Jing, Steven J. Miller, Jaclyn D. Porfilio, Ryan Ronan, Jirapat Samranvedhya, Frederick W. Strauch, and Blaine Talbut. *Benford's law and continuous dependent random variables*. Annals of Physics vol. 388, 2018, pp. 350-381.
- Durmić, Irfan. *Benford Behavior of a Higher-Dimensional Fragmentation Process*. 2022.
- Jang, Dennis et al. *Chains of Distributions, Hierarchical Bayesian Models and Benford's Law*. 2008.