The Bergman Game, SMALL 2021

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Definition

The *Bergman Game* is a turn-based game played on an **doubly infinite tape**.

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...	0	0	1	...

The **Split Move**:

...	0	0	2	0	...

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The **Combine Move:**

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...	0	0	1	...

The **Split Move**:

...	0	0	2	0	...
...	1	0	0	1	...

Playing the Bergman Game

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	

Playing the Bergman Game

SPLIT

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	

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SPLIT

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	

Playing the Bergman Game

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…	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	…
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	

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SPLIT

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	

Playing the Bergman Game

COMBINE

. . .	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	. . .
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	

Playing the Bergman Game

COMBINE

…	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	…
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	

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COMBINE

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	

Playing the Bergman Game

COMBINE

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	

Playing the Bergman Game

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	0	0	1	0	7	1	0	0	0	
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	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	

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	0	0	0	0	9	0	0	0	0	
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	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	

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	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	

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	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	

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…	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	…
	0	0	0	0	9	0	0	0	0	
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	0	0	2	0	4	1	1	0	0	
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	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	

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	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
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	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	
	1	0	0	0	2	0	1	1	0	

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	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
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	1	0	0	0	2	0	1	1	0	

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	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	
	1	0	0	0	2	0	1	1	0	
	1	0	0	0	2	0	0	0	1	

Playing the Bergman Game

SPLIT

...	$?_{-4}$	$?_{-3}$	$?_{-2}$	$?_{-1}$	$?_0$	$?_1$	$?_2$	$?_3$	$?_4$	...
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	
	1	0	0	0	2	0	1	1	0	
	1	0	0	0	2	0	0	0	1	

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	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	
	1	0	0	0	2	0	1	1	0	
	1	0	0	0	2	0	0	0	1	
	1	0	1	0	0	1	0	0	1	

The Wonders of φ

- The polynomial $x^2 - x - 1 = 0$ "works well" with these rules because $x^2 = 1 + x$ and $2x^2 = 1 + x^3$.
- The golden mean, $\varphi = \frac{1+\sqrt{5}}{2}$ is a root of this polynomial.
- Any integer can be uniquely represented as a sum of non-consecutive powers of φ.

Playing the Bergman Game: The Reveal

...	φ^{-4}	φ^{-3}	φ^{-2}	φ^{-1}	φ^{0}	φ^{1}	φ^{2}	φ^{3}	φ^{4}	...
	0	0	0	0	9	0	0	0	0	
	0	0	1	0	7	1	0	0	0	
	0	0	2	0	5	2	0	0	0	
	0	0	2	0	4	1	1	0	0	
	0	0	2	0	4	0	0	1	0	
	1	0	0	1	4	0	0	1	0	
	1	0	0	0	3	1	0	1	0	
	1	0	0	0	2	0	1	1	0	
	1	0	0	0	2	0	0	0	1	
	1	0	1	0	0	1	0	0	1	

History

Definition

The Fibonacci Numbers are a recursively defined sequence so that $F_0 = 1, F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

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Theorem (Zeckendorf, 1972,[2])

Every positive integer may be written uniquely as a sum of non-adjacent Fibonacci numbers .

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Theorem (Zeckendorf, 1972,[2])

Every positive integer may be written uniquely as a sum of non-adjacent Fibonacci numbers .

Example

$$2021 = 1597 + 377 + 34 + 13 = F_{15} + F_{12} + F_7 + F_5$$

The **Zeckendorf Game** (see [1]) uses very similar rules to the Bergman Game, with extra boundary moves that prevent us from using negative indices. It produces Zeckendorf Decomposition.

The Bergman Game is Long

Zeckendorf Game (left) vs Bergman Game (right) on 20 chips:

The Bergman Game is MUCH more complicated!

The Bergman Game Invariants

Take the following four game states from the recent game:

φ^{-2}	φ^{-1}	φ^{0}	φ^{1}	φ^{2}	Value(S)
0	0	9	0	0	$9\varphi^0 = 9$
1	0	7	1	0	$\varphi^{-2} + 7\varphi^0 + 1\varphi^1 = 9$
2	0	5	2	0	$2\varphi^{-2} + 5\varphi^0 + 2\varphi^2 = 9$
2	0	4	1	1	$2\varphi^{-2} + 4\varphi^0 + 1\varphi^1 + 1\varphi^2 = 9$

Definition

Value$(S) = \sum_j S(j)\varphi^j$, that is, the number which the game state S represents as a base φ decomposition.

This is an *invariant*.

Number of Chips

φ^{-2}	φ^{-1}	φ^{0}	φ^{1}	φ^{2}	#chips(S)
0	0	9	0	0	$9 = 9$
1	0	7	1	0	$1 + 7 + 1 = 9$
2	0	5	2	0	$2 + 5 + 2 = 9$
2	0	4	1	1	$2 + 4 + 1 + 1 = 8$

Definition

#chips(S) = the number of chips in game state S

- #chips(S) stays the same when we split and goes down by one when we combine. Bounds # of combines.
- It is a *monovariant*, a quantity which only changes in one direction over the course of the game.

Index Sum

φ^{-2}	φ^{-1}	φ^{0}	φ^{1}	φ^{2}	IndexSum(S)
1	0	7	1	0	$1(-2)+7(0)+1(1)=-1$
2	0	5	2	0	$2(-2)+5(0)+2(1)=-2$
2	0	4	1	1	$2(-2)+4(0)+1(1)+1(2)=-1$

Definition

$\text{IndexSum}(S)=\sum_j S(j)\cdot j$, a weighted sum of the indices in game state S.

- A split decreases this by one. Can bound # of successive splits.
- A combine into index j increases this by $-j+3$

The Bergman Game Terminates: Left/Right Bound

Lemma (Right Bound)

We have a right bound on the game of $\log_\varphi Value(S)$.

The Bergman Game Terminates: Left/Right Bound

Lemma (Right Bound)

We have a right bound on the game of $\log_\varphi \mathit{Value}(S)$.

- We bound the maximum gap size between summands during the game.
- We then perform a worst-case analysis to provide a left bound.
- Together these give a maximum and minimum for $\mathrm{IndexSum}(S)$.

The Bergman Game Terminates

Proposition

The Bergman Game Terminates.

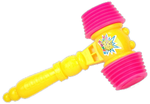
Proof.

- There are at most #chips(S) combines
- Suffices to bound successive splits
- IndexSum(S) is bounded above and below.

□

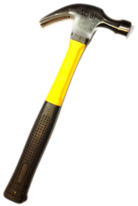
Building Better Hammers

- We first prove slow termination.



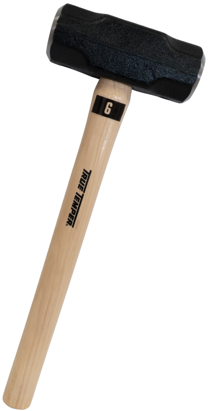
Building Better Hammers

- We first prove slow termination.
- Information about the final state $\rightarrow$ better left bound.



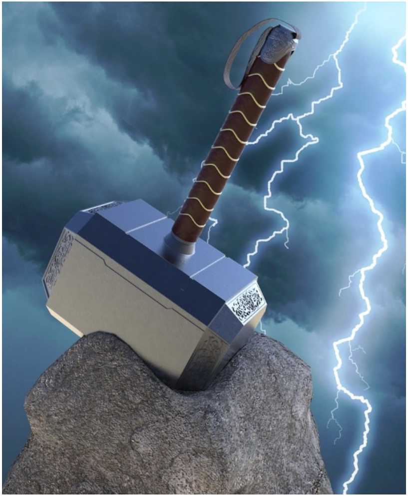
Building Better Hammers

- We first prove slow termination.
- Information about the final state $\rightarrow$ better left bound.
- Better Left Bound $\rightarrow$ Fast Termination depending on Length of Initial State.



Building Better Hammers

- We first prove slow termination.
- Information about the final state → better left bound.
- Better Left Bound → Fast Termination depending on Length of Initial State.
- Fast Termination Depending on Length → Fast Termination only depending on Chips.



Summary of Our Bergman Game Results

Theorem (SMALL, 2021)

There is a tight left bound $-2n - \log_{\varphi} n$ on the left-most used edge of a game that begins with n chips at the 0^{th} index.

Theorem (SMALL, 2021)

The longest Bergman Game with n summands terminates in $\Theta(n^2)$ moves. Furthermore, an $O(n)$ game is achievable from any initial state.

The Generalized Bergman Game

Definition

We say a sequence is a Positive Linear Recurrence Sequence (PLRS) if it is given by a linear recurrence with characteristic polynomial $x^k - c_1 x^{k-1} - \cdots - c_k$ for some c_i with $c_1, c_k > 0$ and $k \geq 2$. We say it is non-increasing if $c_1 \geq c_2 \geq \cdots \geq c_k > 0$. For convenience if $j > k$ we let $c_j = 0$.

Example

Let $a_0 = a_1 = a_2 = 1$, and for $n \geq 3$,
$a_n := 3a_{n-1} + 2a_{n-2} + a_{n-3}$.
1, 1, 1, 6, 21, 76, 276, . . .

Generalized Bergman game

The Bergman Game is based on φ and its recurrence relation. We also define games based on the roots of any non-increasing PLRS. We call such games together the Generalized Bergman Game.

Theorem (SMALL, 2021)

The longest Generalized Bergman Game with n summands terminates in $\Theta(n^2)$ moves.

Further Questions

- Is there a winning strategy for either player?
- How hard is it to determine the winner or winning strategy on the Bergman Game?
- How far can the results on the Bergman Game be pushed beyond non-increasing PLRS games?

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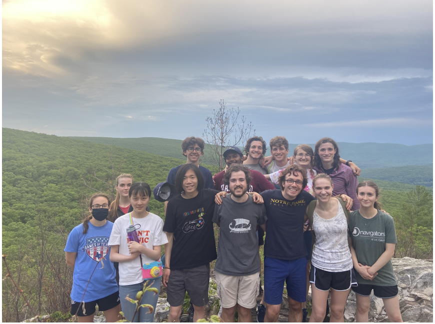

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