The Langlands Program: Beyond Endoscopy

Oscar E. González[1], oscar.gonzalez3@upr.edu
Kevin Kwan[2], kevinkwanch@gmail.com

[1]Department of Mathematics, University of Puerto Rico, Río Piedras.

[2]Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.

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What is the Langlands program?

- Conjectures relating many areas of math.
- Proposed by Robert P. Langlands about 50 years ago.

Automorphic Forms

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1. For any z in $\mathbb{H}$ and any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$, f satisfies the equation $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$.
2. f is a holomorphic (complex analytic) function on $\mathbb{H}$.
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Automorphic forms generalize modular forms.

Langlands' Functoriality Conjecture

Conjecture 1.1

Given two reductive groups H, G with a homomorphism ${}^L H \to {}^L G$, there exists a related transfer of automorphic forms on H to automorphic forms on G $\Pi(H) \to \Pi(G)$.

What are L-functions?

Diverse families of functions attached to arithmetic objects, for example:

- **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad \Re s > 1,$$

useful in studying distribution of primes.

- **Dirichlet L-function**

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for} \quad \Re s > 1,$$

where χ is a Dirichlet's character, i.e. homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{S}^1$. Useful in studying distribution of primes in arithmetic progression.

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$$\zeta_E(s) = \sum_{\mathfrak{a}} \frac{1}{N_{E/\mathbb{Q}}(\mathfrak{a})^s},$$

for $\Re s > 1$, where the sum is over all non-zero integral ideals of E.

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Definition 2.1

Let E be a number field. Let $\rho : \mathrm{Gal}(E/\mathbb{Q}) \to GL(d, \mathbb{C})$ be a finite dimensional Galois representation. The Artin L-function associated to ρ is given in terms of the Euler product:

$$L(s, \rho) = \prod_p \det(I - \rho(Fr_p)p^{-s})^{-1},$$

where Fr_p is the Frobenius element in $\mathrm{Gal}(E/\mathbb{Q})$.

Denote by σ_E the Artin representation obtained from the factorization $\zeta_E(s) = \zeta_{\mathbb{Q}}(s)L(s, \sigma_E)$, where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function.

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$$\zeta_E(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N_{E/\mathbb{Q}}(\mathfrak{p})^s}.$$

Essential Properties of L-functions

- Can be completed by 'gamma factors'. The completed L-functions are meromorphic on $\mathbb{C}$ with at most poles at $s = 0$ and $s = 1$ and they satisfy **functional equations**.

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- Functional equation: $\Lambda(s) = \Lambda(1-s)$.
- $\Lambda(s)$ is analytic on $\mathbb{C}$ except with a simple pole at $s = 1$. Also, $\mathrm{Res}_{s=1}\,\zeta(s) = 1$.
- Riemann Hypothesis: all non-trivial zeros lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Such a result would provide results on the distribution of prime numbers.

Functional Equation of $L(s, \chi)$

For a primitive character χ (i.e., not induced by any character of smaller modulus), we have the following.

- Gamma factor: $\gamma(s) = \pi^{-s/2}\Gamma((s+\delta)/2)$, where $\delta = 0$ if $\chi(-1) = 1$ and $\delta = 1$ if $\chi(-1) = -1$.

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- Functional equation $\Lambda(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}}\Lambda(1-s, \bar{\chi})$.
- $L(s, \chi)$ is entire if $\chi \neq \chi_0$. If $\chi = \chi_0$, then $L(s, \chi)$ is analytic on $\mathbb{C}$ except having a simple pole at $s = 1$.

Functional equation of $\zeta_E(s)$ (Hecke)

- Gamma factor: $\gamma(s) = \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{2r_2}$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, r_1 and $2r_2$ are the number of real and complex places of the number field E.

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- Completion: $\Lambda_E(s) = |D_E|^{s/2}\gamma(s)\zeta_E(s)$, where D_E is the discriminant of the number field E.

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- r_1 is the number of real places of E,
- $2r_2$ is the number of complex places of E ($r_1 + 2r_2 = n$),
- h_E is the class number of E,
- R_E is the regulator of E,
- w_E is the number of roots of unity in E

Theorem 2.2 (Class Number Formula)

$$\operatorname{Res}_{s=1} \zeta_E(s) = \frac{2^{r_1}(2\pi)^{r_2} h_E R_E}{w_E \sqrt{|D_E|}}.$$

Arthur-Selberg Trace Formula

Definition 2.3 (Right Regular Representation of G)

The right regular representation of group G on $L^2(\Gamma \setminus G)$, denoted by R, is defined by

$$R(y)\phi(x) = \phi(xy),$$

where x, $y \in G$, $\phi \in L^2(\Gamma \setminus G)$.

Definition 2.4 (Right Regular Operator)

Let $f \in C_c(G)$. We define the right regular operator $R(f)$ on $L^2(\Gamma \setminus G)$ as follows: for $\phi \in L^2(\Gamma \setminus G)$, we have

$$R(f)\phi(x) = \int_G f(x)R(y)\phi(x)dy = \int_G f(y)\phi(xy)dy.$$

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- The ring of adeles is the **restricted** direct product of all $\mathbb{Q}_p$, where $p \leq \infty$. We denote the ring of adeles by $\mathbb{A}$.
- In this way $\mathbb{A}$ is locally compact and we can assign a Haar measure on $G(\mathbb{A})$, where $G = GL(n)$.

Arthur-Selberg Trace Formula

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- Denote by G_γ the centralizer of γ in G.

Theorem 2.5 (Elliptic Part of Arthur-Selberg Trace Tormula)

$$\begin{aligned}\operatorname{tr} R(f) &= \sum_{\gamma \text{ ell}} \operatorname{meas}(\gamma) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)dg + \cdots \\ &= \sum_{\gamma \text{ ell}} \operatorname{meas}(\gamma) \prod_v \int_{G_\gamma(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)} f_v(g^{-1}\gamma g)dg_v. + \cdots\end{aligned}$$

Idea of Beyond Endoscopy and L-functions

- Given an automorphic form π on G and a representation r of ${}^L G$, one can define an automorphic L-function

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- Let G, G' be reductive groups. If an automorphic form π on G is a functorial transfer from a smaller G', then one expects the L-function $L(s,\pi,r)$ to have a pole at $s=1$ for some representation r of ${}^LG$.

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- Let G, G' be reductive groups. If an automorphic form π on G is a functorial transfer from a smaller G', then one expects the L-function $L(s, \pi, r)$ to have a pole at $s = 1$ for some representation r of ${}^L G$.
- In particular, the order of pole of $L(s, \pi, r)$ at $s = 1$, which we denote by $m_r(\pi)$, should be nonzero if and only if π is a transfer.

Idea of Beyond Endoscopy and L-functions

- Langlands' idea is to weight the spectral terms in the stable trace formula by $m_r(\pi)$, resulting in a trace formula whose spectral side detects only π for which the $m_r(\pi)$ is nonzero.

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- Since in general $L(s, \pi, r)$ is not a priori defined at $s = 1$, we account for the weight factor by taking the residue at $s = 1$ of the logarithmic derivative of $L(s, \pi, r)$.

Outline of Altuğ's work

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- This isolates the contribution from trivial representation, which should give us the major contribution.
- Give other versions of trace formula to be used in detecting the functorial transfer.

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- Obstacle: We cannot directly apply Poisson summation formula!
- There are singularities from the real orbital integrals.

Our Work

By using the Class Number Formula, we have

$$\begin{aligned} \operatorname{meas}(\gamma) &= \operatorname{meas}(Z_+ G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})) \\ &= \operatorname{meas}(E^\times \backslash I_E^1) \\ &= \frac{2^{r_1}(2\pi)^{r_2} h_E R_E}{w_E} \\ &= \sqrt{|D_E|}\; L(1, \sigma_E). \end{aligned}$$

Using the CNF, we can write the elliptic part of the trace formula as follows:

$$\sum_{\pm p^k} \sum_{\mathrm{tr}(\gamma),\ldots,\mathrm{tr}(\gamma^{n-1})} \frac{1}{|s_\gamma|} L(1,\sigma_E)|D_\gamma|^{1/2} \mathrm{Orb}(f_\infty;\gamma) \prod_q \mathrm{Orb}(f_q;\gamma).$$

Approximate functional equation of Artin L-function

Assuming Artin's conjecture, we can write the approximate functional equation of $L(1, \sigma_E)$.

$$L(s,\rho) = \sum_n \frac{\lambda_\rho(n)}{n^s} \frac{1}{2\pi i} \int_{(3)} \left(\frac{n}{X\sqrt{q}}\right)^{-u} G(u)$$

$$\frac{\pi^{-d(s+u)n/2}\ \Gamma\left(\frac{s+u}{2}\right)^{d^+ + 2dr_2} \Gamma\left(\frac{s+u+1}{2}\right)^{d^- + 2dr_2}}{\pi^{-d(s)n/2}\ \Gamma\left(\frac{s}{2}\right)^{d^+ + 2dr_2} \Gamma\left(\frac{s+1}{2}\right)^{d^- + 2dr_2}} \frac{du}{u} + \epsilon(\rho) q^{\frac{1}{2}-s}$$

$$\frac{\pi^{-d(1-s)n/2}\ \Gamma\left(\frac{1-s}{2}\right)^{d^+ + 2dr_2} \Gamma\left(\frac{2-s}{2}\right)^{d^- + 2dr_2}}{\pi^{-d(s)n/2}\ \Gamma\left(\frac{s}{2}\right)^{d^+ + 2dr_2} \Gamma\left(\frac{s+1}{2}\right)^{d^- + 2dr_2}} \sum_n \frac{\bar{\lambda}_\rho(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{(3)} \left(\frac{nX}{\sqrt{q}}\right)^{-u}$$

$$G(u) \frac{\pi^{-d(1-s+u)n/2}\ \Gamma\left(\frac{1-s+u}{2}\right)^{d^+ + 2dr_2} \Gamma\left(\frac{2-s+u}{2}\right)^{d^- + 2dr_2}}{\pi^{-d(1-s)n/2}\ \Gamma\left(\frac{1-s}{2}\right)^{d^+ + 2dr_2} \Gamma\left(\frac{2-s}{2}\right)^{d^- + 2dr_2}} \frac{du}{u}.$$

Our work

Recall our form of the elliptic part of the trace formula

$$\sum_{\pm p^k} \sum_{\mathrm{tr}(\gamma),\ldots,\mathrm{tr}(\gamma^{n-1})} \frac{1}{|s_\gamma|} L(1,\sigma_E)|D_\gamma|^{1/2}\mathrm{Orb}(f_\infty;\gamma)\prod_q \mathrm{Orb}(f_q;\gamma).$$

By applying the AFE and using some results of Shelstad on real orbital integrals [Shelstad 1979], we get

Theorem 1

Assume Artin's conjecture. Then

$$L(1,\sigma_E)|D_\gamma|^{1/2}Orb(f_\infty;\gamma)$$

is smooth. This result is unconditional for $n = 2, 3$.

What we'd like to do

- Work with a general reductive group instead of $GL(n)$.
- Study the product of p-adic orbital integrals.
- Apply Poisson summation to the resulting smooth expression.

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