The Langlands Program: Beyond Endoscopy

Oscar E. González\(^1\), oscar.gonzalez3@upr.edu
Kevin Kwan\(^2\), kevinkwanch@gmail.com

\(^1\)Department of Mathematics, University of Puerto Rico, Río Piedras.
\(^2\)Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.

Young Mathematicians Conference
The Ohio State University
August 21, 2016
What is the Langlands program?

- Conjectures relating many areas of math.
- Proposed by Robert P. Langlands about 50 years ago.
Automorphic Forms

A modular form of weight $k$ is a complex-valued function $f$ on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}, \text{Im}(z) > 0\}$ ($z = a + bi$ with $b > 0$), satisfying the following three conditions:
Automorphic Forms

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1. For any \( z \) in \( \mathbb{H} \) and any matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( SL_2(\mathbb{Z}) \), \( f \) satisfies the equation \( f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \).

2. \( f \) is a holomorphic (complex analytic) function on \( \mathbb{H} \).

3. \( f \) satisfies certain growth conditions at the cusps.
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1. For any $z$ in $\mathbb{H}$ and any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{Z})$, $f$ satisfies the equation $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$.
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Automorphic forms generalize modular forms.
Langlands’ Functoriality Conjecture

Conjecture 1.1

Given two reductive groups $H$, $G$ with a homomorphism $^LH \to ^LG$, there exists a related transfer of automorphic forms on $H$ to automorphic forms on $G \ \Pi(H) \to \Pi(G)$.
What are $L$-functions?

Diverse families of functions attached to arithmetic objects, for example:

- **Riemann zeta function**
  
  \[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } \Re s > 1, \]

  useful in studying distribution of primes.

- **Dirichlet $L$-function**
  
  \[ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \text{ for } \Re s > 1, \]

  where $\chi$ is a Dirichlet’s character, i.e. homomorphism
  \[ \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{S}^1. \]
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\[
\zeta_E(s) = \sum_a \frac{1}{N_{E/\mathbb{Q}}(a)^s},
\]

for $\Re s > 1$, where the sum is over all non-zero integral ideals of $E$. 
- Artin $L$-function: associated to a Galois representation.
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**Definition 2.1**

Let $E$ be a number field. Let $\rho : \text{Gal}(E/\mathbb{Q}) \to \text{GL}(d, \mathbb{C})$ be a finite dimensional Galois representation. The Artin $L$-function associated to $\rho$ is given in terms of the Euler product:

$$L(s, \rho) = \prod_p \det(I - \rho(Fr_p)p^{-s})^{-1},$$

where $Fr_p$ is the Frobenius element in $\text{Gal}(E/\mathbb{Q})$.

Denote by $\sigma_E$ the Artin representation obtained from the factorization $\zeta_E(s) = \zeta_{\mathbb{Q}}(s)L(s, \sigma_E)$, where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function.
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\[ \zeta_E(s) = \prod_p \frac{1}{1 - N_{E/\mathbb{Q}}(p)^s}. \]
Essential Properties of $L$-functions

- Can be completed by ‘gamma factors’. The completed $L$-functions are meromorphic on $\mathbb{C}$ with at most poles at $s = 0$ and $s = 1$ and they satisfy functional equations.
Functional Equations of $\zeta(s)$

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- Completion: $\Lambda(s) = \gamma(s) \zeta(s)$. 

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For a primitive character $\chi$ (i.e., not induced by any character of smaller modulus), we have the following.

- **Gamma factor:** $\gamma(s) = \pi^{-s/2} \Gamma((s + \delta)/2)$, where $\delta = 0$ if $\chi(-1) = 1$ and $\delta = 1$ if $\chi(-1) = -1$. 
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- $L(s, \chi)$ is entire if $\chi \neq \chi_0$. If $\chi = \chi_0$, then $L(s, \chi)$ is analytic on $\mathbb{C}$ except having a simple pole at $s = 1$. 
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- Gamma factor: $\gamma(s) = \Gamma_R(s)^{r_1} \Gamma_C(s)^{2r_2}$, where $\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)$, $\Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s)$, $r_1$ and $2r_2$ are the number of real and complex places of the number field $E$. 
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- $r_1$ is the number of real places of $E$,
- $2r_2$ is the number of complex places of $E$ ($r_1 + 2r_2 = n$),
- $h_E$ is the class number of $E$,
- $R_E$ is the regulator of $E$,
- $w_E$ is the number of roots of unity in $E$

**Theorem 2.2 (Class Number Formula)**

$$\text{Res}_{s=1} \zeta_E(s) = \frac{2^{r_1}(2\pi)^{r_2}h_E R_E}{w_E \sqrt{|D_E|}}.$$
Arthur-Selberg Trace Formula

**Definition 2.3 (Right Regular Representation of $G$)**

*The right regular representation of group $G$ on $L^2(\Gamma \backslash G)$, denoted by $R$, is defined by*

$$R(y)\phi(x) = \phi(xy),$$

*where $x, y \in G$, $\phi \in L^2(\Gamma \backslash G)$.*

**Definition 2.4 (Right Regular Operator)**

*Let $f \in C_c(G)$. We define the right regular operator $R(f)$ on $L^2(\Gamma \backslash G)$ as follows: for $\phi \in L^2(\Gamma \backslash G)$, we have*

$$R(f)\phi(x) = \int_G f(x)R(y)\phi(x)dy = \int_G f(y)\phi(xy)dy.$$
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- The ring of adeles is the **restricted** direct product of all $\mathbb{Q}_p$, where $p \leq \infty$. We denote the ring of adeles by $\mathbb{A}$. 
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In this way $\mathbb{A}$ is locally compact and we can assign a Haar measure on $G(\mathbb{A})$, where $G = GL(n)$. 
Arthur-Selberg Trace Formula

- $\gamma \in G(\mathbb{Q})$ is said to be **elliptic** if its characteristic polynomial is irreducible over $\mathbb{Q}$.
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Denote by $G_\gamma$ the centralizer of $\gamma$ in $G$.

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**Theorem 2.5 (Elliptic Part of Arthur-Selberg Trace Formula)**

$$
\text{tr } R(f) = \sum_{\gamma \text{ ell}} \text{meas}(\gamma) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1} \gamma g) dg + \cdots
$$

$$
= \sum_{\gamma \text{ ell}} \text{meas}(\gamma) \prod_v \int_{G_\gamma(\mathbb{Q}_v) \backslash G(\mathbb{Q}_v)} f_v(g^{-1} \gamma g) dg_v + \cdots
$$
Idea of Beyond Endoscopy and $L$-functions

- Given an automorphic form $\pi$ on $G$ and a representation $r$ of $L^1 G$, one can define an automorphic $L$-function

\[
L(s, \pi, r) = \sum_{n=1}^{\infty} \frac{a_{\pi,r}(n)}{n^s}.
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- Let $G, G'$ be reductive groups. If an automorphic form $\pi$ on $G$ is a functorial transfer from a smaller $G'$, then one expects the $L$-function $L(s, \pi, r)$ to have a pole at $s = 1$ for some representation $r$ of $L^G$. 
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- Let $G, G'$ be reductive groups. If an automorphic form $\pi$ on $G$ is a functorial transfer from a smaller $G'$, then one expects the $L$-function $L(s, \pi, r)$ to have a pole at $s = 1$ for some representation $r$ of $L^*G$.

- In particular, the order of pole of $L(s, \pi, r)$ at $s = 1$, which we denote by $m_r(\pi)$, should be nonzero if and only if $\pi$ is a transfer.
Langlands’ idea is to weight the spectral terms in the stable trace formula by $m_r(\pi)$, resulting in a trace formula whose spectral side detects only $\pi$ for which the $m_r(\pi)$ is nonzero.
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- Langlands’ idea is to weight the spectral terms in the stable trace formula by $m_r(\pi)$, resulting in a trace formula whose spectral side detects only $\pi$ for which the $m_r(\pi)$ is nonzero.
- Since in general $L(s, \pi, r)$ is not a priori defined at $s = 1$, we account for the weight factor by taking the residue at $s = 1$ of the logarithmic derivative of $L(s, \pi, r)$. 
Outline of Altuğ’s work

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- Give other versions of trace formula to be used in detecting the functorial transfer.
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There are singularities from the real orbital integrals.
By using the Class Number Formula, we have

\[
\text{meas}(\gamma) = \text{meas}(Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) \\
= \text{meas}(E^\times \backslash I^1_E) \\
= \frac{2^{r_1}(2\pi)^{r_2} h_E R_E}{w_E} \\
= \sqrt{|D_E|} \ L(1, \sigma_E).
\]
Using the CNF, we can write the elliptic part of the trace formula as follows:

\[
\sum_{\pm p^k \text{ tr}(\gamma), \ldots, \text{tr}(\gamma^{n-1})} \sum \frac{1}{|s_{\gamma}|} L(1, \sigma_E) |D_\gamma|^{1/2} \text{Orb}(f_\infty; \gamma) \prod_q \text{Orb}(f_q; \gamma).
\]
Assuming Artin’s conjecture, we can write the approximate functional equation of $L(1, \sigma_E)$.
\[ L(s, \rho) = \sum_n \frac{\lambda_{\rho}(n)}{n^s} \frac{1}{2\pi i} \int_{(3)} \left( \frac{n}{X\sqrt{q}} \right)^{-u} G(u) \]

\[
\begin{align*}
&\quad \pi^{-d(s+u)n/2} \Gamma\left(\frac{s+u}{2}\right)^{d^++2dr_2} \Gamma\left(\frac{s+u+1}{2}\right)^{d^-+2dr_2} \\
&\quad \pi^{-d(s)n/2} \Gamma\left(\frac{s}{2}\right)^{d^++2dr_2} \Gamma\left(\frac{s+1}{2}\right)^{d^-+2dr_2} \\
&\quad \pi^{-d(1-s)n/2} \Gamma\left(\frac{1-s}{2}\right)^{d^++2dr_2} \Gamma\left(\frac{2-s}{2}\right)^{d^-+2dr_2} \\
&\quad \pi^{-d(1-s+u)n/2} \Gamma\left(\frac{1-s+u}{2}\right)^{d^++2dr_2} \Gamma\left(\frac{2-s+u}{2}\right)^{d^-+2dr_2} \\
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\end{align*}
\]

\[ G(u) \frac{\lambda_{\rho}(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{(3)} \left( \frac{nX}{\sqrt{q}} \right)^{-u} \]

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\[ d\left(1-s+u\right) \int_{(3)} \left( \frac{nX}{\sqrt{q}} \right)^{-u} \]

\[ d\left(1-s\right) \int_{(3)} \left( \frac{nX}{\sqrt{q}} \right)^{-u} \]
Recall our form of the elliptic part of the trace formula

\[ \sum_{\pm p^k \text{ tr}(\gamma),\ldots,\text{tr}(\gamma^{n-1})} \sum \frac{1}{|s_{\gamma}|} L(1, \sigma_E)|D_{\gamma}|^{1/2} \text{Orb}(f_\infty; \gamma) \prod_q \text{Orb}(f_q; \gamma). \]

By applying the AFE and using some results of Shelstad on real orbital integrals [Shelstad 1979], we get

**Theorem 1**

*Assume Artin’s conjecture. Then*

\[ L(1, \sigma_E)|D_{\gamma}|^{1/2} \text{Orb}(f_\infty; \gamma) \]

*is smooth. This result is unconditional for n = 2, 3.*
### What we’d like to do

- Work with a general reductive group instead of $GL(n)$.
- Study the product of $p$-adic orbital integrals.
- Apply Poisson summation to the resulting smooth expression.
The Ohio State University
Rodica Costin, Craig Jackson, Bart Snapp
Williams College
NSF grants DMS1347804, DMS1561945 and DMS1265673
The Chinese University of Hong Kong
Steven J. Miller and Tian An Wong
Roger Van Peski


