Results on GL(2) L-Functions: Biases in Coefficients and Gaps Between Zeros

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http://web.williams.edu/Mathematics/sjmiller/public_html/

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Gaps between Critical Zeros

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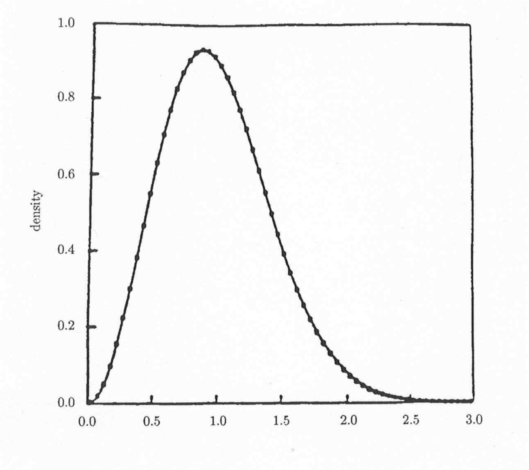
The Random Matrix Theory Connection

Philosophy: Critical-zero statistics of L-functions agree with eigenvalue statistics of large random matrices.

- Montgomery - pair-correlations of zeros of $\zeta(s)$ and eigenvalues of the Gaussian Unitary Ensemble.
- Hejhal, Rudnick and Sarnak - Higher correlations and automorphic L-functions.
- Odlyzko - further evidence through extensive numerical computations.

Consecutive Zero Spacings



Consecutive zero spacings of $\zeta(s)$ vs. GUE predictions (Odlyzko).

Large Gaps between Zeros

Let $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_i \leq \cdots$ be the ordinates of the critical zeros of an L-function.

Conjecture

Gaps between consecutive zeros that are arbitrarily large, relative to the average gap size, appear infinitely often.

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$$\text{Letting} \quad \Lambda = \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\text{average spacing}},$$

this conjecture is equivalent to $\Lambda = \infty$.

- Best unconditional result for the Riemann zeta function is $\Lambda > 2.69$.

Degree 2 Case

Higher degree *L*-functions are mostly unexplored.

Theorem (Turnage-Butterbaugh '14)

Let $T \geq 2$, $\varepsilon > 0$, $\zeta_K(s)$ the Dedekind zeta function attached to a quadratic number field K with discriminant d with $|d| \leq T^\varepsilon$, and $\mathcal{S}_T := \{\gamma_1, \gamma_2, \dots, \gamma_N\}$ be the distinct zeros of $\zeta_K\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let κ_T denote the maximum gap between consecutive zeros in $\mathcal{S}_T$. Then

$$\kappa_T \;\geq\; \sqrt{6}\frac{\pi}{\log\sqrt{|d|}T}\left(1 + O(d^\varepsilon \log T)^{-1}\right).$$

- Assuming GRH, this means $\Lambda \geq \sqrt{6} \approx 2.449$.

A Lower Bound on Large Gaps

We proved the following unconditional theorem for an L-function associated to a holomorphic cusp form f on $\mathrm{GL}(2)$.

Theorem (BMMRTW '14)

Let $\mathcal{S}_T := \{\gamma_1, \gamma_2, ..., \gamma_N\}$ be the set of distinct zeros of $L\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let κ_T denote the maximum gap between consecutive zeros in $\mathcal{S}_T$. Then

$$\kappa_T \geq \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right),$$

where c_f is the residue of the Rankin-Selberg convolution $L(s, f \times \bar{f})$ at $s = 1$.

Assuming GRH, there are infinitely many normalized gaps between consecutive zeros at least $\sqrt{3}$ times the mean spacing, i.e.,

$$\Lambda \geq \sqrt{3} \approx 1.732.$$

An Upper Bound on Small Gaps

Theorem (BMMRTW '14)

L in Selberg class primitive of degree m_L. Assume GRH for $\log L(s) = \sum_{n=1}^{\infty} b_L(n)/n^s$, $\sum_{n \le x} |b_L(n) \log n|^2 = (1 + o(1))x \log x$. Have a computable nontrivial upper bound on μ_L (liminf of smallest average gap) depending on m_L.

m_L	upper bound for μ_L
1	0.606894
2	0.822897
3	0.905604
4	0.942914
5	0.962190
⋮	⋮

($m_L = 1$ due to Carneiro, Chandee, Littmann and Milinovich).

Key idea: use pair correlation analysis.

Results on Gaps and Shifted Second Moments

Shifted Moment Result

To prove our theorem, use a method due to R.R. Hall and the following shifted moment result.

Theorem (BMMRTW '14)

$$\int_T^{2T} L\left(\frac{1}{2}+it+\alpha, f\right) L\left(\frac{1}{2}-it+\beta, f\right) dt$$

$$= c_f T \sum_{n\geq 0} \frac{(-1)^n 2^{n+1} (\alpha+\beta)^n (\log T)^{n+1}}{(n+1)!} + O\left(T (\log T)^{1-\delta}\right),$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|, |\beta| \ll 1/\log T$.

Key idea: differentiate wrt parameters, yields formulas for integrals of products of derivatives.

Shifted Moments Proof Technique

- Approximate functional equation:

$$L(s+\alpha, f) \ = \ \sum_{n\geq 1} \frac{\lambda_f(n)}{n^{s+\alpha}} e^{-\frac{n}{X}} + F(s) \sum_{n\leq X} \frac{\lambda_f(n)}{n^{1-s-\alpha}} + E(s),$$

  where $\lambda_f(n)$ are the Fourier coefficients of $L(s, f)$, $F(s)$ is a functional equation term, and $E(s)$ is an error term.

- We have an analogous expression for $L(1 - s + \beta, f)$.

Shifted Moments Proof Technique

- Analyze product

$$L(s+\alpha,f)L(1-s+\beta,f),$$

  where each factor gives rise to four products (so sixteen products to estimate).

- Use a generalization of Montgomery and Vaughan's mean value theorem and contour integration to estimate product and compute the resulting moments.

Shifted Moment Result for Derivatives

- Shifted moment result yields lower order terms and moments of derivatives of L-functions by differentiation and Cauchy's integral formula.

- Derive an expression for

$$\int_T^{2T} L^{(\mu)}\left(\frac{1}{2}+it, f\right) L^{(\nu)}\left(\frac{1}{2}-it, f\right) dt,$$

  where $T \geq 2$ and $\mu, \nu \in \mathbb{Z}^+$. Use this in Hall's method to obtain the lower bound stated in our theorem.

- Need $(\mu, \nu) \in \{(0,0),(1,0),(1,1)\}$; other cases previously done (Good did $(0,0)$ and Yashiro did $\mu = \nu$).

Modified Wirtinger Inequality

Using Hall's method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg.

Lemma (Bredberg)

Let $y : [a, b] \to \mathbb{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then

$$\int_a^b |y(x)|^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 dx.$$

Proving our Result

- For ρ a real parameter to be determined later, define
$$g(t) \ := \ e^{i\rho t \log T} L\left(\tfrac{1}{2} + it, f\right),$$
Fix f and let $\tilde{\gamma}_f(k)$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.

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- $g(t)$ has same zeros as $L(s, f)$ (at $t = \tilde{\gamma}_f(k)$). Use in the modified Wirtinger's inequality.

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- $g(t)$ has same zeros as $L(s,f)$ (at $t=\tilde{\gamma}_f(k)$). Use in the modified Wirtinger's inequality.

- For adjacent zeros have

$$\sum_{n=1}^{N-1} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g(t)|^2 dt \;\leq\; \sum_{n=1}^{N-1} \frac{\kappa_T^2}{\pi^2} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g'(t)|^2 dt.$$

- Summing over zeros with $n \in \{1,\dots,N\}$ and trivial estimation yields integrals from T to $2T$.

Proving our Result

- $|g(t)|^2 = |L(1/2+it,f)|^2$ and

$$|g'(t)|^2 = |L'(1/2+it,f)|^2 + \rho^2 \log^2 T \cdot |L(1/2+it,f)|^2$$
$$+ 2\rho \log T \cdot \mathrm{Re}\left(L'(1/2+it,f)\overline{L(1/2+it,f)}\right).$$

- Apply sub-convexity bounds to $L(1/2+it,f)$:

$$\int_T^{2T} |g(t)|^2 dt \leq \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |g'(t)|^2 dt + O\left(T^{\frac{2}{3}}(\log T)^{\frac{5}{6}}\right).$$

- As $g(t)$ and $g'(t)$ may be expressed in terms of $L\left(\frac{1}{2}+it,f\right)$ and its derivatives, can write our inequality explicitly in terms of formula given by our mixed moment theorem.

Finishing the Proof

- After substituting our formula, we have

$$\frac{\kappa_T^2}{\pi^2} \ \geq \ \frac{3}{3\rho^2 - 6\rho + 4}(\log T)^{-2}\left(1 + O(\log T)^{-\delta}\right).$$

- The polynomial in ρ is minimized at $\rho = 1$, yielding

$$\kappa_T \ \geq \ \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right).$$

Essential $\mathrm{GL}(2)$ properties

Properties

For primitive f on $\mathrm{GL}(2)$ over $\mathbb{Q}$ (Hecke or Maass) with

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \quad \Re(s) > 1,$$

we isolate needed crucial properties (all are known).

1. $L(s,f)$ has an analytic continuation to an entire function of order 1.

2. $L(s,f)$ satisfies a function equation of the form

$$\begin{aligned} \Lambda(s,f) \quad &:= \quad L(s,f_\infty)L(s,f) \;=\; \epsilon_f \Lambda(1-s,\bar{f}) \\ \text{with } L(s,f_\infty) \quad &= \quad Q^s \Gamma\left(\frac{s}{2}+\mu_1\right)\Gamma\left(\frac{s}{2}+\mu_2\right). \end{aligned}$$

Properties (continued)

3. Convolution L-function $L(s, f \times \bar{f})$,

$$\sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^s}, \quad \Re(s) > 1,$$

is entire except for a simple pole at $s = 1$.

4. The Dirichlet coefficients (normalized so that the critical strip is $0 \leq \Re(s) \leq 1$) satisfy

$$\sum_{n \leq x} |a_f(n)|^2 \ll x.$$

5. For some small $\delta > 0$, we have a subconvexity bound

$$\left| L\left(\frac{1}{2} + it, f\right) \right| \ll |t|^{\frac{1}{2} - \delta}.$$

Properties (status)

- Mœglin and Waldspurger prove the needed properties of $L(s, f \times \overline{f})$ (in greater generality).
- Dirichlet coefficient asymptotics follow for Hecke forms essentially from the work of Rankin and Selberg, and for Maass by spectral theory.
- Michel and Venkatesh proved a subconvexity bound for primitive $GL(2)$ L-functions over $\mathbb{Q}$.
- Other properties are standard and are valid for $\mathrm{GL}(2)$.

Bias Conjecture for Moments of Fourier Coefficients of Elliptic Curve L-functions

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Families and Moments

A *one-parameter family* of elliptic curves is given by

$$\mathcal{E} : y^2 \;=\; x^3 + A(T)x + B(T)$$

where $A(T), B(T)$ are polynomials in $\mathbb{Z}[T]$.

- Each specialization of T to an integer t gives an elliptic curve $\mathcal{E}(t)$ over $\mathbb{Q}$.
- The r^{th} *moment* of the Fourier coefficients is

$$A_{r,\mathcal{E}}(p) \;=\; \sum_{t \mod p} a_{\mathcal{E}(t)}(p)^r.$$

Tate's Conjecture

Tate's Conjecture for Elliptic Surfaces

Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\text{ét}}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$-\mathrm{ord}_{s=2} L_2(\mathcal{E}, s) \ = \ \mathrm{rank}\, NS(\mathcal{E}/\mathbb{Q}),$$

where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $\mathrm{Re}(s) = 2$.

Tate's conjecture is known for rational surfaces: An elliptic surface $y^2 = x^3 + A(T)x + B(T)$ is rational iff one of the following is true:

- $0 < \max\{3\deg A, 2\deg B\} < 12$;
- $3\deg A = 2\deg B = 12$ and $\mathrm{ord}_{T=0} T^{12}\Delta(T^{-1}) = 0$.

Negative Bias in the First Moment

$A_{1,\mathcal{E}}(p)$ and Family Rank (Rosen-Silverman)

If Tate's Conjecture holds for $\mathcal{E}$ then

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \frac{A_{1,\mathcal{E}}(p)\log p}{p} \ = \ -\mathrm{rank}(\mathcal{E}/\mathbb{Q}).$$

- By the Prime Number Theorem, $A_{1,\mathcal{E}}(p) = -rp + O(1)$ implies $\mathrm{rank}(\mathcal{E}/\mathbb{Q}) = r$.

Bias Conjecture

Second Moment Asymptotic (Michel)

For families $\mathcal{E}$ with $j(T)$ non-constant, the second moment is

$$A_{2,\mathcal{E}}(p) \ = \ p^2 + O(p^{3/2}).$$

- The lower order terms are of sizes $p^{3/2}$, p, $p^{1/2}$, and 1.

Bias Conjecture

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- The lower order terms are of sizes $p^{3/2}$, p, $p^{1/2}$, and 1.

In every family we have studied, we have observed:

Bias Conjecture

The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.

Preliminary Evidence and Patterns

Let $n_{3,2,p}$ equal the number of cube roots of 2 modulo p, and set $c_0(p) = \left[\left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)\right] p$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3 - x}{p}\right)\right]^2$, $c_{3/2}(p) = p \sum_{x(p)} \left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2 - 2p & p \equiv 2 \bmod 3 \\ 0 & p \equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2 - 2p & p \equiv 1 \bmod 4 \\ 0 & p \equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and otherwise is 0.

Preliminary Evidence and Patterns

The first family is the family of all elliptic curves; it is a two parameter family and we expect the main term of its second moment to be p^3.

Note that except for our family $y^2 = x^3 + Tx^2 + 1$, all the families $\mathcal{E}$ have $A_{2,\mathcal{E}}(p) = p^2 - h(p)p + O(1)$, where $h(p)$ is non-negative. Further, many of the families have $h(p) = m_{\mathcal{E}} > 0$.

Note $c_1(p)$ is the square of the coefficients from an elliptic curve with complex multiplication. It is non-negative and of size p for $p \not\equiv 3 \bmod 4$, and zero for $p \equiv 1 \bmod 4$ (send $x \mapsto -x \bmod p$ and note $\left(\frac{-1}{p}\right) = -1$).

It is somewhat remarkable that all these families have a correction to the main term in Michel's theorem in the same direction, and we analyze the consequence this has on the average rank. For our family which has a $p^{3/2}$ term, note that on average this term is zero and the p term is negative.

Lower order terms and average rank

$$\frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t}\phi\left(\gamma_t\frac{\log R}{2\pi}\right) = \widehat{\phi}(0)+\phi(0)-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right)a_t(p)$$
$$-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)a_t(p)^2+O\left(\frac{\log\log R}{\log R}\right).$$

If ϕ is non-negative, we obtain a bound for the average rank in the family by restricting the sum to be only over zeros at the central point. The error $O\left(\frac{\log\log R}{\log R}\right)$ comes from trivial estimation and ignores probable cancellation, and we expect $O\left(\frac{1}{\log R}\right)$ or smaller to be the correct magnitude. For most families $\log R \sim \log N^a$ for some integer a.

Lower order terms and average rank (cont)

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

Assume the second moment of $a_t(p)^2$ is $p^2 - m_{\mathcal{E}}p + O(1)$, $m_{\mathcal{E}} > 0$.

We have already handled the contribution from p^2, and $-m_{\mathcal{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathcal{E}}p) \\ &= \frac{2m_{\mathcal{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned}$$

Thus there is a contribution of size $\frac{1}{\log R}$.

Lower order terms and average rank (cont)

A good choice of test functions (see Appendix A of [ILS]) is the Fourier pair

$$\phi(x) = \frac{\sin^2(2\pi\frac{\sigma}{2}x)}{(2\pi x)^2}, \quad \widehat{\phi}(u) = \begin{cases} \frac{\sigma - |u|}{4} & \text{if } |u| \le \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Note $\phi(0) = \frac{\sigma^2}{4}$, $\widehat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma}$, and evaluating the prime sum gives

$$S_2 \sim \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_{\mathcal{E}}}{\log R}\, \phi(0).$$

Lower order terms and average rank (cont)

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned}\frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\leq \frac{\phi(0)}{\sigma} + \left(r + \frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\phi(0) \\ \text{Ave Rank}_{[N,2N]}(\mathcal{E}) &\leq \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}.\end{aligned}$$

$\sigma = 1$, $m_{\mathcal{E}} = 1$: for conductors of size 10^{12}, the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

$\sigma = 2$: lower order correction contributes .02 for conductors of size 10^{12}, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier's bound (already there without the potential correction term!).

Interpretation: Approaching semicircle 2nd moment from below

Sato-Tate Law for Families without CM

For large primes p, the distribution of $a_{\mathcal{E}(t)}(p)/\sqrt{p}$, $t \in \{0, 1, \dots, p-1\}$, approaches a semicircle on $[-2, 2]$.

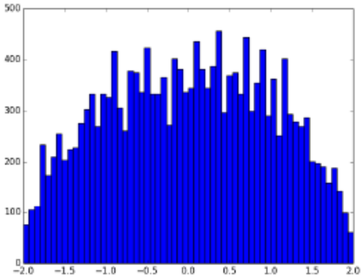

Figure: $a_{\mathcal{E}(t)}(p)$ for $y^2 = x^3 + Tx + 1$ at the 2014th prime.

Implications for Excess Rank

- Katz-Sarnak's one-level density statistic is used to measure the average rank of curves over a family.
- More curves with rank than expected have been observed, though this excess average rank vanishes in the limit.
- Lower-order biases in the moments of families explain a small fraction of this excess rank phenomenon.

Theoretical Evidence

Methods for Obtaining Explicit Formulas

For a family $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, we can write

$$a_{\mathcal{E}(t)}(p) \;=\; -\sum_{x \mod p} \left(\frac{x^3 + A(t)x + B(t)}{p}\right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol mod$p$ given by

$$\left(\frac{x}{p}\right) \;=\; \begin{cases} 1 & \text{if } x \text{ is a non-zero square modulo } p \\ 0 & \text{if } x \equiv 0 \bmod p \\ -1 & \text{otherwise.} \end{cases}$$

Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

$$\sum_{x \bmod p} \left(\frac{ax+b}{p}\right) = 0 \quad \text{if } p \nmid a$$

$$\sum_{x \bmod p} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \nmid b^2-4ac \\ (p-1)\left(\frac{a}{p}\right) & \text{if } p \mid b^2-4ac \end{cases}$$

Lemmas on Legendre Symbols

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Average Values of Legendre Symbols

The value of $\left(\frac{x}{p}\right)$ for $x \in \mathbb{Z}$, when averaged over all primes p, is 1 if x is a non-zero square, and 0 otherwise.

Rank 0 Families

Theorem (MMRW'14): Rank 0 Families Obeying the Bias Conjecture

For families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bx + cT + d$,

$$A_{2,\mathcal{E}}(p) \ = \ p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{a^2-3b}{p}\right)\right).$$

- The average bias in the size p term is -2 or -1, according to whether $a^2 - 3b \in \mathbb{Z}$ is a non-zero square.

Families with Rank

Theorem (MMRW'14): Families with Rank

For families of the form $\mathcal{E} : y^2 = x^3 + aT^2x + bT^2$,

$$A_{2,\mathcal{E}}(p) \ = \ p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{-3a}{p}\right)\right) - \left(\sum_{x(p)} \left(\frac{x^3+ax}{p}\right)\right)^2.$$

- These include families of rank 0, 1, and 2.
- The average bias in the size p terms is -3 or -2, according to whether $-3a \in \mathbb{Z}$ is a non-zero square.

Families with Complex Multiplication

Theorem (MMRW'14): Families with Complex Multiplication

For families of the form $\mathcal{E} : y^2 = x^3 + (aT + b)x$,

$$A_{2,\mathcal{E}}(p) \;=\; (p^2 - p)\left(1 + \left(\frac{-1}{p}\right)\right).$$

- The average bias in the size p term is -1.
- The size p^2 term is not constant, but is on average p^2, and an analogous Bias Conjecture holds.

Families with Unusual Distributions of Signs

Theorem (MMRW'14): Families with Unusual Signs

For the family $\mathcal{E} : y^2 = x^3 + Tx^2 - (T+3)x + 1$,

$$A_{2,\mathcal{E}}(p) \ = \ p^2 - p\left(2 + 2\left(\frac{-3}{p}\right)\right) - 1.$$

- The average bias in the size p term is -2.
- The family has an usual distribution of signs in the functional equations of the corresponding L-functions.

The Size $p^{3/2}$ Term

Theorem (MMRW'14): Families with a Large Error

For families of the form
$\mathcal{E} : y^2 = x^3 + (T + a)x^2 + (bT + b^2 - ab + c)x - bc$,

$$A_{2,\mathcal{E}}(p) = p^2 - 3p - 1 + p \sum_{x \bmod p} \left(\frac{-cx(x+b)(bx-c)}{p} \right)$$

- The size $p^{3/2}$ term is given by an elliptic curve coefficient and is thus on average 0.
- The average bias in the size p term is -3.

General Structure of the Lower Order Terms

The lower order terms appear to always

- have no size $p^{3/2}$ term or a size $p^{3/2}$ term that is on average 0;
- exhibit their negative bias in the size p term;
- be determined by polynomials in p, elliptic curve coefficients, and congruence classes of p (i.e., values of Legendre symbols).

Numerical Investigations

Numerical Methods

- As complexity of coefficients increases, it is much harder to find an explicit formula.
- We can always just calculate the second moment from the explicit formula; if $\mathcal{E}\colon y^2 = f(x)$, we have

$$A_{2,\mathcal{E}}(p) = \sum_{t(p)} \left(\sum_{x(p)} \left(\frac{f(x)}{p} \right) \right)^2 .$$

- Takes an hour for the first 500 primes. Optimizations?

Numerical Methods

Consider the family $y^2 = f(x) = ax^3 + (bT + c)x^2 + (dT + e)x + f$. By similar arguments used to prove special cases,

$$A_{2,\mathcal{E}}(p) \;=\; p^2 - 2p + pC_0(p) - pC_1(p) - 1 + \#_1,$$

where

$$C_0(p) \;=\; \sum_{x(p)} \sum_{y(p):\ \beta(x,y)\equiv 0} \left(\frac{A(x)A(y)}{p}\right),$$

$$C_1(p) \;=\; \sum_{x(p):\ \beta(x,x)\equiv 0} \left(\frac{A(x)^2}{p}\right),$$

$$\#_1 \;=\; p \sum_{x(p)} \sum_{y(p):\ A(x)\equiv 0 \text{ and } A(y)\equiv 0} \left(\frac{B(x)B(y)}{p}\right),$$

and β, A, and B are polynomials.

Numerical Methods

- $C_o(p)$ ordinarily $O(p^2)$ to compute.
- Sum over zeros of $\beta(x,y) \mod p$
- Fixing an x, β is a quadratic in y. So, with the quadratic formula mod p, we know where to look for y to see if there is a zero.
- Now $O(p)$; runs from 6000^{th} to 7000^{th} prime in an hour.

Potential Counterexamples

Families of Rank as Large as 3

$\mathcal{E} : y^2 = x^3 + ax^2 + bT^2x + cT^2$ with $b, c \neq 0$:

$$\begin{aligned} A_{2,\mathcal{E}}(p) \;&=\; p^2 + p \sum_{P(x,y)\equiv 0} \left(\frac{(x^3+bx)(y^3+by)}{p}\right) \\ &\quad + p\left[\sum_{x^3+bx\equiv 0}\left(\frac{ax^2+c}{p}\right)\right]^2 - p\sum_{P(x,x)\equiv 0}\left(\frac{x^3+bx}{p}\right)^2 \\ &\quad - p\left(2+\left(\frac{-b}{p}\right)\right) - \left[\sum_{x \bmod p}\left(\frac{x^3+bx}{p}\right)\right]^2 - 1 \end{aligned}$$

where $P(x, y) = bx^2y^2 + c(x^2 + xy + y^2) + bc(x + y)$.

A Positive Size p Term?

$p\left[\sum_{x^3+bx\equiv 0}\left(\frac{ax^2+c}{p}\right)\right]^2$ can be $+9p$ on average!

- Terms such as $-p\sum_{P(x,x)\equiv 0}\left(\frac{x^3+bx}{p}\right)^2$, $-p\left(2+\left(\frac{-b}{p}\right)\right)$, and $-\left[\sum_{x \bmod p}\left(\frac{x^3+bx}{p}\right)\right]^2$ contribute negatively to the size p bias.

- The term $p\sum_{P(x,y)\equiv 0}\left(\frac{(x^3+bx)(y^3+by)}{p}\right)$ is of size $p^{3/2}$.

$$A_{2,\mathcal{E}}(p) = p^2 + p\sum_{P(x,y)\equiv 0}\left(\frac{(x^3+bx)(y^3+by)}{p}\right) + p\left[\sum_{x^3+bx\equiv 0}\left(\frac{ax^2+c}{p}\right)\right]^2$$
$$-p\sum_{P(x,x)\equiv 0}\left(\frac{x^3+bx}{p}\right)^2 - p\left(2+\left(\frac{-b}{p}\right)\right) - \left[\sum_{x \bmod p}\left(\frac{x^3+bx}{p}\right)\right]^2 - 1$$

where $P(x,y) = bx^2y^2 + c(x^2+xy+y^2) + bc(x+y)$.

Analyzing the Size $p^{3/2}$ Term

We averaged $\sum_{P(x,y)\equiv 0}\left(\frac{(x^3+bx)(y^3+by)}{p}\right)$ over the first 10,000 primes for several rank 3 families of the form $\mathcal{E}: y^2 = x^3 + ax^2 + bT^2x + cT^2$.

Family	Average
$y^2 = x^3 + 2x^2 - 4T^2x + T^2$	-0.0238
$y^2 = x^3 - 3x^2 - T^2x + 4T^2$	-0.0357
$y^2 = x^3 + 4x^2 - 4T^2x + 9T^2$	-0.0332
$y^2 = x^3 + 5x^2 - 9T^2x + 4T^2$	-0.0413
$y^2 = x^3 - 5x^2 - T^2x + 9T^2$	-0.0330
$y^2 = x^3 + 7x^2 - 9T^2x + T^2$	-0.0311

The Right Object to Study

$c_{3/2}(p) := \sum_{P(x,y)\equiv 0}\left(\frac{(x^3+bx)(y^3+by)}{p}\right)$ is not a natural object to study (for us multiply by p).

An example distribution for $y^2 = x^3 + 2x^3 - 4T^2x + T^2$.

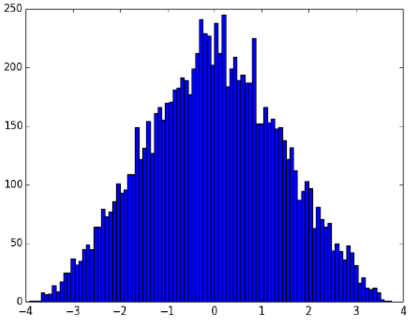

Figure: $c_{3/2}(p)$ over the first 10,000 primes.

In Terms of Elliptic Curve Coefficients

Compare it to the distribution of a sum of 2 elliptic curve coefficients.

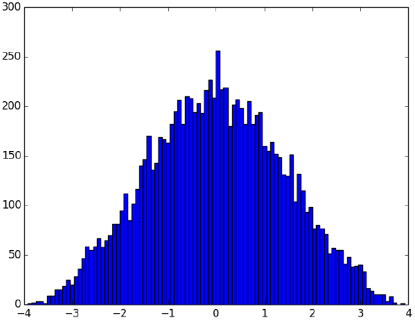

Figure: $-\sum_{x \bmod p} \left(\frac{x^3+x+1}{p}\right) - \sum_{x \bmod p} \left(\frac{x^3+x+2}{p}\right)$ over the first 10,000 primes.

More Error Distributions

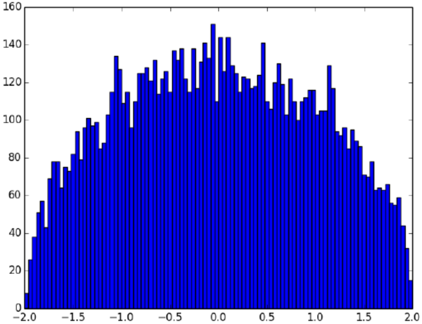

Figure: $c_{3/2}(p)$ for $y^2 = 4x^3 + 5x^2 + (4T - 2)x + 1$, first 10,000 primes.

More Error Distributions

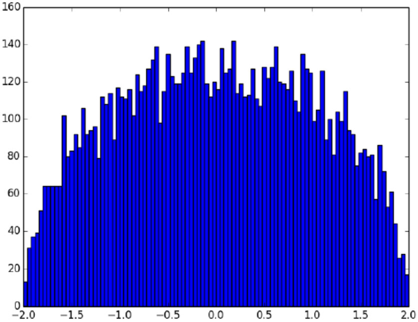

Figure: $-\sum_{x \bmod p} \left(\frac{x^3+x+1}{p}\right)$ distribution, first 10,000 primes.

More Error Distributions

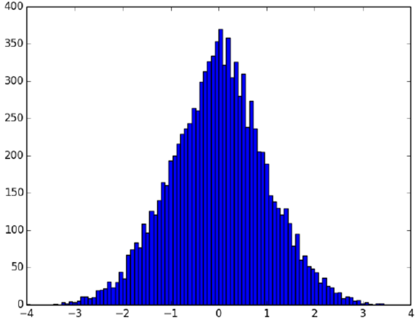

Figure: $c_{3/2}(p)$ over $y^2 = 4x^3 + (4T+1)x^2 + (-4T-18)x + 49$, first 10,000 primes.

More Error Distributions

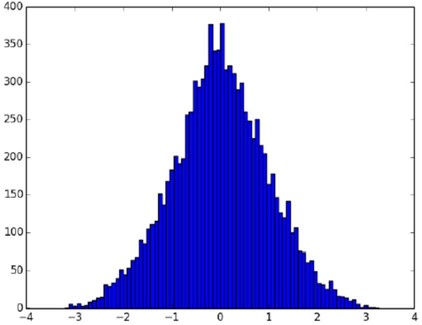

Figure: $-\sum_{x \bmod p} \left(\frac{x^5+x^3+x^2+x+1}{p}\right)$ distribution, first 10,000 primes.

Summary of $p^{3/2}$ Term Investigations

In the cases we've studied, the size $p^{3/2}$ terms

- appear to be governed by (hyper)elliptic curve coefficients;
- may be hiding negative contributions of size p;
- prevent us from numerically measuring average biases that arise in the size p terms.

Future Directions

Questions for Further Study

- Are the size $p^{3/2}$ terms governed by (hyper)elliptic curve coefficients? Or at least other L-function coefficients?

- Does the average bias always occur in the terms of size p?

- Does the Bias Conjecture hold similarly for all higher even moments?

- What other (families of) objects obey the Bias Conjecture? Kloosterman sums? Cusp forms of a given weight and level? Higher genus curves?

References

References

Gaps:

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Biases:

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Thank you!