Results on GL(2) $L$-Functions: Biases in Coefficients and Gaps Between Zeros

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http://web.williams.edu/Mathematics/sjmiller/public_html/

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Gaps between Critical Zeros

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The Random Matrix Theory Connection

Philosophy: Critical-zero statistics of $L$-functions agree with eigenvalue statistics of large random matrices.

- Montgomery - pair-correlations of zeros of $\zeta(s)$ and eigenvalues of the Gaussian Unitary Ensemble.

- Hejhal, Rudnick and Sarnak - Higher correlations and automorphic $L$-functions.

- Odlyzko - further evidence through extensive numerical computations.
Consecutive zero spacings of $\zeta(s)$ vs. GUE predictions (Odlyzko).
Large Gaps between Zeros

Let $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_i \leq \cdots$ be the ordinates of the critical zeros of an $L$-function.

**Conjecture**

Gaps between consecutive zeros that are arbitrarily large, relative to the average gap size, appear infinitely often.
Large Gaps between Zeros

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**Conjecture**

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Letting $\Lambda = \limsup_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{\text{average spacing}}$, this conjecture is equivalent to $\Lambda = \infty$.

- Best unconditional result for the Riemann zeta function is $\Lambda > 2.69$. 
Higher degree $L$-functions are mostly unexplored.

**Theorem (Turnage-Butterbaugh ’14)**

Let $T \geq 2$, $\varepsilon > 0$, $\zeta_K(s)$ the Dedekind zeta function attached to a quadratic number field $K$ with discriminant $d$ with $|d| \leq T^\varepsilon$, and $S_T := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ be the distinct zeros of $\zeta_K\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let $\kappa_T$ denote the maximum gap between consecutive zeros in $S_T$. Then

$$\kappa_T \geq \sqrt{6} \frac{\pi}{\log \sqrt{|d|} T} \left(1 + O(d^\varepsilon \log T)^{-1}\right).$$

- Assuming GRH, this means $\Lambda \geq \sqrt{6} \approx 2.449.$
A Lower Bound on Large Gaps

We proved the following unconditional theorem for an $L$-function associated to a holomorphic cusp form $f$ on $\text{GL}(2)$.

**Theorem (BMMRTW ’14)**

Let $S_T := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ be the set of distinct zeros of $L \left( \frac{1}{2} + it, f \right)$ in the interval $[T, 2T]$. Let $\kappa_T$ denote the maximum gap between consecutive zeros in $S_T$. Then

$$
\kappa_T \geq \frac{\sqrt{3}\pi}{\log T} \left( 1 + O \left( \frac{1}{c_f \left( \log T \right)^{-\delta}} \right) \right),
$$

where $c_f$ is the residue of the Rankin-Selberg convolution $L(s, f \times \bar{f})$ at $s = 1$.

Assuming GRH, there are infinitely many normalized gaps between consecutive zeros at least $\sqrt{3}$ times the mean spacing, i.e.,

$$
\Lambda \geq \sqrt{3} \approx 1.732.
$$
An Upper Bound on Small Gaps

**Theorem (BMMRTW ’14)**

$L$ in Selberg class primitive of degree $m_L$. Assume GRH for
\[ \log L(s) = \sum_{n=1}^{\infty} b_L(n)/n^s, \quad \sum_{n \leq x} |b_L(n) \log n|^2 = (1 + o(1)) x \log x. \]

Have a computable nontrivial upper bound on $\mu_L$ (liminf of smallest average gap) depending on $m_L$.

<table>
<thead>
<tr>
<th>$m_L$</th>
<th>upper bound for $\mu_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.606894</td>
</tr>
<tr>
<td>2</td>
<td>0.822897</td>
</tr>
<tr>
<td>3</td>
<td>0.905604</td>
</tr>
<tr>
<td>4</td>
<td>0.942914</td>
</tr>
<tr>
<td>5</td>
<td>0.962190</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

($m_L = 1$ due to Carneiro, Chandee, Littmann and Milinovich).

**Key idea:** use pair correlation analysis.
Results on Gaps and Shifted Second Moments
To prove our theorem, use a method due to R.R. Hall and the following shifted moment result.

**Theorem (BMMRTW ’14)**

\[
\int_T^{2T} L \left( \frac{1}{2} + it + \alpha, f \right) L \left( \frac{1}{2} - it + \beta, f \right) dt
= c_f T \sum_{n \geq 0} \frac{(-1)^n 2^{n+1} (\alpha + \beta)^n (\log T)^{n+1}}{(n + 1)!} + O \left( T(\log T)^{1-\delta} \right),
\]

where \( \alpha, \beta \in \mathbb{C} \) and \( |\alpha|, |\beta| \ll 1/\log T \).

Key idea: differentiate wrt parameters, yields formulas for integrals of products of derivatives.
Shifted Moments Proof Technique

- Approximate functional equation:

\[
L(s + \alpha, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{s+\alpha}} e^{-\frac{n}{X}} + F(s) \sum_{n \leq X} \frac{\lambda_f(n)}{n^{1-s-\alpha}} + E(s),
\]

where \( \lambda_f(n) \) are the Fourier coefficients of \( L(s, f) \), \( F(s) \) is a functional equation term, and \( E(s) \) is an error term.

- We have an analogous expression for \( L(1 - s + \beta, f) \).
Shifted Moments Proof Technique

- Analyze product

\[ L(s + \alpha, f)L(1 - s + \beta, f), \]

where each factor gives rise to four products (so sixteen products to estimate).

- Use a generalization of Montgomery and Vaughan’s mean value theorem and contour integration to estimate product and compute the resulting moments.
Shifted Moment Result for Derivatives

- Shifted moment result yields lower order terms and moments of derivatives of $L$-functions by differentiation and Cauchy’s integral formula.

- Derive an expression for

$$
\int_{T}^{2T} L^{(\mu)} \left( \frac{1}{2} + it, f \right) L^{(\nu)} \left( \frac{1}{2} - it, f \right) \, dt,
$$

where $T \geq 2$ and $\mu, \nu \in \mathbb{Z}^+$. Use this in Hall’s method to obtain the lower bound stated in our theorem.

- Need $(\mu, \nu) \in \{ (0, 0), (1, 0), (1, 1) \}$; other cases previously done (Good did $(0, 0)$ and Yashiro did $\mu = \nu$).
Using Hall’s method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg.

**Lemma (Bredberg)**

Let $y : [a, b] \to \mathbb{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then

$$
\int_a^b |y(x)|^2 dx \leq \left( \frac{b - a}{\pi} \right)^2 \int_a^b |y'(x)|^2 dx.
$$
For $\rho$ a real parameter to be determined later, define

$$g(t) := e^{i\rho t \log T} L \left( \frac{1}{2} + it, f \right),$$

Fix $f$ and let $\tilde{\gamma}_f(k)$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$. 
Proving our Result

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$g(t)$ has same zeros as $L(s, f)$ (at $t = \tilde{\gamma}_f(k)$). Use in the modified Wirtinger’s inequality.
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- $g(t)$ has same zeros as $L(s, f)$ (at $t = \tilde{\gamma}_f(k)$). Use in the modified Wirtinger’s inequality.

- For adjacent zeros have

$$\sum_{n=1}^{N-1} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g(t)|^2 dt \leq \sum_{n=1}^{N-1} \frac{k^2 T}{\pi^2} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g'(t)|^2 dt.$$

- Summing over zeros with $n \in \{1, \ldots, N\}$ and trivial estimation yields integrals from $T$ to $2T$. 
Proving our Result

- \(|g(t)|^2 = |L(1/2 + it, f)|^2\) and
  \[|g'(t)|^2 = |L'(1/2 + it, f)|^2 + \rho^2 \log^2 T \cdot |L(1/2 + it, f)|^2\]
  \[+ 2\rho \log T \cdot \text{Re} \left( L'(1/2 + it, f) L(1/2 + it, f) \right).\]

- Apply sub-convexity bounds to \(L(1/2 + it, f)\):
  \[\int_T^{2T} |g(t)|^2 dt \leq \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |g'(t)|^2 dt + O \left( T^\frac{2}{3} (\log T)^{\frac{5}{6}} \right).\]

- As \(g(t)\) and \(g'(t)\) may be expressed in terms of \(L \left( \frac{1}{2} + it, f \right)\) and its derivatives, can write our inequality explicitly in terms of formula given by our mixed moment theorem.
Finishing the Proof

- After substituting our formula, we have

\[
\frac{\kappa_T^2}{\pi^2} \geq \frac{3}{3\rho^2 - 6\rho + 4} (\log T)^{-2} \left(1 + O(\log T)^{-\delta}\right).
\]

- The polynomial in \(\rho\) is minimized at \(\rho = 1\), yielding

\[
\kappa_T \geq \frac{\sqrt{3}\pi}{\log T} \left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right).
\]
Essential $GL(2)$ properties
Properties

For primitive $f$ on $GL(2)$ over $\mathbb{Q}$ (Hecke or Maass) with

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \quad \Re(s) > 1,$$

we isolate needed crucial properties (all are known).

1. $L(s, f)$ has an analytic continuation to an entire function of order 1.

2. $L(s, f)$ satisfies a function equation of the form

$$\Lambda(s, f) := L(s, f_{\infty})L(s, f) = \epsilon_f \Lambda(1 - s, \bar{f})$$

with $L(s, f_{\infty}) = Q^s \Gamma \left( \frac{s}{2} + \mu_1 \right) \Gamma \left( \frac{s}{2} + \mu_2 \right)$. 
Properties (continued)

3. Convolution $L$-function $L(s, f \times \bar{f})$, 
\[
\sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^s}, \quad \Re(s) > 1,
\]
is entire except for a simple pole at $s = 1$.

4. The Dirichlet coefficients (normalized so that the critical strip is $0 \leq \Re(s) \leq 1$) satisfy 
\[
\sum_{n \leq x} |a_f(n)|^2 \ll x.
\]

5. For some small $\delta > 0$, we have a subconvexity bound 
\[
\left| L \left( \frac{1}{2} + it, f \right) \right| \ll |t|^{\frac{1}{2} - \delta}.
\]
Properties (status)

- Mœglin and Waldspurger prove the needed properties of $L(s, f \times \overline{f})$ (in greater generality).

- Dirichlet coefficient asymptotics follow for Hecke forms essentially from the work of Rankin and Selberg, and for Maass by spectral theory.

- Michel and Venkatesh proved a subconvexity bound for primitive $GL(2)$ $L$-functions over $\mathbb{Q}$.

- Other properties are standard and are valid for $GL(2)$. 
Bias Conjecture for Moments of Fourier Coefficients of Elliptic Curve $L$-functions

Joint with students Blake Mackall (Williams), Christina Rapti (Bard) and Karl Winsor (Michigan)

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A one-parameter family of elliptic curves is given by

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$$

where $A(T), B(T)$ are polynomials in $\mathbb{Z}[T]$. 

- Each specialization of $T$ to an integer $t$ gives an elliptic curve $\mathcal{E}(t)$ over $\mathbb{Q}$.
- The $r^{th}$ moment of the Fourier coefficients is

$$A_{r,\mathcal{E}}(p) = \sum_{t \mod p} a_{\mathcal{E}(t)}(p)^r.$$
Tate’s Conjecture

Tate’s Conjecture for Elliptic Surfaces

Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the $L$-series attached to $\mathbb{H}_{\text{et}}^2(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$-\text{ord}_{s=2} L_2(\mathcal{E}, s) = \text{rank } \text{NS}(\mathcal{E}/\mathbb{Q}),$$

where $\text{NS}(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $\text{Re}(s) = 2$.

Tate’s conjecture is known for rational surfaces: An elliptic surface $y^2 = x^3 + A(T)x + B(T)$ is rational iff one of the following is true:

- $0 < \max\{3\deg A, 2\deg B\} < 12$;
- $3\deg A = 2\deg B = 12$ and $\text{ord}_{T=0} T^{12} \Delta(T^{-1}) = 0$. 
Negative Bias in the First Moment

\( A_{1,\mathcal{E}}(p) \) and Family Rank (Rosen-Silverman)

If Tate’s Conjecture holds for \( \mathcal{E} \) then

\[
\lim_{X \to \infty} \frac{1}{X} \sum_{p \leq X} \frac{A_{1,\mathcal{E}}(p) \log p}{p} = -\text{rank}(\mathcal{E}/\mathbb{Q}).
\]

By the Prime Number Theorem,

\( A_{1,\mathcal{E}}(p) = -rp + O(1) \) implies \( \text{rank}(\mathcal{E}/\mathbb{Q}) = r. \)
Second Moment Asymptotic (Michel)

For families $\mathcal{E}$ with $j(T)$ non-constant, the second moment is

$$A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2}).$$

- The lower order terms are of sizes $p^{3/2}, p, p^{1/2},$ and 1.
Bias Conjecture

Second Moment Asymptotic (Michel)

For families $\mathcal{E}$ with $j(T)$ non-constant, the second moment is

$$A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2}).$$

- The lower order terms are of sizes $p^{3/2}$, $p$, $p^{1/2}$, and 1.

In every family we have studied, we have observed:

Bias Conjecture

The largest lower term in the second moment expansion which does not average to 0 is on average negative.
Preliminary Evidence and Patterns

Let \( n_{3,2,p} \) equal the number of cube roots of 2 modulo \( p \), and set
\[
    c_0(p) = \left[ \left( \frac{-3}{p} \right) + \left( \frac{3}{p} \right) \right] p,
    c_1(p) = \left[ \sum_{x \mod p} \left( \frac{x^3 - x}{p} \right) \right]^2,
    c_{3/2}(p) = p \sum_{x(p)} \left( \frac{4x^3 + 1}{p} \right).
\]

<table>
<thead>
<tr>
<th>Family</th>
<th>( A_{1,\varepsilon}(p) )</th>
<th>( A_{2,\varepsilon}(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y^2 = x^3 + Sx + T )</td>
<td>0</td>
<td>( p^3 - p^2 )</td>
</tr>
<tr>
<td>( y^2 = x^3 + 2^4(-3)^3(9T + 1)^2 )</td>
<td>0</td>
<td>( \begin{cases} 2p^2 - 2p &amp; p \equiv 2 \pmod{3} \ 0 &amp; p \equiv 1 \pmod{3} \end{cases} )</td>
</tr>
<tr>
<td>( y^2 = x^3 \pm 4(4T + 2)x )</td>
<td>0</td>
<td>( \begin{cases} 2p^2 - 2p &amp; p \equiv 1 \pmod{4} \ 0 &amp; p \equiv 3 \pmod{4} \end{cases} )</td>
</tr>
<tr>
<td>( y^2 = x^3 + (T + 1)x^2 + Tx )</td>
<td>0</td>
<td>( p^2 - 2p - 1 )</td>
</tr>
<tr>
<td>( y^2 = x^3 + x^2 + 2T + 1 )</td>
<td>0</td>
<td>( p^2 - 2p - \left( \frac{-3}{p} \right) )</td>
</tr>
<tr>
<td>( y^2 = x^3 + Tx^2 + 1 )</td>
<td>(-p)</td>
<td>( p^2 - n_{3,2,p}p - 1 + c_{3/2}(p) )</td>
</tr>
<tr>
<td>( y^2 = x^3 - T^2x + T^2 )</td>
<td>(-2p)</td>
<td>( p^2 - p - c_1(p) - c_0(p) )</td>
</tr>
<tr>
<td>( y^2 = x^3 - T^2x + T^4 )</td>
<td>(-2p)</td>
<td>( p^2 - p - c_1(p) - c_0(p) )</td>
</tr>
</tbody>
</table>

where \( c_{p,a;m} = 1 \) if \( p \equiv a \pmod{m} \) and otherwise is 0.
Preliminary Evidence and Patterns

The first family is the family of all elliptic curves; it is a two parameter family and we expect the main term of its second moment to be $p^3$.

Note that except for our family $y^2 = x^3 + Tx^2 + 1$, all the families $\mathcal{E}$ have $A_{2,\mathcal{E}}(p) = p^2 - h(p)p + O(1)$, where $h(p)$ is non-negative. Further, many of the families have $h(p) = m_{\mathcal{E}} > 0$.

Note $c_1(p)$ is the square of the coefficients from an elliptic curve with complex multiplication. It is non-negative and of size $p$ for $p \not \equiv 3 \mod 4$, and zero for $p \equiv 1 \mod 4$ (send $x \mapsto -x \mod p$ and note $(\frac{-1}{p}) = -1$).

It is somewhat remarkable that all these families have a correction to the main term in Michel’s theorem in the same direction, and we analyze the consequence this has on the average rank. For our family which has a $p^{3/2}$ term, note that on average this term is zero and the $p$ term is negative.
Lower order terms and average rank

\[
\frac{1}{N} \sum_{t=N}^{2N} \sum_{\gamma_t} \phi \left( \gamma_t \frac{\log R}{2\pi} \right) = \hat{\phi}(0) + \phi(0) - \frac{2}{N} \sum_{t=N}^{2N} \sum_{p} \frac{\log p}{\log R} \frac{1}{p} \phi \left( \frac{\log p}{\log R} \right) a_t(p)
\]

\[
- \frac{2}{N} \sum_{t=N}^{2N} \sum_{p} \frac{\log p}{\log R} \frac{1}{p^2} \phi \left( \frac{2 \log p}{\log R} \right) a_t(p)^2 + O \left( \frac{\log \log R}{\log R} \right).
\]

If \( \phi \) is non-negative, we obtain a bound for the average rank in the family by restricting the sum to be only over zeros at the central point. The error \( O \left( \frac{\log \log R}{\log R} \right) \) comes from trivial estimation and ignores probable cancellation, and we expect \( O \left( \frac{1}{\log R} \right) \) or smaller to be the correct magnitude. For most families \( \log R \sim \log N^a \) for some integer \( a \).
The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

Assume the second moment of $a_t(p)^2$ is $p^2 - m_\varepsilon p + O(1)$, $m_\varepsilon > 0$.

We have already handled the contribution from $p^2$, and $-m_\varepsilon p$ contributes

\[
S_2 \sim \frac{-2}{N} \sum_p \frac{\log p}{\log R} \hat{\phi} \left(2 \frac{\log p}{\log R} \right) \frac{1}{p^2} \frac{N}{p} (-m_\varepsilon p)
\]

\[
= \frac{2m_\varepsilon}{\log R} \sum_p \hat{\phi} \left(2 \frac{\log p}{\log R} \right) \frac{\log p}{p^2}.
\]

Thus there is a contribution of size $\frac{1}{\log R}$. 
A good choice of test functions (see Appendix A of [ILS]) is the Fourier pair

\[
\phi(x) = \frac{\sin^2\left(2\pi \frac{\sigma}{2} x\right)}{(2\pi x)^2}, \quad \hat{\phi}(u) = \begin{cases} 
\frac{\sigma - |u|}{4} & \text{if } |u| \leq \sigma \\
0 & \text{otherwise.}
\end{cases}
\]

Note \( \phi(0) = \frac{\sigma^2}{4} \), \( \hat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma} \), and evaluating the prime sum gives

\[
S_2 \sim \left( \frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R} \right) \frac{m_\varepsilon}{\log R} \phi(0).
\]
Lower order terms and average rank (cont)

Let $r_t$ denote the number of zeros of $E_t$ at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$
\frac{1}{N} \sum_{t=N}^{2N} r_t \phi(0) \leq \frac{\phi(0)}{\sigma} + \left(r + \frac{1}{2}\right) \phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_\mathcal{E}}{\log R} \phi(0)
$$

Ave Rank$_{[N,2N]}(\mathcal{E}) \leq \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_\mathcal{E}}{\log R}$.

$\sigma = 1$, $m_\mathcal{E} = 1$: for conductors of size $10^{12}$, the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier’s observed $r + \frac{1}{2} + .40$.

$\sigma = 2$: lower order correction contributes $.02$ for conductors of size $10^{12}$, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier’s bound (already there without the potential correction term!).
Interpretation: Approaching semicircle 2nd moment from below

**Sato-Tate Law for Families without CM**

For large primes $p$, the distribution of $a_{\mathcal{E}(t)}(p)/\sqrt{p}$, $t \in \{0, 1, \ldots, p - 1\}$, approaches a semicircle on $[-2, 2]$.

**Figure:** $a_{\mathcal{E}(t)}(p)$ for $y^2 = x^3 + Tx + 1$ at the 2014th prime.
Implications for Excess Rank

- Katz-Sarnak’s one-level density statistic is used to measure the average rank of curves over a family.

- More curves with rank than expected have been observed, though this excess average rank vanishes in the limit.

- Lower-order biases in the moments of families explain a small fraction of this excess rank phenomenon.
Theoretical Evidence
Methods for Obtaining Explicit Formulas

For a family $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, we can write

$$a_{\mathcal{E}(t)}(p) = -\sum_{x \mod p} \left( \frac{x^3 + A(t)x + B(t)}{p} \right)$$

where $\left( \frac{x}{p} \right)$ is the Legendre symbol mod $p$ given by

$$\left( \frac{x}{p} \right) = \begin{cases} 1 & \text{if } x \text{ is a non-zero square modulo } p \\ 0 & \text{if } x \equiv 0 \mod p \\ -1 & \text{otherwise.} \end{cases}$$
Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

\[
\sum_{x \mod p} \left( \frac{ax + b}{p} \right) = 0 \quad \text{if} \quad p \nmid a
\]

\[
\sum_{x \mod p} \left( \frac{ax^2 + bx + c}{p} \right) = \begin{cases} 
- \left( \frac{a}{p} \right) & \text{if} \quad p \nmid b^2 - 4ac \\
(p - 1) \left( \frac{a}{p} \right) & \text{if} \quad p \mid b^2 - 4ac
\end{cases}
\]
Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

\[ \sum_{x \mod p} \left( \frac{ax + b}{p} \right) = 0 \text{ if } p \nmid a \]

\[ \sum_{x \mod p} \left( \frac{ax^2 + bx + c}{p} \right) = \begin{cases} 
- \left( \frac{a}{p} \right) & \text{if } p \nmid b^2 - 4ac \\
(p - 1) \left( \frac{a}{p} \right) & \text{if } p \mid b^2 - 4ac
\end{cases} \]

Average Values of Legendre Symbols

The value of \( \left( \frac{x}{p} \right) \) for \( x \in \mathbb{Z} \), when averaged over all primes \( p \), is 1 if \( x \) is a non-zero square, and 0 otherwise.
Theorem (MMRW’14): Rank 0 Families Obeying the Bias Conjecture

For families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bx + cT + d$,

\[
A_{2,\mathcal{E}}(p) = p^2 - p \left( 1 + \left( \frac{-3}{p} \right) + \left( \frac{a^2 - 3b}{p} \right) \right).
\]

- The average bias in the size $p$ term is $-2$ or $-1$, according to whether $a^2 - 3b \in \mathbb{Z}$ is a non-zero square.
Theorem (MMRW’14): Families with Rank

For families of the form $\mathcal{E} : y^2 = x^3 + aT^2 x + bT^2$,

$$A_{2,\mathcal{E}}(p) = p^2 - p \left( 1 + \left( \frac{-3}{p} \right) + \left( \frac{-3a}{p} \right) \right) - \left( \sum_{x(p)} \left( \frac{x^3 + ax}{p} \right) \right)^2.$$

- These include families of rank 0, 1, and 2.
- The average bias in the size $p$ terms is $-3$ or $-2$, according to whether $-3a \in \mathbb{Z}$ is a non-zero square.
Theorem (MMRW’14): Families with Complex Multiplication

For families of the form \( E : y^2 = x^3 + (aT + b)x, \)

\[
A_{2,E}(p) = (p^2 - p) \left( 1 + \left( \frac{-1}{p} \right) \right).
\]

- The average bias in the size \( p \) term is \(-1\).

- The size \( p^2 \) term is not constant, but is on average \( p^2 \), and an analogous Bias Conjecture holds.
Families with Unusual Distributions of Signs

Theorem (MMRW’14): Families with Unusual Signs

For the family $\mathcal{E} : y^2 = x^3 + Tx^2 - (T + 3)x + 1,$

$$A_{2,\mathcal{E}}(p) = p^2 - p \left( 2 + 2 \left( \frac{-3}{p} \right) \right) - 1.$$

- The average bias in the size $p$ term is $-2.$
- The family has an usual distribution of signs in the functional equations of the corresponding $L$-functions.
The Size $p^{3/2}$ Term

Theorem (MMRW’14): Families with a Large Error

For families of the form
$$E : y^2 = x^3 + (T + a)x^2 + (bT + b^2 - ab + c)x - bc,$$

$$A_{2,E}(p) = p^2 - 3p - 1 + p \sum_{x \mod p} \left( \frac{-cx(x + b)(bx - c)}{p} \right)$$

- The size $p^{3/2}$ term is given by an elliptic curve coefficient and is thus on average 0.

- The average bias in the size $p$ term is $-3$. 
General Structure of the Lower Order Terms

The lower order terms appear to always

- have no size $p^{3/2}$ term or a size $p^{3/2}$ term that is on average 0;

- exhibit their negative bias in the size $p$ term;

- be determined by polynomials in $p$, elliptic curve coefficients, and congruence classes of $p$ (i.e., values of Legendre symbols).
Numerical Investigations
Numerical Methods

- As complexity of coefficients increases, it is much harder to find an explicit formula.

- We can always just calculate the second moment from the explicit formula; if $\mathcal{E}: y^2 = f(x)$, we have

$$A_{2,\mathcal{E}}(p) = \sum_{t(p)} \left( \sum_{x(p)} \left( \frac{f(x)}{p} \right) \right)^2.$$

- Takes an hour for the first 500 primes. Optimizations?
Numerical Methods

Consider the family $y^2 = f(x) = ax^3 + (bT + c)x^2 + (dT + e)x + f$. By similar arguments used to prove special cases,

$$A_{2,\varepsilon}(p) = p^2 - 2p + pC_0(p) - pC_1(p) - 1 + \#_1,$$

where

$$C_0(p) = \sum_{x(p) y(p): \beta(x,y)\equiv0} \left( \frac{A(x)A(y)}{p} \right),$$

$$C_1(p) = \sum_{x(p): \beta(x,x)\equiv0} \left( \frac{A(x)^2}{p} \right),$$

$$\#_1 = p \sum_{x(p) y(p): A(x)\equiv0 \text{ and } A(y)\equiv0} \left( \frac{B(x)B(y)}{p} \right),$$

and $\beta$, $A$, and $B$ are polynomials.
Numerical Methods

- $C_o(p)$ ordinarily $O(p^2)$ to compute.

- Sum over zeros of $\beta(x, y) \mod p$

- Fixing an $x$, $\beta$ is a quadratic in $y$. So, with the quadratic formula mod $p$, we know where to look for $y$ to see if there is a zero.

- Now $O(p)$; runs from 6000$^{th}$ to 7000$^{th}$ prime in an hour.
Potential Counterexamples

Families of Rank as Large as 3

\[ E : y^2 = x^3 + ax^2 + bT^2x + cT^2 \text{ with } b, c \neq 0: \]

\[ A_{2,E}(p) = p^2 + p \sum_{P(x,y) \equiv 0} \left( \frac{(x^3 + bx)(y^3 + by)}{p} \right) \]

\[ + p \left[ \sum_{x^3 + bx \equiv 0} \left( \frac{ax^2 + c}{p} \right) \right]^2 - p \sum_{P(x,x) \equiv 0} \left( \frac{x^3 + bx}{p} \right)^2 \]

\[ - p \left( 2 + \left( \frac{-b}{p} \right) \right) - \left[ \sum_{x \mod p} \left( \frac{x^3 + bx}{p} \right) \right]^2 - 1 \]

where \( P(x, y) = bx^2y^2 + c(x^2 + xy + y^2) + bc(x + y). \)
A Positive Size $\rho$ Term?

$$p \left[ \sum_{x^3 + bx \equiv 0} \left( \frac{ax^2 + c}{\rho} \right) \right]^2$$ can be $+9\rho$ on average!

- Terms such as $-p \sum_{P(x,x)\equiv 0} \left( \frac{x^3 + bx}{\rho} \right)^2$, $-p \left( 2 + \left( \frac{-b}{\rho} \right) \right)$, and $- \left[ \sum_{x \mod \rho} \left( \frac{x^3 + bx}{\rho} \right) \right]^2$ contribute negatively to the size $\rho$ bias.

- The term $p \sum_{P(x,y)\equiv 0} \left( \frac{(x^3 + bx)(y^3 + by)}{\rho} \right)$ is of size $\rho^{3/2}$.

$$A_{2,\epsilon}(\rho) = p^2 + p \sum_{P(x,y)\equiv 0} \left( \frac{(x^3 + bx)(y^3 + by)}{\rho} \right) + p \left[ \sum_{x^3 + bx \equiv 0} \left( \frac{ax^2 + c}{\rho} \right) \right]^2$$

$$- p \sum_{P(x,x)\equiv 0} \left( \frac{x^3 + bx}{\rho} \right)^2 - p \left( 2 + \left( \frac{-b}{\rho} \right) \right) - \left[ \sum_{x \mod \rho} \left( \frac{x^3 + bx}{\rho} \right) \right]^2 - 1$$

where $P(x, y) = bx^2y^2 + c(x^2 + xy + y^2) + bc(x + y)$. 
Analyzing the Size $p^{3/2}$ Term

We averaged $\sum_{P(x,y) \equiv 0} \left( \frac{(x^3+bx)(y^3+by)}{p} \right)$ over the first 10,000 primes for several rank 3 families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bT^2x + cT^2$.

<table>
<thead>
<tr>
<th>Family</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y^2 = x^3 + 2x^2 - 4T^2x + T^2$</td>
<td>$-0.0238$</td>
</tr>
<tr>
<td>$y^2 = x^3 - 3x^2 - T^2x + 4T^2$</td>
<td>$-0.0357$</td>
</tr>
<tr>
<td>$y^2 = x^3 + 4x^2 - 4T^2x + 9T^2$</td>
<td>$-0.0332$</td>
</tr>
<tr>
<td>$y^2 = x^3 + 5x^2 - 9T^2x + 4T^2$</td>
<td>$-0.0413$</td>
</tr>
<tr>
<td>$y^2 = x^3 - 5x^2 - T^2x + 9T^2$</td>
<td>$-0.0330$</td>
</tr>
<tr>
<td>$y^2 = x^3 + 7x^2 - 9T^2x + T^2$</td>
<td>$-0.0311$</td>
</tr>
</tbody>
</table>
The Right Object to Study

c_{3/2}(p) := \sum_{P(x,y) \equiv 0} \left( \frac{(x^3+bx)(y^3+by)}{p} \right)\text{ is not a natural object to study (for us multiply by } p).\text{ }

An example distribution for \( y^2 = x^3 + 2x^3 - 4T^2x + T^2 \).

\textbf{Figure: } c_{3/2}(p) \text{ over the first 10,000 primes.}
In Terms of Elliptic Curve Coefficients

Compare it to the distribution of a sum of 2 elliptic curve coefficients.

Figure: \[-\sum_{x \mod p} \left( \frac{x^3+x+1}{p} \right) - \sum_{x \mod p} \left( \frac{x^3+x+2}{p} \right)\] over the first 10,000 primes.
More Error Distributions

**Figure:** $c_{3/2}(\rho)$ for $y^2 = 4x^3 + 5x^2 + (4T-2)x + 1$, first 10,000 primes.
More Error Distributions

Figure: $- \sum_{x \mod p} \left( \frac{x^3 + x + 1}{p} \right)$ distribution, first 10,000 primes.
More Error Distributions

**Figure**: $c_{3/2}(p)$ over $y^2 = 4x^3 + (4T + 1)x^2 + (-4T - 18)x + 49$, first 10,000 primes.
More Error Distributions

Figure:  $- \sum_{x \mod p} \left( \frac{x^5 + x^3 + x^2 + x + 1}{p} \right)$ distribution, first 10,000 primes.
In the cases we’ve studied, the size $p^{3/2}$ terms

- appear to be governed by (hyper)elliptic curve coefficients;

- may be hiding negative contributions of size $p$;

- prevent us from numerically measuring average biases that arise in the size $p$ terms.
Future Directions
Questions for Further Study

- Are the size $p^{3/2}$ terms governed by (hyper)elliptic curve coefficients? Or at least other $L$-function coefficients?

- Does the average bias always occur in the terms of size $p$?

- Does the Bias Conjecture hold similarly for all higher even moments?

- What other (families of) objects obey the Bias Conjecture? Kloosterman sums? Cusp forms of a given weight and level? Higher genus curves?
References
References

Gaps:


Biases:


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Thank you!