Second Moment Distributions Associated to Elliptic Surfaces

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Elliptic Surface

- Define an elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ (proper, flat, etc.).
- The generic fiber $\pi^{-1}(\eta)$ gives an elliptic curve over the function field $\mathbb{Q}(T)$.
- This curve has discriminant $\Delta(T)$, finitely many singular specializations over the base.
- Define $U = \{t \in \mathbb{P}^1 : \Delta(t) \neq 0\}$. Then $\pi : \mathcal{E} \to U$ is smooth locus.

Trace of Frobenius

- For a a fixed $E/\mathbb{Q}$, we care about $a(p) = p + 1 - \#E(\mathbb{F}_p)$.
- Why? Because they encode a ton of arithmetic information, both algebraically (cohomology) and analytically (coefficients of Hasse-Weil L-function).

Moments

- For the surface $\pi : \mathcal{E} \to U$, we may care to understand $a_t(p)$ as $t \in U$ varies over the base.
- First, reduce mod p by a base change $\mathcal{E}' \to \mathbb{P}^1(\mathbb{F}_p)$, then there are only finitely many fibers.
- Now define

$$A_1(p) = \sum_{t \in \mathbb{P}^1(\mathbb{F}_p)} a_t(p) - a_\infty(p), \quad A_2(p) = \sum_{t \in \mathbb{P}^1(\mathbb{F}_p)} a_t(p)^2 - a_\infty^2(p).$$

Michel's Result

Theorem ([Mic95])

For an elliptic surface $\pi : \mathcal{E}_T \to \mathbb{P}^1$ with non-constant j-invariant, we have

$$A_2(p) = p^2 + O(p^{3/2}).$$

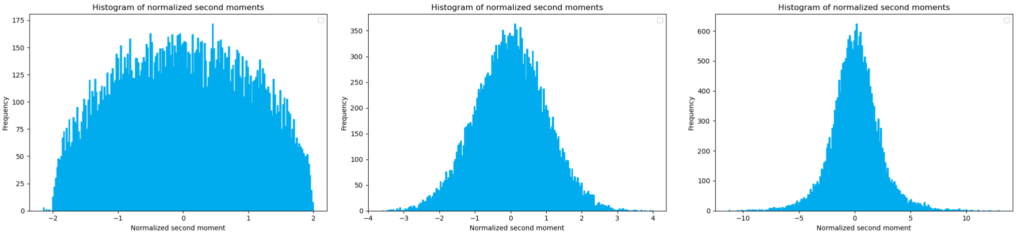
Moreover, there are exactly four lower-order terms; they are $O(p^{3/2})$, $O(p)$, $O(p^{1/2})$, and $O(1)$ respectively and each has a cohomological interpretation.

This suggests that we can normalize to isolate some weight-3 contribution, defining

$$B_2(p) := \frac{A_2(p) - p^2}{p^{3/2}}.$$

$B_2(p)$ as a random variable

- Cheek et al. [CGJ$^+$24] plotted $B_2(p)$ as $p \to \infty$ over the primes.
- Treating $B_2(p)$ as a *random variable*, we may be interested in the moments of its distribution.



Conjecture

They conjectured the following:

Conjecture

For an elliptic surface $\mathcal{E}_T$, the variance of $B_2(p)$ is always an integer.

After collecting more data, we believe that all moments should be integral.

Pencils of Cubics

- We say an elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ is a *pencil of cubics* if the generic fiber $E/\mathbb{Q}(T)$ has a model of the form

$$E : y^2 = P(x)T + Q(x)$$

  where $\deg P(x), \deg Q(x) \leq 3$ That is, we need the Weierstrass equation to be linear in the varying parameter.
- Professors Matija Kazalicki and Bartosz Naskręcki recently confirmed Steven J. Miller's *bias conjecture* for the class of pencils of cubics [KN25].
- In doing so, they give explicit formulae for the second moment $A_2(p)$ of such families.

Confirmation of the integral moment conjecture

- Using their work, we were able to confirm our conjecture in this case.
- Moreover, the distribution of $B_2(p)$ always ended up being distributed as $-b(p)$, where $b(p)$ is the trace on a fixed genus one or two curve.
- The only issue arises in the so-called *typical* case.

Example (Typical Case)

$$\mathcal{A}_{2,\mathcal{E}}(p) = p^2 - p \cdot d_p - p \cdot \#S(\mathbb{F}_p) - a_\infty^2(p)$$

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$$\mathcal{B}_{2,\mathcal{E}}(p) = \frac{\mathcal{A}_{2,\mathcal{E}}(p) - p^2}{p^{3/2}} = -\frac{d_p}{\sqrt{p}} - \frac{\#S(\mathbb{F}_p)}{\sqrt{p}} - \frac{a_\infty^2}{p^{3/2}}.$$

Average over p to compute moments. We find that $B_2(p)$ is distributed exactly as $d_p/\sqrt{p}$.

Distribution of $d_p/\sqrt{p}$

$\mathcal{B}_{2,\mathcal{E}}(p) \sim d_p/\sqrt{p}$, so it suffices to show all moments are integers.

- In the "typical" case considered in their paper, $\overline{D}$ is genus 2.

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- For generic curves, whose Jacobian has small endomorphism group, the Sato-Tate conjecture has been verified and so we're done.
- In general, assuming Sato-Tate for genus two curves, we have the result. ■
- It turns out that all other (non-typical) cases can be proven unconditionally.

Algebraic Approach

Let $i : U \hookrightarrow \mathbb{P}^1_{\mathbb{Q}}$ be the inclusion.

- For any $\ell \neq p$, form the ℓ-adic lisse sheaf $\mathcal{F} := R^1\pi_*\mathbb{Q}_\ell$ on the base U.
- By Deligne, $i_*\mathrm{Sym}^m\mathcal{F}$ is rank two and pure of weight m. [Del80]
- We have the following results [KN25]:

Algebraic Approach

Putting $W_{m,\ell} = H^1_{\text{ét}}(\mathbb{P}^1 \otimes \overline{\mathbb{Q}}, i_*\mathrm{Sym}^m\mathcal{F}_\ell)$ and letting $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote a geometric Frobenius, we can compute traces fiber by fiber:

$$\mathrm{Trace}(\mathrm{Frob}_p | W_{m,\ell}) = -\sum_{t \in \mathbb{P}^1(\mathbb{F}_p)} \mathrm{Trace}(\mathrm{Frob}_p | (i_*\mathrm{Sym}^m\mathcal{F}_\ell)_t). \tag{1}$$

Algebraic Approach

Secondly, we have the following formulas. If the fiber $E_t = \pi^{-1}(t)$ is smooth, then

$$\mathrm{Trace}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^1\mathcal{F}_\ell)_t) = p + 1 - \#E_t(\mathbb{F}_p) = a_t(p) \tag{2}$$

and

$$\mathrm{Trace}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^2\mathcal{F}_\ell)_t) = \mathrm{Trace}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^1\mathcal{F}_\ell)_t)^2 - p = a_t(p)^2 - p \tag{3}$$

If the fiber E_t is singular, then

$$\mathrm{Trace}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^1\mathcal{F}_\ell)_t) = \begin{cases} 1 & \text{if the fiber is split multiplicative} \\ -1 & \text{if the fiber is nonsplit multiplicative} \\ 0 & \text{if the fiber is additive.} \end{cases}$$

Analogy

- Putting this all together, we have

$$-\,\mathrm{Trace}(\mathrm{Frob}_p | W_{2,\ell}) = \sum_{t \notin S} a_t(p)^2 - p^2 + O(p). \tag{4}$$

- Averaging over primes, this gives that the moments of $B_2(p)$ are the same as those of $-\mathrm{Trace}(\mathrm{Frob}_p | W_{2,\ell})/p^{3/2}$.
- This is analogous with our above results, assuming some equidistribution on the left.

The idea

- Condier an elliptic surface with fibers $\mathcal{E}_T : y^2 = x^3 + A(T)x + B(T)$.
- We can form a sequence of families by considering

$$\mathcal{E}_{T^n} : y^2 = x^3 + A(T^n)x + B(T^n).$$

- How does the variance of $B_2(p)$ change as n varies? There are some nice patterns.

Examples

- The family $\mathcal{E}_T : y^2 = x^3 + (T+1)x + (T+1)$ has nice properties.
- When $n = q^m$ is a prime power, we noticed numerically that the variance of $B_2(p)$ over $\mathcal{E}_{T^{q^m}}$ seems to always be m, the exponent in the prime power.
- There aren't a ton of tools to prove observations like this hold in general. There are heuristics. The map $t \mapsto t^n$ has degree $\gcd(n, p-1)$ on $\mathbb{F}_p$. We average
$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} \gcd(q^m, p-1) = m+1.$$
Using this and assuming the achieved fibers are *random enough*, we can prove the observation.

Modular Example

1. Another tractable case is 2-covers of universal elliptic curves.
2. Example: consider the universal elliptic curve over $\Gamma_1(6)$.
3. Deligne: traces on $\mathrm{Sym}^2\mathcal{F} \longleftrightarrow$ coefficients of unique newform $f \in S_4(\Gamma_1(6))$. [Del71]
4. Rankin-Selberg method and "guessing representations": Let $g \in S_4(\Gamma_1(12))$ be the unique newform, then the trace we want to compute is

$$= \begin{cases} 2a_f(p) + a_g(p) & p \equiv 1 \mod 4 \\ -a_g(p) & p \equiv 3 \mod 4. \end{cases}$$

Averaging over primes gives variance of original family 1 with variance of 2-cover 3. This matches our data.

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