Getting Hype About Hyper-Bishops: Probabilities of safe squares on higher-dimensional chess boards.

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1. Bishops

1.1 Introduction

Among various chess-based problems in mathematics, there are a number based around the n-queens problem, concerning the number of configurations of n queens on a chess-board such that none can take any others (see, for example [Rou60]). The full behaviour of these is only understood for small n. We consider variations of a problem introduced by Miller, Sheng, and Turek showed in 2020 ([MST21]) about the percentage of safe squares when n rooks are placed randomly on an $n \times n$ chessboard. They showed that in the limit $n \to \infty$, the percentage of safe squares converges to $1/e^2$. Inspired by this paper, we generalize to bishops, queens, and various higher-dimensional interpretations of the standard pieces.

Definition 1.1 (Seen and Safe)**.** *We say that a space is safe if there are no pieces that in a single move could move onto it in one move. Similarly, a piece sees a space if it could enter the space in one move.*

1.2 Combinatorial Tools

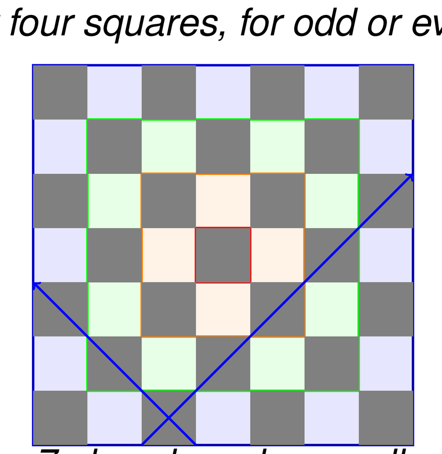
Frequently, we use variations on the ratio (from [MST21])

$$\binom{n^2 - an - b}{n} / \binom{n^2}{n},$$

which represents the percentage of ways to place n pieces on an $n \times n$ chessboard that leave a certain target square, where the target square is seen pieces in $an + b$ squares. The first binomial coeficient is the number of ways to place the n pieces in the $n^2 - an - b$ squares that don't see the target square, while the second is the total number of ways to place all n pieces.

1.3 Bishops and Rings

Definition 1.2 (Chessboard Rings)**.** *We define the k^{th} ring on a chessboard to be the set of squares that lie the same distance from the edge, beginning with the 1st ring as either the center square or the center four squares, for odd or even chess boards respectively.*



Depiction of the rings on a 7 by 7 chessboard, as well as the attacking path of a bishop at (3,1)

From our definition of Rings, we have that a bishop attacks $2n - 2r$ squares, where r is its ring, defining the centermost ring as $r = 0$. This means that the percentage of safe squares on an $n \times n$ board with n bishops is

$$\sum_{r=1}^{(n-1)/2} \left(\frac{4(2r)}{n^2} \cdot \frac{\binom{n^2-2n+2r+3}{n}}{\binom{n^2}{n}} \right). \tag{1}$$

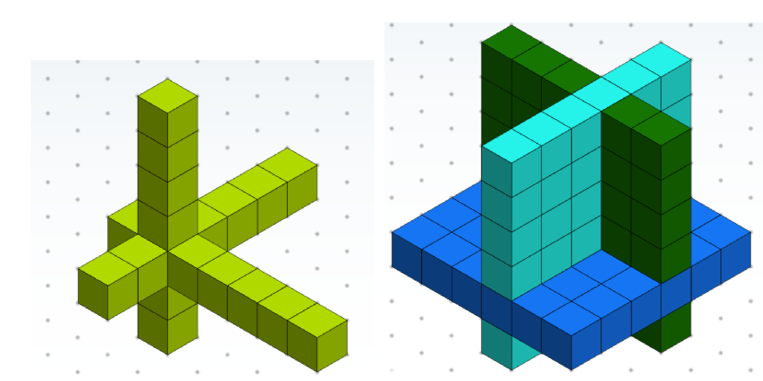
Then, we take the limit, finding that the average percentage of safe squares is

$$\lim_{n\to\infty} \sum_{r=1}^{(n-1)/2} \left(\frac{4(2r)}{n^2} \cdot \frac{\binom{n^2-2n+2r+3}{n}}{\binom{n^2}{n}} \right) = \frac{2}{e^2} \approx 27.06\%.$$

2. Hyper-Rooks

2.1 Hyper-Rooks Definitions

We consider two generalisations of rooks in a higher dimension d. The first, which we call Hyper-Rooks, can move into any square on d orthogonal spaces of dimension $d-1$, which for 3 dimensions means that they move in 3 orthogonal planes. The second, that we call "Line-Rooks" keeps them moving in d orthogonal lines.



3-dimensional Line-Rook and Hyper-Rook movement patterns

	Hyper-Rooks	Line-Rooks
Pieces to Dominate	n	$\approx n^{k-1}$
Squares Seen	$\approx kn^{k-1}$	$kn - k + 1$

2.2 Rook Averages

Then, we consider two variations on our basic combinatorial tool. For Hyper-Rooks, the probability a square on a k-dimensional chessboard of size n being safe is approximately

$$\binom{n^k - kn^{k-1}}{n} / \binom{n^k}{n}. \tag{2}$$

Meanwhile Line Rooks give us a probability of

$$\binom{n^k - kn + k - 1}{n^{k-1}} / \binom{n^k}{n^{k-1}}. \tag{3}$$

Using the same techniques that [MST21] used of bounding the probability with the extreme possibilities, we get that

$$\lim_{n\to\infty} \binom{n^k - kn^{k-1}}{n} / \binom{n^k}{n} = \frac{1}{e^k},$$
$$\lim_{n\to\infty} \binom{n^k - kn + k - 1}{n^{k-1}} / \binom{n^k}{n^{k-1}} = \frac{1}{e^k}.$$

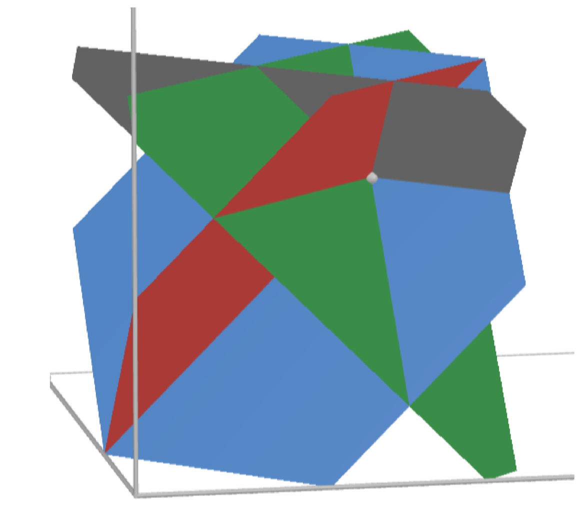
These align incredibly well with the [MST21] probability of $\frac{1}{e^2}$. We also showed that the variance of the probability a square was safe tends towards 0 as n approaches infinity.

3. Hyper-Bishops

We define a Hyper-Bishop in the 3^{rd} dimension at (i, j, k) as being able to attack along four diagonal planes given by the following:

$$(x - i) + (y - j) + (z - k) = 0,$$
$$(x - i) + (y - j) - (z - k) = 0,$$
$$(x - i) - (y - j) + (z - k) = 0,$$
$$(x - i) - (y - j) - (z - k) = 0.$$

These four planes represent planes where any movement in one direction must be made up for in the other two. The following graph gives a graphical representation.



A 3-dimensional Hyper-Bishop.

This definition has several important properties, notably the fact that slices of the space look like regular bishops, and that it maintains the ability to only attack squares of one color. It can also be generalized to higher dimensions, as can be seen by noticing that a 2d bishop at (i, j) moves along the lines $(x - i) + (y - j) = 0$ and $(x - i) - (y - j) = 0$.

4. Future Work

This Hyper-Bishop definition's strength lies in its ability to describe bishop movement in higher dimensions in a way that projects down to lower dimensions accurately. However, this strength leads to a lack of symmetry, making counting the number of squares a given bishop attacks challenging. We hope that future work will be successful in determining the expected percentage of safe squares for randomly placed Hyper-Bishops.

References

[MST21] Steven J. Miller, Haoyu Sheng, and Daniel Turek. *When Rooks Miss: Probability through Chess.* 2021. DOI: 10.1080/07468342.2021.1886774.

[Rou60] W.W. Rouse Ball. *Mathematical Recreations and Essays.* New York: Macmillan, 1960, pp. 165–171.

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