The Katz-Sarnak Density Conjecture and Bounding Central Point Vanishing of *L*-Functions

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Polymath Jr

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Introduction

Goals

- Determine correct scale / statistics to study zeros.
- Discuss the tools / techniques for proofs.
- Highlight calculations for Dirichlet L-functions (simplest case).
- New records for bounding vanishing at central point, lowest zero in cuspidal families.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha \bmod 1$.
- Spacings b/w Zeros of L-functions.

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Better estimate for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s), L(s,\chi) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

Finding correct statistic can be hard!

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

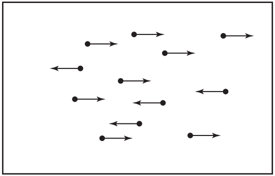
$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Classical Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}.$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

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$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Not most general case, gives flavor.

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

Introduction to *L*-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

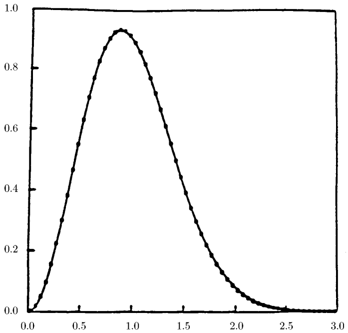
$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \;=\; -\frac{d}{ds}\log\zeta(s) \;=\; -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p\cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

May the Fourier transform be $\widehat{\phi}(y) := \int_{-\infty}^{\infty}\phi(x)e^{-2\pi ixy}dx$.

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

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Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.
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Explicit Formula (Contour Integration)

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Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Riemann Zeta Function: Let $\sum_\rho$ denote the sum over the zeros of $\zeta(s)$ in the critical strip, g an even Schwartz function of compact support and $\phi(r) = \int_{-\infty}^{\infty} g(u)e^{iru}du$. Then

$$\sum_\rho \phi(\gamma_\rho) \;=\; 2\phi\left(\frac{i}{2}\right) - \sum_p \sum_{k=1}^{\infty} \frac{2\log p}{p^{k/2}} g\left(k \log p\right) + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{iy - \frac{1}{2}} + \frac{\Gamma'(\frac{iy}{2}+\frac{5}{4})}{\Gamma(\frac{iy}{2}+\frac{5}{4})} - \frac{1}{2}\log \pi\right)\phi(y)\,dy.$$

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$$\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} h(y)dy$$
$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}}$$
$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\Big(\frac{1}{\log m}\Big).$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: *n*-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: *n*-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{\pm j_1,\dots,\pm j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

Correspondences

Similarities between *L*-Functions and Nuclei:

$$\text{Zeros} \longleftrightarrow \text{Energy Levels}$$

$$\text{Schwartz test function} \longrightarrow \text{Neutron}$$

$$\text{Support of test function} \longleftrightarrow \text{Neutron Energy.}$$

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.

2. Explicit Formula: Relates sums over zeros to sums over primes.

3. Averaging Formulas: Orthogonality of characters for Dirichlet L-functions, Petersson formula for cusp forms.

Applications of n-level density

Bounding the order of vanishing at the central point:
Average rank $\cdot\ \phi(0) \ \le\ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

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Bounding the order of vanishing at the central point:
$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.
Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_n such that probability of at least r zeros at the central point is at most c_n/r^n.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Example:
Dirichlet L-functions

Dirichlet Characters (m prime)

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g. Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & (k,m) = 1 \\ 0 & (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an ℓ such that $\chi(g) = \zeta_{m-1}^{l}$. Hence for each $\ell, 1 \le \ell \le m-2$:

$$\chi_\ell(k) \ = \begin{cases} \zeta_{m-1}^{\ell a} & k \equiv g^a(m) \\ 0 & (k,m) > 0. \end{cases}$$

Dirichlet L-Functions

Let χ be a primitive character mod m. Gauss sum

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m},$$

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\begin{aligned} \epsilon &= \begin{cases} 0 & \text{if } \chi(-1) = \ \ 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \\ \Lambda(s,\chi) &= (-i)^{\epsilon} c(m,\chi) m^{-1/2} \Lambda(1-s,\bar{\chi}). \end{aligned}$$

Explicit Formula

Let ϕ be an even Schwartz function with compact support $(-\sigma, \sigma)$, let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right)$$

$$= \int_{-\infty}^{\infty} \phi(y)dy$$

$$- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}}$$

$$- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1}$$

$$+ O\left(\frac{1}{\log m}\right).$$

Expansion

$\{\chi_0\} \cup \{\chi_\ell\}_{1 \le \ell \le m-2}$ are all the characters mod m.
Consider the family of primitive characters mod a prime m ($m - 2$ characters):

$$\int_{-\infty}^{\infty} \phi(y)dy$$
$$- \quad \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}}$$
$$- \quad \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1}$$
$$+ \quad O\Big(\frac{1}{\log m}\Big).$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise.} \end{cases}$$

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For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \;=\; \begin{cases} -1 + m - 1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

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Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}.$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

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$$\ll \quad \frac{1}{m}\sum_{p}^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$\ll\quad \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}+\sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}}\ll\frac{1}{m}\sum_{k}^{m^\sigma}k^{-1/2}+\sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-1/2}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma} \frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$+ \quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$\ll \quad \frac{1}{m}\sum_{p}^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}} \ll \frac{1}{m}\sum_{k}^{m^\sigma} k^{-1/2} + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma} k^{-1/2}$$

$$\ll \quad \frac{1}{m}\sum_{k}^{m^\sigma} k^{-1/2} + \frac{1}{m}\sum_{k}^{m^\sigma} k^{-1/2} \ll \frac{1}{m}m^{\sigma/2}.$$

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)+\bar{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\bar{\chi}^2(p)] = \begin{cases} 2(m-2) & p\equiv\pm1(m) \\ -2 & p\not\equiv\pm1(m). \end{cases}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\ll \frac{1}{m-2}\sum_p^{m^{\sigma/2}} p^{-1} + \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}} p^{-1}$$

$$\ll \frac{1}{m-2}\sum_k^{m^{\sigma/2}} k^{-1} + \sum_{k\equiv1(m)}^{m^{\sigma/2}} k^{-1} + \sum_{k\equiv-1(m)}^{m^{\sigma/2}} k^{-1} \ll \frac{\log m}{m}.$$

Cuspidal Newforms

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case and talk about extensions (joint with Chris Hughes, then much further with SMALL REUs).

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \;\; f(\gamma z) = (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dxdy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) = \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) = 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right) + \delta(m,n).$$

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Modular Form Preliminaries:Fourier Coefficient Review

$$\begin{aligned} \lambda_f(n) &= a_f(n) n^{\frac{k-1}{2}} \\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d \mid (m,n) \\ (d,M)=1}} \lambda_f\left(\frac{mn}{d}\right). \end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k \mu(N) \lambda_f(N) \sqrt{N}.$$

The above will allow us to split into even and odd families: $1 \pm \epsilon_f$.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) \;=\; \sum_{a \bmod q}^{*} e(an/q) \;=\; \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\text{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m\le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{dy}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$ and φ is Euler's totient function.

Limited Support ($\sigma < 1$): Sketch of proof

- Estimate Kloosterman-Bessel terms trivially.
  ◇ Kloosterman sum: $d\overline{d} \equiv 1 \bmod q$, $\tau(q)$ is the number of divisors of q,

$$S(m,n;q) \;=\; \sum_{d \bmod q}^{*} e\left(\frac{md}{q} + \frac{n\overline{d}}{q}\right)$$

$$|S(m,n;q)| \;\le\; (m,n,q)\sqrt{\min\left\{\frac{q}{(m,q)}, \frac{q}{(n,q)}\right\}}\;\tau(q).$$

  ◇ Bessel function: integer $k \geq 2$, $J_{k-1}(x) \ll \min\left(x, x^{k-1}, x^{-1/2}\right)$.

- Use Fourier Coefficients to split by sign: N fixed: $\pm \sum_f \lambda_f(N) * (\cdots)$.

Increasing Support ($\sigma < 2$): Sketch of the proof

- Using Dirichlet Characters, handle Kloosterman terms.
- Have terms like

$$\int_0^\infty J_{k-1}\left(4\pi \frac{\sqrt{m^2 yN}}{c}\right) \widehat{\phi}\left(\frac{\log y}{\log R}\right) \frac{dy}{\sqrt{y}}$$

  with arithmetic factors to sum outside.

- Works for support up to $(-2, 2)$.

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Stating in greater generality for later use.

Gauss sum: χ a character modulo q: $|G_\chi(n)| \le \sqrt{q}$ with

$$G_\chi(n) \ = \sum_{a \bmod q} \chi(a) \exp(2\pi i a n/q).$$

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Kloosterman expansion:

$$S(m^2, p_1 \cdots p_n N; Nb) = \frac{-1}{\varphi(b)} \sum_{\chi (\text{mod } b)} \chi(N) G_\chi(m^2) G_\chi(1) \overline{\chi}(p_1 \cdots p_n).$$

Lemma: Assuming GRH for Dirichlet L-functions, $\text{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, non-principal characters negligible.
Proof: use $J_{k-1}(x) \ll x$ and see

$$\ll \frac{1}{\sqrt{N}} \sum_{m \le N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b < N^{2006}}} \frac{1}{b} \frac{1}{\varphi(b)} \sum_{\substack{\chi (\text{mod } b) \\ \chi \neq \chi_0}} \left| G_\chi(m^2) G_\chi(1) \right|$$

$$\times \frac{m}{b\sqrt{N}} \prod_{j=1}^{n} \left| \sum_{p_j \neq N} \overline{\chi}(p_j) \log p_j \cdot \frac{1}{\log R} \widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \right|.$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma} \widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right) J_{k-1}\left(4\pi\frac{\sqrt{m^2 x_1 x_2 N}}{c}\right)\frac{dx_1\, dx_2}{\sqrt{x_1 x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1 x_2 & \quad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int \widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}} J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\frac{du_1\, du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}.$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

Bounding Vanishing

Approaches for Bounding Vanishing

- Increasing the support for the 1-level.
- Optimizing the test function.
- Using *n*-level densities.

Previous Work

Optimizing:

- *Determining Optimal Test Functions for Bounding the Average Rank in Families of L-Functions* (Jesse Freeman, Steven J. Miller), in SCHOLAR – a Scientific Celebration Highlighting Open Lines of Arithmetic Research, Conference in Honour of M. Ram Murty's Mathematical Legacy on his 60th Birthday (A. C. Cojocaru, C. David and F. Pappaardi, editors), Contemporary Mathematics **655**, AMS and CRM, 2015. https://arxiv.org/pdf/1507.03598.pdf
- *Determining optimal test functions for* 2*-level densities* (Elżbieta Bołdyriew, Fangu Chen, Charles Devlin VI, Steven J. Miller, Jason Zhao), Research in Number Theory **9** (2023), article number 32, https://doi.org/10.1007/s40993-022-00367-0.

Increasing support:

- *Extending support for the centered moments of the low lying zeroes of cuspidal newforms* (Peter Cohen, Justine Dell, Oscar E. Gonzalez, Geoffrey Iyer, Simran Khunger, Chung-Hang Kwan, Steven J. Miller, Alexander Shashkov, Alicia Smith Reina, Carsten Sprunger, Nicholas Triantafillou, Nhi Truong, Roger Van Peski, Stephen Willis, and Yingzi Yang), to appear in Algebra & Number Theory.

New Work

◇ *Bounding Vanishing at the Central Point of Cuspidal Newforms* (with Jiahui (Stella) Li), Journal of Number Theory (Computational Section) **244** (2023), 279–307.

◇ *Bounding Excess Rank of Cuspidal Newforms via Centered Moments* (Sohom Dutta, Steven J. Miller), Research in Number Theory **10** (2024), no. 76, https://doi.org/10.1007/s40993-024-00567-w.

Test Functions

Naive Test Function

The naive test functions are the Fourier test function pair

$$\phi_{\text{naive}}(x) = \left(\frac{\sin(\pi v_n x)}{(\pi v_n x)}\right)^2 \quad , \quad \widehat{\phi}_{\text{naive}}(y) = \frac{1}{v_n}\left(y - \frac{|y|}{v_n}\right)$$

for $|y| < v_n$ where v_n is the support.

Test Functions

Naive close but not optimal; optimal for 1-level is

$$\widehat{\phi}_{\text{optimal}}(y) \;:=\; (f_0 * \overline{f_0})(y)$$

for

$$f_0(x) \;:=\; \frac{\cos\left(\frac{|x|}{2} - \frac{\pi+1}{4}\right)}{\sqrt{2}\sin\left(\frac{1}{4}\right) + \sin\left(\frac{\pi+1}{4}\right)}\;, \quad 0 \le |x| \le 1$$

for $G = \mathrm{SO(even)}$ and

$$f_0(x) \;:=\; \frac{\cos\left(\frac{|x|}{2} + \frac{\pi-1}{4}\right)}{3\sin\left(\frac{\pi+1}{4}\right) - 2\sin\left(\frac{\pi-1}{4}\right)}\;, \quad 0 < |x| < 1$$

for $G = \mathrm{SO(odd)}$.

1-Level Bound

$$D_1(\mathcal{F}_N, \phi) \ := \ \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log c_f\right).$$

1-Level Bound

$$D_1(\mathcal{F}_N, \phi) \;:=\; \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log c_f\right).$$

From [ILS]: Let ϕ be a non-negative, even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ for some finite σ. Let $\mathcal{G}$ be the group associated to the family $\mathcal{F}_N$ (i.e., Unitary, Symplectic, Orthogonal, $\mathrm{SO(even)}$, $\mathrm{SO(odd)}$). *Set*

$$g_{\mathcal{F}}(\phi) \;:=\; \frac{1}{\phi(0)} \int_{-\infty}^{\infty} \widehat{\phi}(y) \widehat{W}_{\mathcal{G}(\mathcal{F})}(y) dy.$$

As $N \to \infty$ the percent of forms in the family $\mathcal{F}_N$ that vanish to order exactly (or at least) r is bounded by

$$p_r \;\leq\; \frac{1}{r}(g_{\mathcal{F}}(\phi)).$$

Naive vs Optimal for 1-Level (1 is 100%)

Comparison of Bounds for the 1-level density for $G = \mathrm{SO(even)}$		
Rank	Bound From naive test function	Bound From optimal test function
2	0.43750000	0.43231300
4	0.21875000	0.21615700
6	0.14583333	0.14410400
8	0.10937500	0.10807800
10	0.08750000	0.08646260
12	0.07291670	0.07205220
14	0.06250000	0.06175900
16	0.05468750	0.05403910
18	0.04861110	0.04803848
20	0.04375000	0.04323130

4-Level Bounds

Joint with Jiahui (Stella) Li, combinatorics from generalizing Hughes-Miller.

Order vanish	1-level	2-level	4^{th} centered moment*
6	0.144090	0.01576870	0.00853841
8	0.108067	0.00788434	0.00081336
10	0.086454	0.00473060	0.00018684
20	0.043227	0.00105125	$4.49988 \cdot 10^{-6}$
50	0.017290	0.00015768	$7.13387 \cdot 10^{-8}$

TABLE 2. Comparison of order of vanishing bounds from various approaches. These are upper bounds for vanishing at least r (number in order vanishing column). For the 1-level column, we calculated the bound using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2, 2)$. For the 2-level column, we calculated the bound using the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions used is $(-1, 1)$.
For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test function used is $(-1/3, 1/3)$.

Bounds from n-Level: From Cohen et. al.

Let $n \geq 2$ and $\text{supp}(\phi) \subset (-\frac{2}{n}, \frac{2}{n})$. Define

$$\sigma_\phi^2 := 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2 dy$$

$$R(m,i;\phi) := 2^{m-1}(-1)^{m+1}\sum_{l=0}^{i-1}(-1)^l \binom{m}{l} \left(-\frac{1}{2}\phi^m(0) + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_{l+1}) \int_{-\infty}^{\infty}\phi^{m-l}(x_1)\frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{l+1}|))}{2\pi x_1}dx_1\cdots dx_{l+1}\right)$$

$$S(n,a,\phi) := \sum_{l=0}^{\lfloor\frac{a-1}{2}\rfloor}\frac{n!}{(n-2l)!l!}R(n-2l,a-2l,\phi)\left(\frac{\sigma_\phi^2}{2}\right)^l \text{ then}$$

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}}\left\langle\left(D(f;\phi)-\langle D(f;\phi)\rangle_\pm\right)^n\right\rangle_\pm = (n-1)!!\sigma_\phi^n 1_{n \text{ even}} \pm S(n,a;\phi).$$

Key Expansion

Using binomial theorem and Cauchy-Schwartz, can replace the mean from finite N with the limit

$$\begin{aligned}\lim_{\substack{N\to\infty\\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\left(\sum_j \phi\left(\gamma_{f,j}c_n\right) - \mu(\phi,\mathcal{F})\right)^n \\ = 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi),\end{aligned}$$

and main term of the mean of the 1-level density of $\mathcal{F}_N$ is

$$\mu(\phi,\mathcal{F}) \ := \ \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(y)dy.$$

Bound from n-Level for r Sufficiently Large

Theorem (Dutta-Miller)

For an even n with $r \geq \mu(\phi, \mathcal{F})/\phi(0)$,

$$p_r(\mathcal{F}) \ \leq \ \frac{(n-1)!!\sigma_\phi^n \pm S(n, \frac{n}{2}; \phi)}{(r\phi(0) - \mu(\phi, \mathcal{F}))^n}.$$

Sketch of Proof: $r \geq \mu(\phi, \mathcal{F})/\phi(0)$

For even-level densities, contributions always positive, drop forms with fewer than r zeros:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}} (r\phi(0) + B_f(\phi) - \mu(\phi,\mathcal{F}))^n \ \leq \ \cdots .$$

Have $(r\phi(0) + B_f(\phi) - \mu(\phi, \mathcal{F}_N))^n$; dropping $B_f(\phi)$ can increase sum if first two terms are less than the third.

By assumption on r, sum with and without $B_f(\phi)$ is positive; dropping gives bound

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}} (r\phi(0) - \mu(\phi,\mathcal{F}))^n \leq 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi)$$

Comparison of different n from Dutta-Miller (1 is 100%)

Using support up to $2/n$:

	1-level	2-level	4-th centered moment	6-th centered moment	8-th centered moment	10-th centered moment
Mean	1.5	2.	3.	4.	5.	6.
2	0.4375	Invalid Bound	Invalid Bound	Invalid Bound	Invalid Bound	Invalid Bound
4	0.21875	0.0666667	0.0733686	Invalid Bound	Invalid Bound	Invalid Bound
6	0.145833	0.0205761	0.00247516	0.00234651	0.0509282	Invalid Bound
8	0.109375	0.00986193	0.000405905	0.0000689901	0.0000579621	0.000408389
10	0.0875	0.00576701	0.00011739	7.59594×10^{-6}	1.55879×10^{-6}	1.1438×10^{-6}
12	0.0729167	0.00377929	0.0000456017	1.51897×10^{-6}	1.30376×10^{-7}	2.89295×10^{-8}
14	0.0625	0.00266667	0.0000212365	4.2749×10^{-7}	1.96743×10^{-8}	1.97827×10^{-9}
16	0.0546875	0.00198177	0.0000111825	1.50176×10^{-7}	4.26683×10^{-9}	2.39101×10^{-10}
18	0.0486111	0.00153046	6.435×10^{-6}	6.16387×10^{-8}	1.18309×10^{-9}	4.18191×10^{-11}
20	0.04375	0.00121743	3.96025×10^{-6}	2.83895×10^{-8}	3.91773×10^{-10}	9.47969×10^{-12}

FIGURE 1. Approximate Bounds for the Percent of Vanishing to exact order r for the case G=SO(even) with support $v = 2$ for the 1-level and $v = 2/n$ for the n-level with n going from 1 to 10 and r from 2 through 20 obtained using the naive test function.

Comparison of different n from Dutta-Miller (1 is 100%)

Using support up to $2/n$:

	1-level	2-level	4-th centered moment	6-th centered moment	8-th centered moment	10-th centered moment
Mean	1.01055	1.5211	2.54219	3.56329	4.58439	5.60548
2	0.432313	1.90775	Invalid Bound	Invalid Bound	Invalid Bound	Invalid Bound
4	0.216157	0.0712029	0.0857764	84.2903	Invalid Bound	Invalid Bound
6	0.144104	0.0218109	0.00270998	0.0027933	0.0818933	43 045.8
8	0.108078	0.0104235	0.000436612	0.0000766611	0.0000712948	0.000634483
10	0.0864626	0.00608608	0.000125234	8.22163×10^{-6}	1.78495×10^{-6}	1.46382×10^{-6}
12	0.0720522	0.0039846	0.000048418	1.62145×10^{-6}	1.44421×10^{-7}	3.43972×10^{-8}
14	0.061759	0.00280973	0.0000224782	4.52439×10^{-7}	2.13806×10^{-8}	2.26279×10^{-9}
16	0.0540391	0.00208711	0.0000118106	1.58019×10^{-7}	4.57946×10^{-9}	2.67029×10^{-10}
18	0.0480385	0.00161124	6.78542×10^{-6}	6.45851×10^{-8}	1.25871×10^{-9}	4.59521×10^{-11}
20	0.0432313	0.00128134	4.17071×10^{-6}	2.96519×10^{-8}	4.14116×10^{-10}	1.02951×10^{-11}

FIGURE 5. Approximate Bounds for the Percent of Vanishing to exact order r for the case G=SO(even) with support $v = 2$ for the 1-level and $v = 2/n$ for the n-level with n going from 1 to 10 and r from 2 through 20 obtained using the optimal test function.

Best Results (1 is 100%)

Lowest Bounds for Each Rank for G=SO(even)		
Rank	Level Used	Bound
2	1	0.43231300
4	2	0.066666667
6	6	0.003346510
8	8	0.000579210
10	10	1.14380×10^{-6}
12	12	1.85901×10^{-8}
14	14	2.59310×10^{-10}
16	16	3.09185×10^{-12}
18	18	3.26332×10^{-14}
20	20	3.08920×10^{-16}

Best Results (1 is 100%)

Lowest Bounds for Each Rank for G=SO(odd)		
Rank	Level Used	Bound
1	N/A	1.0000000
3	2	0.111111111
5	2	0.020408300
7	6	0.000292790
9	8	7.65596×10^{-6}
11	10	1.53302×10^{-7}
13	12	2.50956×10^{-9}
15	16	3.03362×10^{-11}
17	18	3.10549×10^{-13}
19	20	4.18402×10^{-17}

Bounds on Lowest Zero in Cuspidal Families

Previous Results

Question

Assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?

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- S. D. Miller: *L*-functions of real archimedian type has $\gamma < 14.13$.
- J. Bober, J. B. Conrey, D. W. Farmer, A. Fujii, S. Koutsoliotas, S. Lemurell, M. Rubinstein, H. Yoshida: General *L*-function has $\gamma < 22.661$.

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- J. Mestre: Elliptic curves: first zero occurs by $O(1/\log\log N_E)$, where N_E is the conductor (expect order $1/\log N_E$).

- J. Goes and S. J. Miller: One-Parameter Family of Elliptic Curves of rank r: $r + \frac{1}{2}$ normalized zeros on average within the band $\approx (-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma})$.

New Results: M– and Tang; Arora, Bruda, Fang, Marquez, M–, Prapashtica, Sharan, Son, Tang and Waheed

Upper Bound Lowest 1st Zero in Even Cusp Families

Let ϕ_ω be non-negative and non-increasing for $|x| < \omega$ and non-positive for $|x| > \omega$, $n = 2m+1$, if ω satisfies

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n < 1_{n \text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n, a; \phi_\omega),$$

at least one form with at least one normalized zero in $(-\omega, \omega)$. Can take

$$\omega_{\min}(\sigma, h) > \left(-\frac{\sigma\int_0^1 h(u)^2\,du + \frac{\sigma^2}{4}\int_{-1}^1\int_0^{2/\sigma} h(u)h(v-u)\,dv\,du}{\frac{1}{\sigma}\int_0^1 h(u)h''(u)\,du + \frac{1}{4}\int_{-1}^1\int_0^{2/\sigma} h(u)h''(v-u)\,dv\,du}\right)^{-\frac{1}{2}} \frac{1}{\pi}.$$

Only know for $\sigma < 2$ (under GRH).

If h even, twice cont diff, monotonically decreasing and supported on $[-1, 1]$, $f(y) := h(2yn/\sigma)$, $g(y) := (f * f)(y)$, $\phi_\omega(x) = (1 - (x/\omega)^2)\widehat{g}(x)$ then ϕ_ω satisfies above requirements.

New Results: S. J. Miller and Tang

Theorem: Normalized Zeros Near the Central Point

$P_{r,\rho}(\mathcal{F})$: percent of forms with at least r normalized zeros in $(-\rho, \rho)$.

For even n and $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$:

$$P_{r,\rho}(\mathcal{F}) \ \leq \ \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}.$$

Explicit Bounds

Number of zeros	2-level	4-level	6-level
6	N/A	10.849910	48.154279
16	N/A	0.004235	$2.83230 \cdot 10^{-4}$
26	N/A	$3.541901 \cdot 10^{-4}$	$6.716802 \cdot 10^{-6}$
28	420.045063	$2.486819 \cdot 10^{-4}$	$3.943864 \cdot 10^{-6}$
30	20.991406	$1.796948 \cdot 10^{-4}$	$2.418466 \cdot 10^{-6}$
32	6.651738	$1.330555 \cdot 10^{-4}$	$1.538761 \cdot 10^{-6}$
34	3.220871	$1.006126 \cdot 10^{-4}$	$1.010576 \cdot 10^{-6}$

Table: Upper bound on probability of forms with at least r normalized zeros within 0.8 average spacing from central point, using naive test function with support $2/n$.
"N/A" means restriction in our theorem not met.

Preliminaries for Constructions and Proofs

- Convolution:

$$(A * B)(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt.$$

- Fourier Transform:

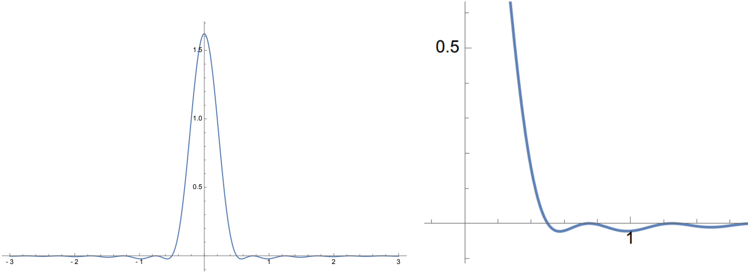
$$\begin{aligned} \widehat{A}(y) &= \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy}dx \\ \widehat{A''}(y) &= -(2\pi y)^2\widehat{A}(y). \end{aligned}$$

- Lemma: $\widehat{(A * B)}(y) = \widehat{A}(y) \cdot \widehat{B}(y)$;
in particular, $\widehat{(A * A)}(y) = \widehat{A}(y)^2 \geq 0$ if A is even.

Construction of Test Function

Create compactly supported $\widehat{\phi}(y)$.

- Choose $h(y)$ even, twice continuously differentiable, supported on $(-1, 1)$, monotonically decreasing.
- $f(y) := h\left(\frac{2y}{\sigma/n}\right)$.
- $g(y) := (f * f)(y), \quad \widehat{g}(x) = \widehat{f}(x)^2 \geq 0$.
- $\widehat{\phi}_\omega(y) := g(y) + (2\pi\omega)^{-2} g''(y)$ thus $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$.



Plot of $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, for $h = \cos\left(\frac{\pi y}{2}\right)$, $\sigma = 2$, $\omega = .5$, and $n = 1$.

Sketch of Proof: Key Expansion

Replace mean from finite N with the limit:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \left(\sum_j \phi\left(\gamma_{f,j} c_n\right) - \mu(\phi, \mathcal{F}) \right)^n$$
$$= 1_{n \text{ even}} (n-1)!!\sigma_\phi^n \pm S(n, a; \phi),$$

and main term of the mean of the 1-level density of $\mathcal{F}_N$ is

$$\mu(\phi, \mathcal{F}) \; := \; \widehat{\phi}(0) + \frac{1}{2} \int_{-1}^{1} \widehat{\phi}(y) dy.$$

Key Observation

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_j \phi(\widetilde{\gamma}_{f,j} c_n) - \mu(\phi, \mathcal{F}) \right)^n = 1_{n \text{ even}} (n-1)!!\sigma_\phi^n \pm S(n, a; \phi).$$

$$\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2).$$

- $\phi_\omega(x) \geq 0$ when $|x| \leq \omega$, and $\phi_\omega(x) \leq 0$ when $|x| > \omega$.
- Contribution of zeroes for $|x| \geq \omega$ is non-positive.
- As n odd, doesn't decrease if drop these non-positive contributions: why we restrict to odd n.

Sketch of Proof: Proof by Contradiction

Dropping negative contributions:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_{|\gamma_{f,j}|\le\omega} \phi_\omega(\gamma_{f,j} c_n) - \mu(\phi_\omega, \mathcal{F}) \right)^n \ \geq \ S(n, a; \phi_\omega).$$

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Assume no forms have a zero on the interval $(-\omega, \omega)$:

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} (-\mu(\phi_\omega, \mathcal{F}))^n \ \ge\ S(n, a; \phi_\omega),$$

$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \ \ge\ S(n, a; \phi_\omega).$$

Sketch of Proof: Proof by Contradiction

Dropping negative contributions:

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$$(-\mu(\phi_\omega,\mathcal{F}))^n \lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \;\ge\; S(n,a;\phi_\omega).$$

As $\lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 = 1$, get

$$(-\mu(\phi_\omega,\mathcal{F}))^n \;\ge\; S(n,a;\phi_\omega).$$

Sketch of Proof: Continued

Because of the compact support of $\widehat{\phi}_\omega$,

$$-\left(\widehat{\phi}_\omega(0)+\frac{1}{2}\int_{-\sigma/n}^{\sigma/n}\widehat{\phi}_\omega(y)dy\right)^n \geq S(n,a;\phi_\omega).$$

Thus, if ω satisfies the following inequality

$$-\left(\widehat{\phi}_\omega(0)+\frac{1}{2}\int_{-\sigma/n}^{\sigma/n}\widehat{\phi}_\omega(y)dy\right)^n < S(n,a;\phi_\omega),$$

we get a contradiction, so at least one form has a normalized zero in $(-\omega,\omega)$.

Quadratic vs Sextic

Upper Bound for Lowest Largest Normalized Zero in Family:

Degree of test function:

$$\text{Quadratic} \qquad 0.21864.$$

Quadratic vs Sextic

Upper Bound for Lowest Largest Normalized Zero in Family:

Degree of test function:

Quadratic	0.21864.
Sextic	0.21850.

Current / Future Work

- Bounds on next few zeros, percentages with low zeros (ongoing).
- Optimize test function.
- Increase support of test function.
- Recent studies increased the support to 4 (Baluyot, Chandee, and Li) for a certain group of L-functions....

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