Large Subsets are Sumsets

Benjamin Baily
bmb2@williams.edu

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With Justine Dell, Sophia Dever, Adam Dionne, Henry Fleischmann, Leo Goldmakher, Gal Gross, Faye Jackson, Steven J. Miller, Ethan Pesikoff, Huy Tuan Pham, Luke Reifenberg, and Vidya Venkatesh.

Williams College

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Set Addition

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	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4
4	4	5	6

Irreducibility

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A set $S \subseteq \mathbb{Z}$ *is* ***reducible*** *if* $S = A + B$ *for two sets* A, B *such that* $|A|, |B| \geq 2$. *Otherwise,* S *is* ***irreducible***.

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The set $\{0, 1, 2\} = \{0, 1\} + \{0, 1\}$ is reducible. In contrast, $\{0, 1, 3\}$ is irreducible.

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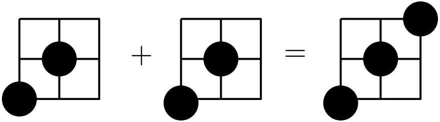
Let $S = \{(0,0),(1,1),(2,2)\}$. Then,

$$S = \{(0,0),(1,1)\} + \{(0,0),(1,1)\}$$

so S is reducible.



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For $d, n \geq 1$ *we define* $f_d(n)$ *as follows.*

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Theorem

For all d*, we have have* $f_d(n) = \Theta(\log n)$.

What is the largest size of an irreducible subset of $[n]^d$?

Lemma

Fix $S \subset \mathbb{Z}^d$ such that $|S| \geq 3$ and $0 \in S$. Let $A = \{0, r\}$ for $r \in \mathbb{Z}^d \setminus \{0\}$. $S = A + B$ for some $B \subset S$ iff for all $s \in S$, $s - r \in S$ or $s + r \in S$.

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- We consider the minimum size of the complement of an irreducible subset of $[n]^d$.
- For each $r \in [n]^d$, in order for the set $\{0, r\}$ to not be a summand of S, there must exist $s \in S$ such that both $r - s, r + s \notin S$.

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- We consider the minimum size of the complement of an irreducible subset of $[n]^d$.
- For each $r \in [n]^d$, in order for the set $\{0, r\}$ to not be a summand of S, there must exist $s \in S$ such that both $r - s, r + s \notin S$.
- If $|[n]^d \setminus S| \ll \log n$, the complement of S is too small for this to hold for all $r \in [n]^d$.

The Largest Irreducible Subset of $[n]^d$

Theorem

Let $S \subseteq [n]^d$. Let $k = \left|[n]^d \setminus S\right|$. Then, S is reducible if

$$\frac{k}{d}\log 2 + H_{\lceil \frac{k}{d} \rceil} + H_{\lceil \binom{k}{2}/d \rceil} < H_{n-1}$$

where H_n is the nth Harmonic number ($H_n \approx \log(n)$).

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Corollary

We have $f_d(n) = \Omega(\log n)$.

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Proposition (Local Irreducibility)

Suppose $S \subset \mathbb{Z}^{\geq 0}$ with $0 \in S$ satisfies the following.

1. *If m is the smallest nonzero element of S, then $2m \notin S$.*
2. *For each $s \in S \setminus \{0, m\}$ there is some $t \in S$ with $t < s$ and $s + t \notin S$.*

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- Irreducibility follows by an iterative argument.

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Suppose $S = \{0, 1, 3\} = A + B$.

- WLOG, $0 \in A, B$ and $A, B \subset S$ by shifting the sets.
- 1 must be in A or B. Suppose $1 \in B$.
- Then, since $1 + 1 \notin S$, $1 \notin A$.
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- So, $A = \{0\}$ and S is irreducible.

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Local Irreducibility in Higher Dimensions

Proposition (Local Irreducibility)

Suppose a set $S \subset \mathbb{Z}^d_{\geq 0}$ with $0 \in S$ satisfies the following.

1. *If m is the lexicographically first nonzero element of S, then $2m \notin S$.*
2. *For each $s \in S \setminus \{0, m\}$ there is some $t \in S$ with $t \prec s$ and $s + t \notin S$.*

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$$\mathcal{Q}_4 = \{0, 1, 3, 5, 6, 7, 9, 10, 12, 13, 14, 15\}$$

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- For $i \geq 10$, set $N_i = N_{i-2} + N_{i-3}$ and let

$$\mathcal{P}_i = \mathcal{P}_{i-1} \cup \{N_{i-1} + 1, \ldots, N_i - 1\}$$

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- We are able to estimate the number of terms in a linear recurrence sequence up to n.
- Let λ denote the largest complex root of the polynomial $x^3 - x - 1 = 0$.

Proposition

Let $\lambda = 1.325\ldots$ denote the largest complex root of the polynomial $x^3 - x - 1 = 0$. Then $|[n] \setminus \mathcal{P}| \sim \log_\lambda n$.

Recap

1. We let $f_d(n)$ denote the minimum size complement of an irreducible subset of $[n]^d$.
2. By showing that every sufficiently large subset $S \subset [n]^d$ admits a factorization of the form $S = A + B$ with $A = \{0, r\}$, we proved that $f_d(n) = \Omega(\log n)$.
3. Define **local irreducibility**, a strong condition on a subset of $\mathbb{Z}^d$ such that every locally irreducible set is also irreducible.
4. Construct large locally irreducible subsets of $[n]^d$ to prove that $f_d(n) = O(\log n)$.
5. Conclude that $f_d(n) = \Theta(\log n)$.

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References

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