CLT and Lower Order Terms

random var X, density f or f_X

$$\text{Prob}(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

Moments: $\mu_k' := \int_{-\infty}^{\infty} x^k f_X(x)\,dx$

Centred Moments $\mu_k := \int_{-\infty}^{\infty} (x-\mu)^k f_X(x)\,dx, \quad \mu = \mu_1'$

$$\mathbb{E}[X^k] \quad \text{or} \quad \mathbb{E}[(X-\mu)^k]$$

HOPE: know $\{\mu_n\} \Rightarrow$ know f_X

Danger: $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$(Tf)(x) \equiv 0$ as $f^{(n)}(0) = 0$

If know f_X and f_Y give same moments for seq $\{k_n\}$ with $k_n \to k$ then $f_X = f_Y$

real analysis: $\begin{cases} x^3 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Moment Generating Function

$$M_X(t) := \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

assume sufficient decay so converge in $(-\delta, \delta) \ni 0$.

$$\Phi_X(t) = M_X(it) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx$$

$\hookrightarrow$ characteristic function: always exists as $|e^{itx}| = 1$

$\hookrightarrow$ if $f(x)$ is Cauchy, $\frac{1}{\pi} \frac{1}{1+x^2}$, no δ works

$$M_X(t) := \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{x^n}{n!} f_X(x)\, t^n\,dx$$

$$= \sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} x^n f_X(x)\,dx\right] \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\mu_n'}{n!} t^n$$

$$\Longrightarrow M_X^{(n)}(0) = \mu_n'$$

when $t=0$

$$M_X(0)$$

$$= \int_{-\infty}^{\infty} e^{0x} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= 1$$

Convolutions

X, Y indep rv with densities f, g

Then $X+Y$ has density $(f*g)(z) = \int_{-\infty}^{\infty} f(t)\, g(z-t)\, dt$

want $X+Y = Z$

$x + y = z$

if $x = t$ need $y = z - t$

rigorously with CDF

$$\text{Prob}(Z \le z) = \int_{t=-\infty}^{\infty} \int_{s=-\infty}^{z-t} f(t)\, g(s)\, ds\, dt = CDF_Z(z)$$

$$= \int_{t=-\infty}^{\infty} f(t) \left[G(z-t) - \underbrace{G(-\infty)}_{0} \right] dt$$

take deriv wrt z,

$CDF' = $ density

Properties of MGF

$$Y = aX + b$$

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{t(aX+b)}\right]$$

$$= E\left[e^{atX} e^{bt}\right] = \int_{-\infty}^{\infty} e^{atx} \underline{e^{bt}} f_X(x)\,dx$$

$$= \left[\int_{-\infty}^{\infty} e^{(at)x} f_X(x)\,dx\right] e^{bt}$$

$$= M_X(at)\, e^{bt} \quad \text{or} \quad e^{bt} M_X(at)$$

$$\boxed{\text{Thm: } M_{aX+b}(t) = e^{bt} M_X(at)}$$

$Y = X_1 + X_2$, assume indep

$$M_Y(t) = M_{X_1+X_2}(t)$$

$$= E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1} e^{tX_2}\right]$$

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right]$$

$$= M_{X_1}(t)\, M_{X_2}(t)$$

$$\boxed{\text{Thm: } M_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t)}$$

What is the MGF of a Gaussian?

$X \sim N(\mu, \sigma^2)$ if $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$

$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2}\,r\,dr\,d\theta$$

$$= 2\pi\, e^{-r^2/2}\Big|_{\infty}^{0} = 2\pi \implies I = \pm\sqrt{2\pi}$$

as $I > 0$, $I = \sqrt{2\pi}$

$X \sim N(0,1)$

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - 2tx + t^2 - t^2)/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} e^{t^2/2}\, dx$$

$$= e^{t^2/2} \int_{u=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

$u = x - t$

$du = dx$

$x: -\infty \to \infty \Rightarrow u: -\infty \to \infty$

$$= e^{t^2/2}$$

$$\boxed{\text{Thm: } X \sim N(0,1) \text{ Then } M_X(t) = e^{t^2/2}}$$

Central Limit Theorem

If $X_1, \ldots, X_n$ are "nice" iid rv (say MGF exist in $(-\delta, \delta)$)

Then $$Z_n := \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

speed depends on higher moments

Proof: Compute $M_{Z_n}(t)$

Show converges to MGF of standard normal

* Wizard of Oz That means densities converge

↳ why does MGF convergence imply densities converge?

Recall: $Z_n = \dfrac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \sum_{k=1}^{n} \dfrac{1}{\sigma\sqrt{n}} X_k - \dfrac{\mu\sqrt{n}}{\sigma}$

$M_{Z_n}(t) = M_{\sum \frac{1}{\sigma\sqrt{n}} X_k - \frac{\mu\sqrt{n}}{\sigma}}(t)$ and $M_{aX+b}(t) = e^{bt} M_X(at)$

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$

$= e^{-\mu\sqrt{n}t/\sigma} M_{\sum \frac{1}{\sigma\sqrt{n}} X_k}(t)$

$= e^{-\mu\sqrt{n}t/\sigma} \prod_{k=1}^{n} M_{\frac{1}{\sigma\sqrt{n}} X_k}(t)$

$= e^{-\mu\sqrt{n}t/\sigma} \prod_{k=1}^{n} M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$ where $X, X_1, \ldots, X_n$ all have same distribution

$= e^{-\mu\sqrt{n}t/\sigma} \left(M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n = M_{Z_n}(t)$

Study $M_{Z_n}(t) = e^{-\mu\sqrt{n}t/\sigma} M_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n$

Take logarithms!

$$\log M_{Z_n}(t) = -\frac{\mu\sqrt{n}t}{\sigma} + n \log M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$M_X(u) = \sum_{k=0}^{\infty} \frac{\mu_k'}{k!} u^k \qquad u = \frac{t}{\sigma\sqrt{n}} \qquad \log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} \cdots$$

$$M_X\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \mu\frac{t}{\sigma\sqrt{n}} + \frac{\mu_2'}{2}\frac{t^2}{\sigma^2 n} + \text{Order}\left(t^3/n^{3/2}\right)$$

$$\sigma^2 = E[X^2] - E[X]^2 = E[(X-\mu)^2]$$

$$\sigma^2 = \mu_2' - \mu^2$$

or $\mu_2' = \sigma^2 + \mu^2$

negligible
first piece
$$\frac{\mu_3'}{3!} \frac{t^3}{\sigma^3 n^{3/2}}$$

$$M_{\bar{X}}\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \mu \frac{t}{\sigma\sqrt{n}} + \frac{\mu_2'}{2}\frac{t^2}{\sigma^2 n} + \text{Order}\left(t^3/n^{3/2}\right)$$

$$n \log M_{\bar{X}}\left(\frac{t}{\sigma\sqrt{n}}\right) = n \log\left(1 + \left(\frac{\mu}{\sigma\sqrt{n}} t + \frac{\sigma^2+\mu^2}{2\sigma^2 n} t^2 + O\left(t^3/n^{3/2}\right)\right)\right)$$

$$= n\left[\frac{\mu}{\sigma\sqrt{n}} t + \frac{\sigma^2+\mu^2}{2\sigma^2 n} t^2\right] - \frac{n}{2}\left[\frac{\mu^2}{\sigma^2 n} t^2\right] + O\left(\frac{1}{\sqrt{n}}\right)$$

"arithmetic" kills moments $\mu_3', \mu_4', \ldots$

$$\log M_{Z_n}(t) = -\frac{\mu\sqrt{n}\,t}{\sigma} + \frac{\mu\sqrt{n}\,t}{\sigma} + \frac{\sigma^2+\mu^2}{2\sigma^2} t^2 - \frac{\mu^2}{2\sigma^2} t^2 + O\left(\frac{1}{\sqrt{n}}\right)$$

$$= t^2/2 + O\left(1/\sqrt{n}\right)$$

$$\text{So } M_{Z_n}(t) = e^{t^2/2}\, e^{O(1/\sqrt{n})}$$

higher moments (shape of density) lives