Random Matrix Ensembles with Split Limiting Behavior

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Introduction

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\mathrm{Prob}(A) = \prod_{1\le i\le j\le N} p(a_{ij}).$$

This means

$$\mathrm{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1\le i\le j\le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

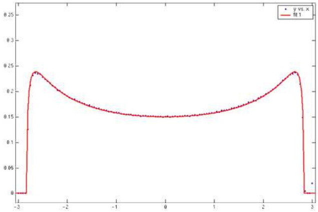
$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

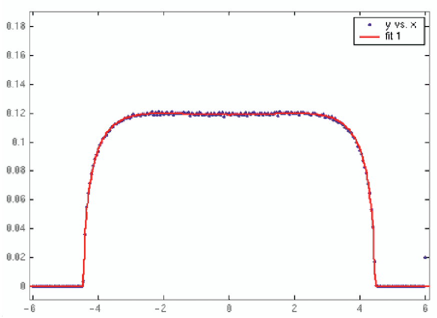
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

The Ensemble of m-Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right).$$

Choose distinct entries i.i.d.r.v.

Results

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

Fixed m equals $m \times m$ GOE, as $m \to \infty$ converges to the semicircle distribution.

Results (continued)

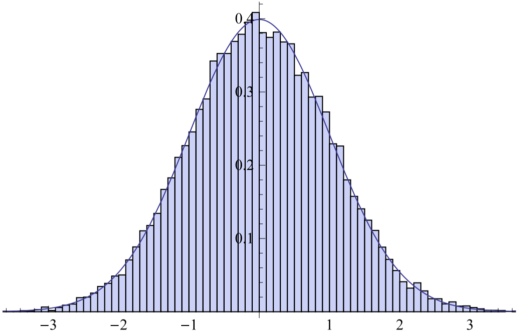


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

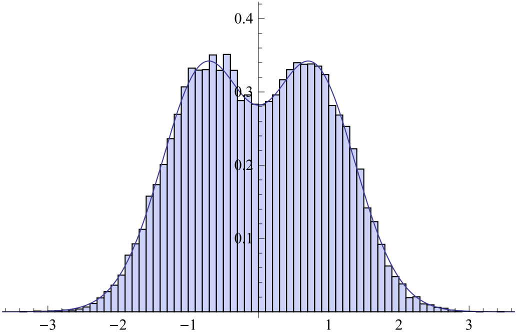


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results (continued)

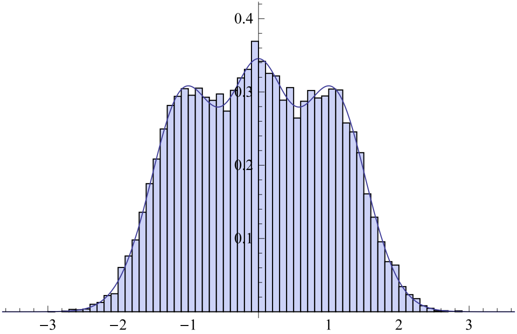


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results (continued)

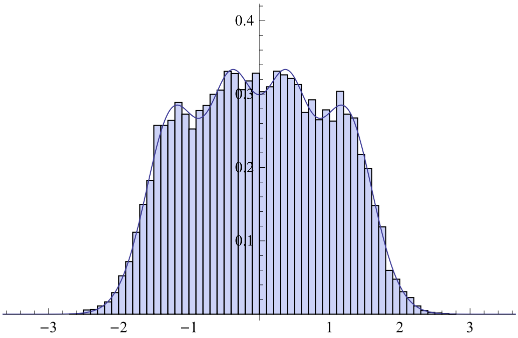


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results (continued)

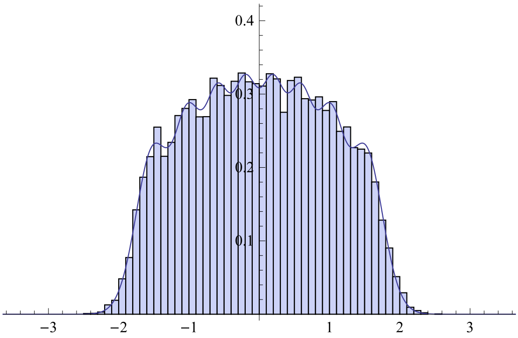


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results (continued)

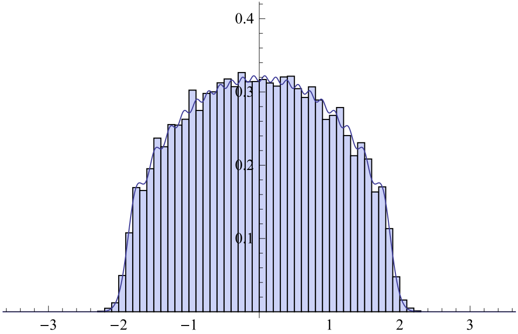


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results (continued)

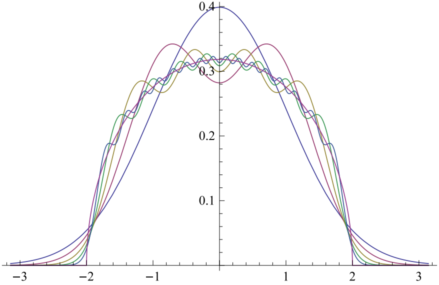


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

Current Research, Broadly

Some topics of interest in random matrix theory:

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Some topics of interest in random matrix theory:

- Universality: Many ensembles' eigenvalues have semicircular behavior.
- Study different classes of ensembles with nonstandard behavior
- Methods to study different quantities: spectral distribution, eigenvalue spacings, distribution of largest eigenvalue, etc.

Previous Work

Capitaine, Donati-Martin and Féral study ensembles where a Wigner ensemble is ‘deformed’ by adding another ensemble, resulting in two separated families of eigenvalues. They prove that

- Under fairly general conditions, the separated ‘blip’ of eigenvalues converges in distribution to a that of a finite GOE/GUE (universality).

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Overview

k-Checkerboard Ensembles

Checkerboard Matrices

Definition

The $N \times N$ (k, w)-checkerboard ensemble is the ensemble of matrices $M = (m_{ij})$ given by

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \bmod k \\ w & \text{if } i \equiv j \bmod k \end{cases}$$

where the $a_{ij} = a_{ji}$ are iid with mean 0, variance 1, and finite higher moments, and w is constant.

Example

A $(3, w)$-checkerboard matrix is of the form

$$\begin{pmatrix} w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\ a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\ a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w \end{pmatrix}$$

Split Eigenvalue Distribution

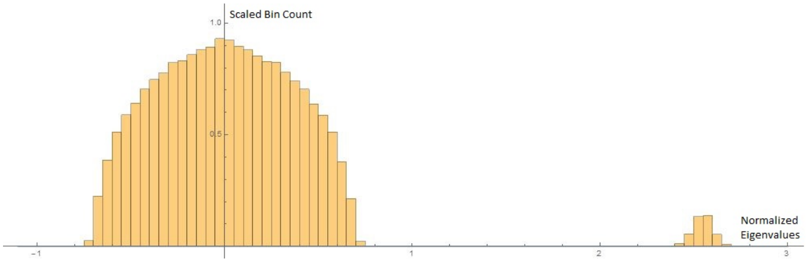

Figure: Histogram of normalized eigenvalues for 500 100 × 100 2-checkerboard matrices.

Eigenvalue Regimes

Theorem

Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of (k, w)-checkerboard matrices. Then almost surely as $N \to \infty$ the eigenvalues of A_N fall into two regimes: $N - k$ of the eigenvalues are $O(N^{1/2+\epsilon})$ and k eigenvalues are of magnitude $Nw/k + O(N^{1/2+\epsilon})$.

Normalized Empirical Spectral Measure

Definition

Given an $N \times N$ Hermitian matrix M_N with eigenvalues $\{\lambda_i\}_{i=1}^N$, the **normalized empirical spectral measure** is

$$\nu_{\frac{1}{\sqrt{N}}M_N}(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - \lambda_i/\sqrt{N})$$

Theorem

Let $\{M_N\}_{N\in\mathbb{N}}$ be a sequence of real $N \times N$ k-checkerboard matrices. Then, the normalized empirical spectral measures $\mu_{\frac{1}{\sqrt{N}}M_N}$ converge weakly almost surely to the semi-circle distribution.

Notion of Convergence

Definition

A sequence of random measures $\{\mu_N\}_{N\in\mathbb{N}}$ converges **weakly almost-surely** to a fixed measure μ if, with probability 1 over the (infinite) product probability space, we have

$$\lim_{N\to\infty}\int f\, d\mu_N = \int f\, d\mu$$

for all continuous and bounded f.

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- If $\{X_N\}_{N\in\mathbb{N}}$ is a sequence of random variables and, for all m, the moments converge $\mathbb{E}[X_N^m] \to \mathbb{E}[X^m]$ as $N \to \infty$, then $\{X_N\}_{N\in\mathbb{N}}$ converges in distribution to X.

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- Compute expected moments of ensemble's empirical spectral distribution and show convergence to desired moments.

Bulk Distribution: Obstructions

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- Recall that there are k eigenvalues of magnitude $Nw/k + O(N^{1/2+\epsilon})$.
- Because of these high magnitude eigenvalues, the limiting expected moments of the normalized ESD do not exist.
- This obstructs the standard application of the method of moments.

Perturbation Theorem

Theorem (Tao)

Let $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$ be a sequence of random Hermitian matrix ensembles such that $\{\nu_{\mathcal{A}_N,N}\}_{N\in\mathbb{N}}$ converges weakly almost surely to a limit ν. Let $\{\tilde{\mathcal{A}}_N\}_{N\in\mathbb{N}}$ be another sequence of random matrix ensembles such that $\frac{1}{N}\,\mathrm{rank}(\tilde{\mathcal{A}}_N)$ converges almost surely to zero. Then $\{\nu_{\mathcal{A}_N+\tilde{\mathcal{A}}_N,N}\}_{N\in\mathbb{N}}$ converges weakly almost surely to ν.

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- Because $w = 0$, the blip eigenvalues are centered at zero.
- This avoids the divergence of limiting expected moments of the normalized ESD.

Examining the Blip I

- To understand the limiting distribution of the blip, we localize our measure to the blip regime.

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- To do this, define a new empirical spectral measure by

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

with f a function ≈ 0 on the bulk and ≈ 1 on the blip.

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- Candidates for f must be roughly 1 near the blip and 0 over the bulk of eigenvalues.
- Candidates for f must be amenable to Eigenvalue-Trace Lemma arguments (so we must either choose a polynomial or deal with Taylor series convergence).
- Any given polynomial does not vanish to a high enough order at $x = 0$ as $N \to \infty$, so we choose family of polynomials.

The Weighting Function

The weighting function used is

$$f_n(x) = x^{2n}(x-2)^{2n}.$$

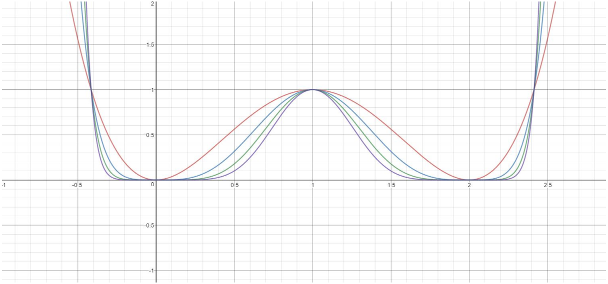

Figure: $f_n(x)$ plotted for $n = 1$ to $n = 4$.

The New Spectral Measure I

Using the weighting function $f_n(x)$ we form a new empirical spectral measure.

Definition

The **empirical blip spectral measure** associated to an $N \times N$ k-checkerboard matrix A is

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f_{n(N)}\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

where $n(N)$ is a function for which there exists some ϵ so that $N^{\epsilon} \ll n(N) \ll N^{1-\epsilon}$.

Computing Expected Moments

Classically, the m^{th} moment of empirical spectral measure

$$\nu_{\frac{1}{\sqrt{N}}M_N}(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - \lambda_i/\sqrt{N})$$

is

$$\mathbb{E}[\int_{\mathbb{R}} x^m d\nu_{\frac{1}{\sqrt{N}}M_N}] = \frac{1}{N^{1+m/2}}\mathbb{E}[\sum_{i=1}^{N}\lambda_i^m] = \frac{1}{N^{1+m/2}}\mathbb{E}\ \text{Tr}\ M_N^m.$$

The expected m^{th} moment of the blip empirical spectral measure

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f_{n(N)}\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

is

$$\mathbb{E}[\mu_{A,N}^{(m)}] = \frac{1}{k}\left(\frac{k}{N}\right)^{2n} \sum_{j=0}^{2n} \binom{2n}{j} \sum_{i=0}^{m+j} \binom{m+j}{i} \left(-\frac{N}{k}\right)^{m-i} \mathbb{E} \text{ Tr } A^{2n+i}.$$

Sorting Cyclic Products

Recall:

$$\mathbb{E}\ \mathrm{Tr}\ M^n = \sum_{1 \le i_1, \ldots, i_n \le N} \mathbb{E}[m_{i_1 i_2} m_{i_2 i_3} \cdots m_{i_n i_1}].$$

Problem: sort cyclic products into similar groups by structure.

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Example

$w \cdots waaaaw \cdots waaw \cdots waw \cdots w$
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Classes

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Two configurations in the same **class** must have the same block sizes but they may be ordered differently and have different numbers of w's between them.

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Indexings & Matchings

Look at all ways to index a's in a configuration such that expectation is nonzero.

Definition

A **matching** is an equivalence relation $\sim$ on the a's which constrains the ways of indexing the a's by forcing some to have the same indices (in either order).

Example

For $a_{i_1 i_2} w_{i_2 i_3} a_{i_3 i_4} w_{i_4 i_5} a_{i_5 i_6} w_{i_6 i_7} a_{i_7 i_8} w_{i_8 i_1}$, if $a_{i_1 i_2} \sim a_{i_5 i_6}$ then $\{i_1, i_2\} = \{i_5, i_6\}$.

Indexings

Definition

Given a configuration, matching, and length of the cyclic product, an **indexing** is a choice of

- the (positive) number of w's between each pair of adjacent blocks (in the cyclic sense), and
- the integer indices of each a and w in the cyclic product.

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- the integer indices of each a and w in the cyclic product.

The definitions of class, configuration, and matching do *not* fix the length of the cyclic product, but indexings do.

Cyclic Product Recap

$$\mathbb{E}\ \mathrm{Tr}\ A^{\eta} = \sum_{\substack{S\text{-classes} \\ C}} \ \sum_{\substack{\text{configurations} \\ \mathscr{C} \in C}} \ \sum_{\text{matchings } M} \ \sum_{\substack{\text{indexings} \\ I \text{ given } M, \mathscr{C}, \eta}} \mathbb{E}[\Pi]$$

where Π is the cyclic product given by the choice of indexing.

Which Classes Contribute?

Lemma

In the limit as $N \to \infty$, the only classes which contribute are those with only 1- or 2-blocks, 1-blocks are matched with exactly one other 1-block, and both a's in any 2-block are matched with their adjacent entry and no others.

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Proof. Fix number of a's. Big blocks or more matchings force indices of a's to be the same so number of indexings is a lower power of N which disappears in the limit.

Computing a class's contribution

Compute the total contribution to $\mathbb{E}\ \mathrm{Tr}\ A^{\eta}$ of a class C with s blocks, v 1-blocks and $(s-v)$ 2-blocks.

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- $\binom{s}{v}$ ways to order the blocks.
- $p(\eta) = \frac{\eta^s}{s!} + O(\eta^{s-1})$ ways to place them among the w's (just choosing vertices on a polygon).

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- Congruence classes 'propagate' through strings of w's.
- 2-blocks have a's matched $a_{ij}a_{ji}$ so congruence class propagates through.
- Each 1-block has non-congruent indices and these therefore determine all other congruence classes (except inner index of 2-blocks).

Hollow GOE

- Fixing congruence classes $[i_1], \dots, [i_v]$
- Number of ways to match

$$a_{[i_1][i_2]} w \dots w a_{[i_2][i_3]} w \dots w a_{[i_v][i_1]} w \dots$$

- This is the same as

$$\mathbb{E}[b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_v i_1}]$$

with each $b_{ij} \sim \mathcal{N}(0, 1)$ iid. and $b_{ij} = b_{ji}$ and $b_{ii} = 0$.

Hollow GOE

Definition

The **hollow Gaussian Orthogonal Ensemble** is given by $B = (b_{ij}) = B^T$ with

$$b_{ij} = \begin{cases} \mathcal{N}_{\mathbb{R}}(0, 1) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Continuing to count indexings

- Once we have chosen matching, $(k-1)^{s-v}$ choices of congruence classes mod k for indices.
- $\left(\left(\frac{N}{k}\right)^{\eta-s} + O\left(\left(\frac{N}{k}\right)^{\eta-s-1}\right)\right)$ choices of indices in these congruence classes.

Putting this all together

Proposition

The total contribution to $\mathbb{E}\ \mathrm{Tr}\ A^\eta$ *of a class with s blocks, v 1-blocks and* $(s-v)$ *2-blocks*

$$p(\eta)\binom{s}{v}(k-1)^{s-v}\mathbb{E}_k\ \mathrm{Tr}\ B^v\left(\left(\frac{N}{k}\right)^{\eta-s}+O\left(\left(\frac{N}{k}\right)^{\eta-s-1}\right)\right)$$

where

$$p(\eta) = \frac{\eta^s}{s!}+O(\eta^{s-1})$$

and the expectation $\mathbb{E}_k\ \mathrm{Tr}\ B^v$ *is taken over the* $k\times k$ *hollow GOE.*

Main theorem

Theorem

Denote the centered moments of the empirical blip spectral measure of the $N \times N$ k-checkerboard ensemble by $\overline{\mu}_{A,N}^{(m)}$. Then

$$\lim_{N\to\infty} \mathbb{E}[\overline{\mu}_{A,N}^{(m)}] \;=\; \frac{1}{k}\mathbb{E}_k \;\mathrm{Tr}\; B^m.$$

Extensions

Theorem

If the a_{ij} are changed to complex (resp. quaternion), ceteris paribus, we have that the expected m^{th} centered moments satisfy

$$\lim_{N\to\infty} \mathbb{E}[\overline{\mu}_{A,N}^{(m)}] = \frac{1}{k}\mathbb{E}_k \operatorname{Tr} B^m$$

where B^m is the complex (resp. quaternion) analogue of the hollow GOE.

Almost-Sure Weak Convergence

Moment convergence theorem

Theorem (Moment Convergence Theorem)

Let μ be a measure on $\mathbb{R}$ with finite moments $\mu^{(m)}$ for all $m \in \mathbb{Z}_{\geq 0}$, and $\mu_1, \mu_2, \dots$ a sequence of measures with finite moments $\mu_n^{(m)}$ such that $\lim_{n\to\infty} \mu_n^{(m)} = \mu^{(m)}$ for all $m \in \mathbb{Z}_{\geq 0}$. If in addition the moments $\mu^{(m)}$ uniquely characterize a measure (Carleman's condition), then the sequence μ_n converges weakly to μ.

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Remark

If the moments converge almost-surely, then the measures almost-surely converge weakly.

Standard arguments and their ultimate downfall

We wish to show m^{th} moments $X_{m,N}$ of empirical spectral measure of $N \times N$ ensemble converge a.s. to desired M_m as $N \to \infty$.

Show

$$|X_{m,N} - M_m| \ \leq \ |X_{m,N} - \mathbb{E}[X_{m,N}]| + |\mathbb{E}[X_{m,N}] - M_m| \, .$$

converges a.s. to 0 as $N \to \infty$.

Standard arguments and their ultimate downfall

Usual approach to show $X_{m,N} \to \mathbb{E}[X_{m,N}]$ a.s.:

Chebyshev:

$$\Pr(|X_{m,N} - \mathbb{E}[X_{m,N}]| > \epsilon) \leq \frac{\mathbb{E}\left[(X_{m,N} - \mathbb{E}[X_{m,N}])^r\right]}{\epsilon^r} = O\left(\frac{1}{N^2}\right)$$

Apply Borel-Cantelli to show

$$\Pr\left(\exists m \text{ such that } X_{m,N} \neq \mathbb{E}[X_{m,N}] \text{ for infinitely many } N\right) = 0.$$

Standard arguments and their ultimate downfall

- With many eigenvalues, all empirical spectral measures look alike.
- With a finite number in the blip, empirical spectral measures look different.
- Hence variance (and higher moments) over ensemble of empirical spectral measure's moments **does not** go to 0.

Solution: Averaged Blip Empirical Spectral Measure

Definition

Fix a function $g : \mathbb{N} \to \mathbb{N}$. The **averaged empirical blip spectral measure** associated to $\overline{A} \in \Omega^{\mathbb{N}}$ is

$$\mu_{N,g,\overline{A}} := \frac{1}{g(N)} \sum_{i=1}^{g(N)} \mu_{A_N^{(i)},N},$$

where $\Omega = \prod_{N=1}^{\infty} \Omega_N$ and Ω_N is the probability space of $N \times N$ k-checkerboard matrices.

Main Result

Theorem

Let $g : \mathbb{N} \to \mathbb{N}$ be such that there exists an $\delta > 0$ for which $g(N) = \omega(N^\delta)$. Then, as $N \to \infty$, the averaged empirical spectral measures $\mu_{N,g,\overline{A}}$ of the k-checkerboard ensemble converge weakly almost-surely to the measure with moments $M_{k,m} = \frac{1}{k}\mathbb{E}_k \operatorname{Tr}[B^m]$.

Proof Sketch

- Because $g(N)$ is growing, by LLN arguments each random centered moment $\mathbb{E}[(\mu^{(m)}_{N,g,\overline{A}} - \mathbb{E}[\mu^{(m)}_{N,g,\overline{A}}])^r]$ converges to 0 at some rate.
- For higher moments of r.v. these converge faster.
- Choose r sufficiently high so that

$$\mathbb{E}[(\mu^{(m)}_{N,g,\overline{A}} - \mathbb{E}[\mu^{(m)}_{N,g,\overline{A}}])^r] = O\left(\frac{1}{N^2}\right).$$

Spectral distribution of hollow GOE

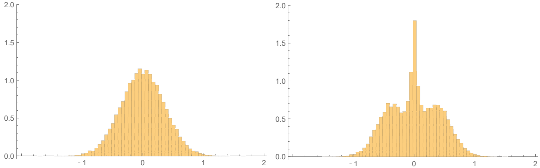

Figure: Hist. of eigenvals of 32000 (Left) 2×2 hollow GOE matrices, (Right) 3×3 hollow GOE matrices.

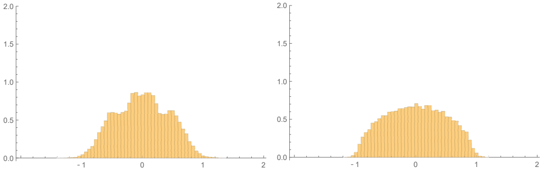

Figure: Hist. of eigenvals of 32000 (Left) 4×4 hollow GOE matrices, (Right) 16×16 hollow GOE matrices.

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  `https://arxiv.org/abs/1609.03120`
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Questions?

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