

Random Matrix Ensembles with Split Limiting Behavior

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Slides available at https://web.williams.edu/Mathematics/sjmillier/public_html

Introduction

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

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Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p , define

$$\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i < j \leq N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A .

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0 :

$$\int f(x)\delta(x - x_0)dx = f(x_0).$$

To each A , attach a probability measure:

$$\begin{aligned}\mu_{A,N}(x) &= \frac{1}{N} \sum_{i=1}^N \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\ \int_a^b \mu_{A,N}(x) dx &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b]\right\}}{N} \\ \text{k}^{\text{th}} \text{ moment} &= \frac{\sum_{i=1}^N \lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.\end{aligned}$$

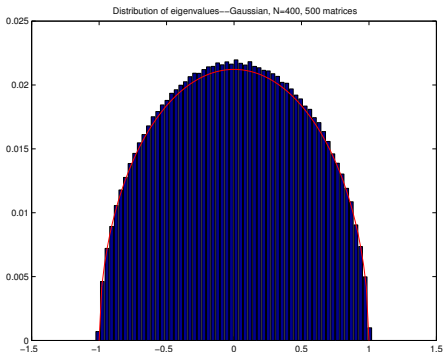
Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A , as $N \rightarrow \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

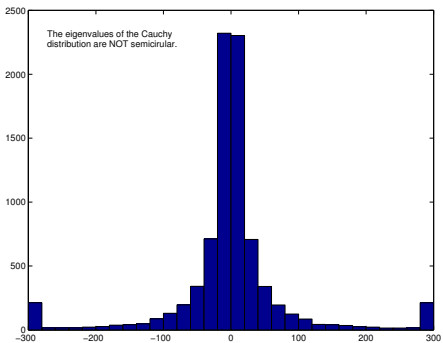
Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical examples



$$\text{Cauchy Distribution: } p(x) = \frac{1}{\pi(1+x^2)}$$

I. Zakharevich, *A generalization of Wigner's law*, Comm. Math. Phys. **268** (2006), no. 2, 403–414.

http://web.williams.edu/Mathematics/sjmillier/public_html/book/papers/innaz.pdf

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A , but choose the matrix elements randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^N \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\text{Trace}(\mathbf{A}^2) = \sum_{i=1}^N \lambda_i(\mathbf{A})^2.$$

By the Central Limit Theorem:

$$\text{Trace}(\mathbf{A}^2) = \sum_{i=1}^N \sum_{j=1}^N a_{ij} a_{ji} = \sum_{i=1}^N \sum_{j=1}^N a_{ij}^2 \sim N^2$$

$$\sum_{i=1}^N \lambda_i(\mathbf{A})^2 \sim N^2$$

Gives $N \text{Ave}(\lambda_i(\mathbf{A})^2) \sim N^2$ or $\text{Ave}(\lambda_i(\mathbf{A})) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k -th moment of $\mu_{A,N}(x)$ is $\text{Trace}(A^k)/2^k N^{k/2+1}$.

Average k -th moment is

$$\int \cdots \int \frac{\text{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k -th moments converge to moments of semi-circle as $N \rightarrow \infty$;
- Control variance (show it tends to zero as $N \rightarrow \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^N \sum_{j=1}^N a_{ij}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \cdot \prod_{\substack{(k,l) \neq (i,j) \\ k < l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} = 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} \rho(a_{ij}) da_{ij}.$$

Main contribution when the $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).

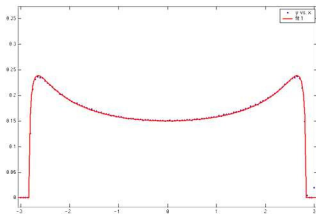
Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices (with Adam Massey and John Sinsheimer), *Journal of Theoretical Probability* **20** (2007), no. 3, 637–662.

<http://arxiv.org/abs/math/0512146>

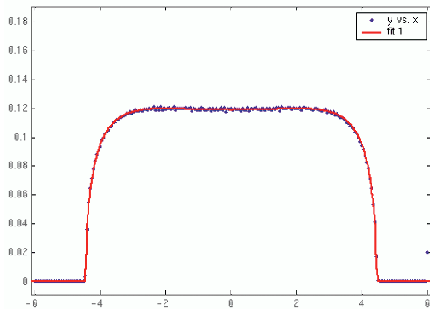
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d -regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)} \sqrt{4(d-1)-x^2} & |x| \leq 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \rightarrow \infty$ recover semi-circle).

The Ensemble of m -Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{pmatrix}.$$

Choose distinct entries i.i.d.r.v.

Results

Theorem: Kolođlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m -block circulant ensemble is given by

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^m \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

Fixed m equals $m \times m$ GOE, as $m \rightarrow \infty$ converges to the semicircle distribution.

Results (continued)

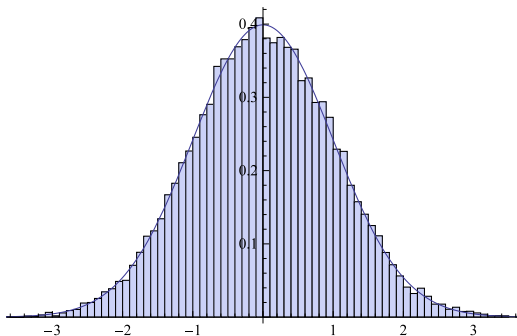


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400 .

Results (continued)

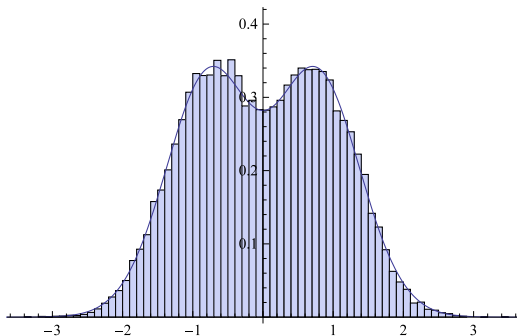


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400 .

Results (continued)

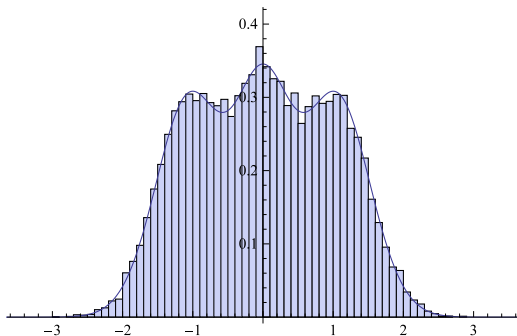


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402 .

Results (continued)

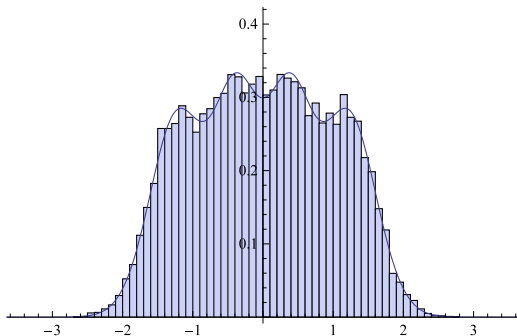


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400 .

Results (continued)

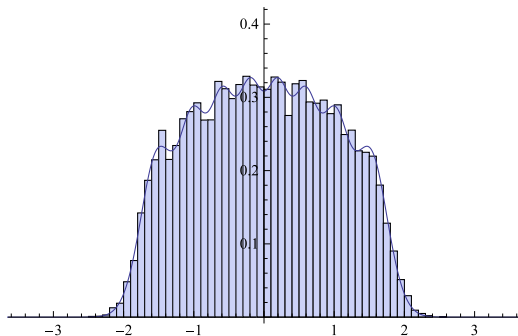


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400 .

Results (continued)

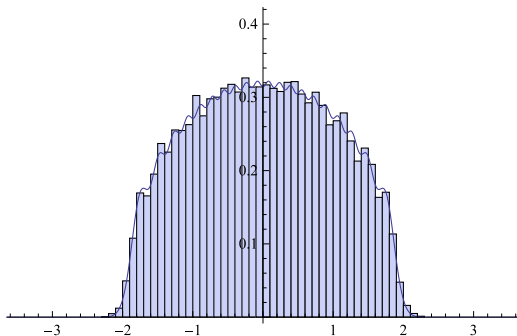


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400 .

Results (continued)

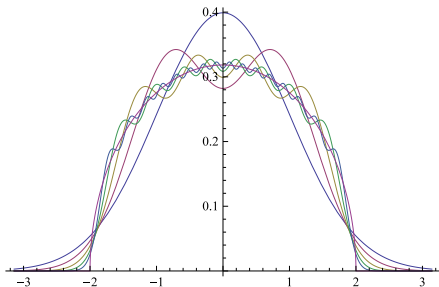


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), *Journal of Theoretical Probability* **26** (2013), no. 4, 1020–1060. <http://arxiv.org/abs/1008.4812>

k-Checkerboard Ensembles

Checkerboard Matrices: $N \times N$ (k, w)-checkerboard ensemble

Matrices $M = (m_{ij}) = M^T$ with a_{ij} iidrv, mean 0, variance 1, finite higher moments, w fixed and

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \pmod{k} \\ w & \text{if } i \equiv j \pmod{k}. \end{cases}$$

Example: $(3, w)$ -checkerboard matrix:

$$\begin{pmatrix} w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\ a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\ a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w \end{pmatrix}$$

Split Eigenvalue Distribution

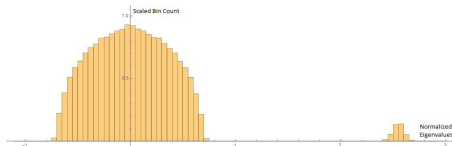


Figure: Histogram of normalized eigenvalues for 500 100×100 2-checkerboard matrices.

Eigenvalue Regimes

Theorem

Let $\{A_N\}_{N \in \mathbb{N}}$ be a sequence of (k, w) -checkerboard matrices. Then almost surely as $N \rightarrow \infty$ the eigenvalues of A_N fall into two regimes: $N - k$ of the eigenvalues are $O(N^{1/2+\epsilon})$ and k eigenvalues are of magnitude $Nw/k + O(N^{1/2+\epsilon})$.

Normalized Empirical Spectral Measure

Definition

Given an $N \times N$ Hermitian matrix M_N with eigenvalues $\{\lambda_i\}_{i=1}^N$, the **normalized empirical spectral measure** is

$$\nu_{\frac{1}{\sqrt{N}}M_N}(\mathbf{x}) := \frac{1}{N} \sum_{i=1}^N \delta(\mathbf{x} - \lambda_i/\sqrt{N})$$

Theorem

Let $\{M_N\}_{N \in \mathbb{N}}$ be a sequence of real $N \times N$ k -checkerboard matrices. Then, the normalized empirical spectral measures $\mu_{\frac{1}{\sqrt{N}}M_N}$ converge weakly almost surely to the semi-circle distribution.

Moment convergence theorem

Theorem (Moment Convergence Theorem)

Let μ be a measure on \mathbb{R} with finite moments $\mu^{(m)}$ for all $m \in \mathbb{Z}_{\geq 0}$, and μ_1, μ_2, \dots a sequence of measures with finite moments $\mu_n^{(m)}$ such that $\lim_{n \rightarrow \infty} \mu_n^{(m)} = \mu^{(m)}$ for all $m \in \mathbb{Z}_{\geq 0}$. If in addition the moments $\mu^{(m)}$ uniquely characterize a measure (Carleman's condition), then the sequence μ_n converges weakly to μ .

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Remark

If the moments converge almost-surely, then the measures almost-surely converge weakly.

Standard arguments

We wish to show m^{th} moments $X_{m,N}$ of empirical spectral measure of $N \times N$ ensemble converge a.s. to desired M_m as $N \rightarrow \infty$.

Show

$$|X_{m,N} - M_m| \leq |X_{m,N} - \mathbb{E}[X_{m,N}]| + |\mathbb{E}[X_{m,N}] - M_m|.$$

converges a.s. to 0 as $N \rightarrow \infty$.

Bulk Distribution: Obstructions

- There are $N - k$ eigenvalues of order $O(N^{1/2+\epsilon})$ in the bulk.

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- Recall that there are k eigenvalues of magnitude $Nw/k + O(N^{1/2+\epsilon})$.
- Because of these high magnitude eigenvalues, the limiting expected moments of the normalized ESD do not exist.
- This obstructs the standard application of the method of moments.

Perturbation Theorem

Theorem (Tao)

Let $\{\mathcal{A}_N\}_{N \in \mathbb{N}}$ be a sequence of random Hermitian matrix ensembles such that $\{\nu_{\mathcal{A}_N, N}\}_{N \in \mathbb{N}}$ converges weakly almost surely to a limit ν . Let $\{\tilde{\mathcal{A}}_N\}_{N \in \mathbb{N}}$ be another sequence of random matrix ensembles such that $\frac{1}{N} \text{rank}(\tilde{\mathcal{A}}_N)$ converges almost surely to zero. Then $\{\nu_{\mathcal{A}_N + \tilde{\mathcal{A}}_N, N}\}_{N \in \mathbb{N}}$ converges weakly almost surely to ν .

Examining the Blip I

- To understand the limiting distribution of the blip, we localize our measure to the blip regime.

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- To understand the limiting distribution of the blip, we localize our measure to the blip regime.
- To do this, define a new empirical spectral measure by

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

with f a function ≈ 0 on the bulk and ≈ 1 on the blip.

Examining the Blip II

- Candidates for f must be amenable to Eigenvalue-Trace Lemma arguments (so we must either choose a polynomial or deal with Taylor series convergence).

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- Candidates for f must be amenable to Eigenvalue-Trace Lemma arguments (so we must either choose a polynomial or deal with Taylor series convergence).
- Any given polynomial does not vanish to a high enough order at $x = 0$ as $N \rightarrow \infty$, so we choose family of polynomials.

The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x - 2)^{2n}$.

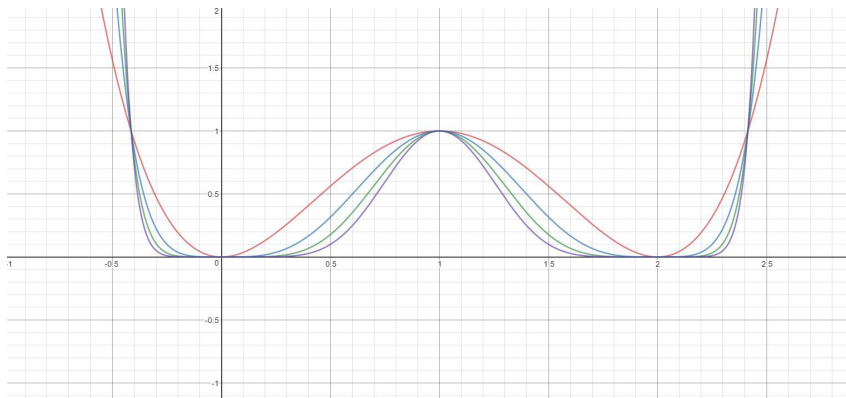


Figure: $f_n(x)$ plotted for $n = 1$ to $n = 4$.

The New Spectral Measure I

Using the weighting function $f_n(x)$ we form a new empirical spectral measure.

Definition

The **empirical blip spectral measure** associated to an $N \times N$ k -checkerboard matrix A is

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f_{n(N)}\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

where $n(N)$ is a function for which there exists some ϵ so that $N^\epsilon \ll n(N) \ll N^{1-\epsilon}$.

Main theorem

Definition

The **hollow Gaussian Orthogonal Ensemble** is given by $B = (b_{ij}) = B^T$ with

$$b_{ij} = \mathcal{N}_{\mathbb{R}}(0, 1)(1 - \delta_{ij})$$

Theorem

We have

$$\lim_{N \rightarrow \infty} \mathbb{E}[\bar{\mu}_{A,N}^{(m)}] = \frac{1}{k} \mathbb{E}_k \operatorname{Tr} B^m,$$

where $\bar{\mu}_{A,N}^{(m)}$ is the centered moments of the empirical blip spectral measure of the $N \times N$ k -checkerboard ensemble and B is in the hollow GOE.

Main Result

Issue: Can't look at blip of just one matrix as only fixed number eigenvalues; average over $g(N)$ such matrices.

Theorem

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be such that there exists an $\delta > 0$ for which $g(N) = \omega(N^\delta)$. Then, as $N \rightarrow \infty$, the averaged empirical spectral measures $\mu_{N,g,\bar{A}}$ of the k -checkerboard ensemble converge weakly almost-surely to the measure with moments $M_{k,m} = \frac{1}{k} \mathbb{E}_k \text{Tr} [B^m]$.

Spectral distribution of hollow GOE

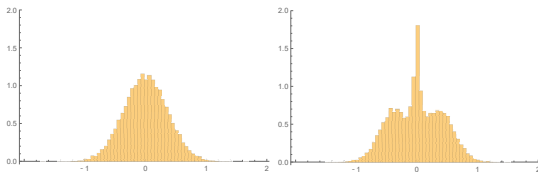


Figure: Hist. of eigenvals of 32000 (Left) 2×2 hollow GOE matrices, (Right) 3×3 hollow GOE matrices.

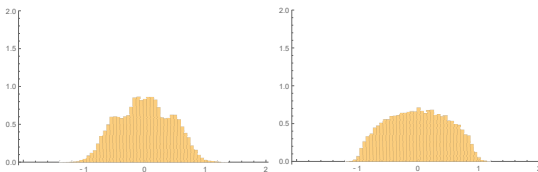


Figure: Hist. of eigenvals of 32000 (Left) 4×4 hollow GOE matrices, (Right) 16×16 hollow GOE matrices.

References / Acknowledgements

Acknowledgments

- Full paper available on arXiv:
<https://arxiv.org/abs/1609.03120>
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Other Random Matrix Theory Papers

- 1 *Distribution of eigenvalues for the ensemble of real symmetric Toeplitz matrices* (with Christopher Hammond), *Journal of Theoretical Probability* **18** (2005), no. 3, 537–566.
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- 2 *Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices* (with Adam Massey and John Sinsheimer), *Journal of Theoretical Probability* **20** (2007), no. 3, 637–662.
<http://arxiv.org/abs/math/0512146>
- 3 *The distribution of the second largest eigenvalue in families of random regular graphs* (with Tim Novikoff and Anthony Sabelli), *Experimental Mathematics* **17** (2008), no. 2, 231–244.
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<http://arxiv.org/abs/1003.2010>
- 6 *The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices* (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), *Journal of Theoretical Probability* **26** (2013), no. 4, 1020–1060. <http://arxiv.org/abs/1008.4812>
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- 8 *The expected eigenvalue distribution of large, weighted d -regular graphs* (with Leo Goldmahker, Cap Khoury and Kesinee Ninsuwan), *Random Matrices: Theory and Applications* **3** (2014), no. 4, 1450015 (22 pages).