Deviations from Large Eigenvalues of a Special Matrix Ensemble

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Introduction

Goals

- See similar behavior in different systems.
- Determine correct scale and statistics to study eigenvalues.
- Discuss the tools and techniques needed to prove the results.

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

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Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system
E_n : energy levels
ψ_n : energy eigenfunctions

Origins of Random Matrix Theory

- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\mathrm{P}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\mathrm{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

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Classical:

- Wigner matrices are much more general and display universal behavior: the semi-circle law.

Wigner's Semi-Circle Law

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$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but choose the matrix elements randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \sim N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2$$

Gives $N\,\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\text{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\text{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2p(a_{ij})da_{ij}\quad\cdot\prod_{\substack{(k,l)\neq(i,j)\\ k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl}\;=\;1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Checkerboard Martrices

Checkerboard Matrices

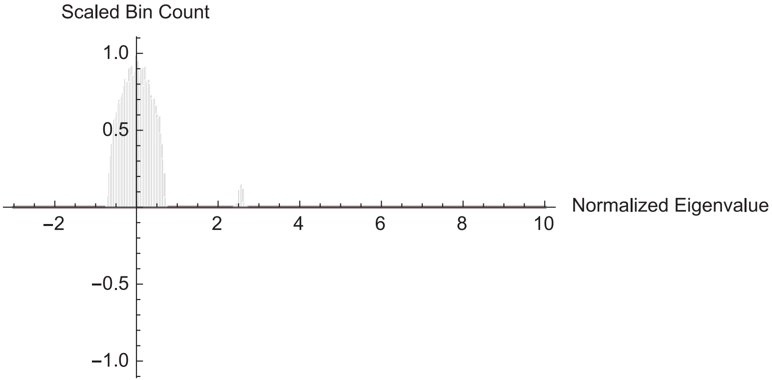
An $N \times N$ matrix A is a *random real symmetric 2-checkerboard matrix* if

$$A = \begin{pmatrix} w & a_{0,1} & w & a_{0,3} & w & \cdots & a_{0,N-1} \\ a_{0,1} & w & a_{1,2} & w & a_{1,4} & \cdots & w \\ w & a_{1,2} & w & a_{2,3} & w & \cdots & a_{2,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & w & a_{2,N-1} & w & a_{4,N-1} & \cdots & w \end{pmatrix},$$

where the $a_{i,j}$ are i.i.d.r.v, and w is a fixed nonzero value (we take $w = 1$), value largely irrelevant.

It is k-checkerboard if w occurs every k entries in row starting with the entry at the index $(i \bmod k)$ of the ith row.

Results



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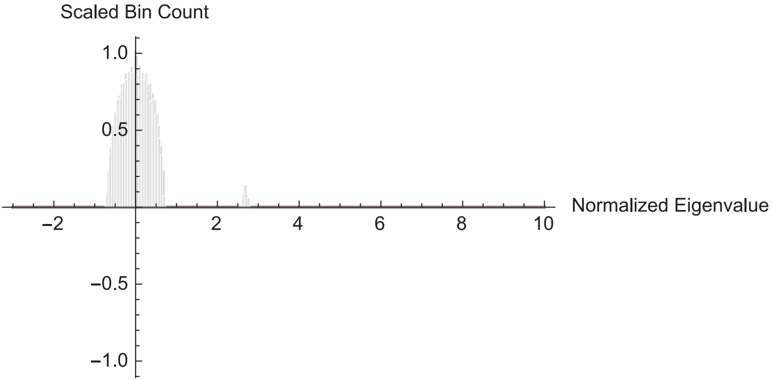

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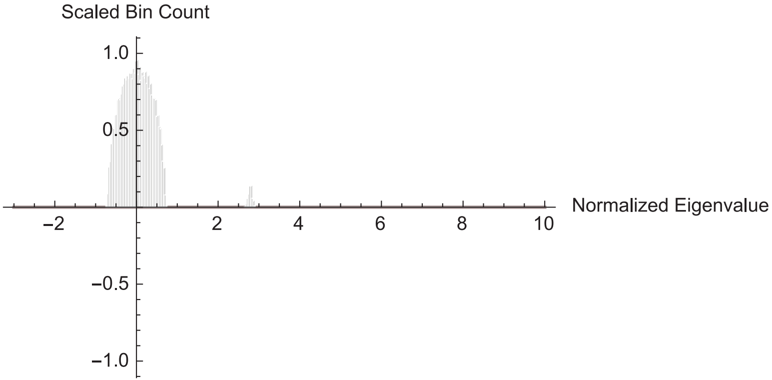

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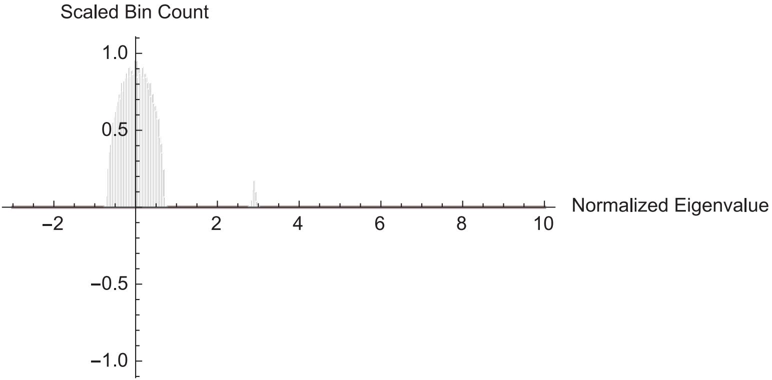

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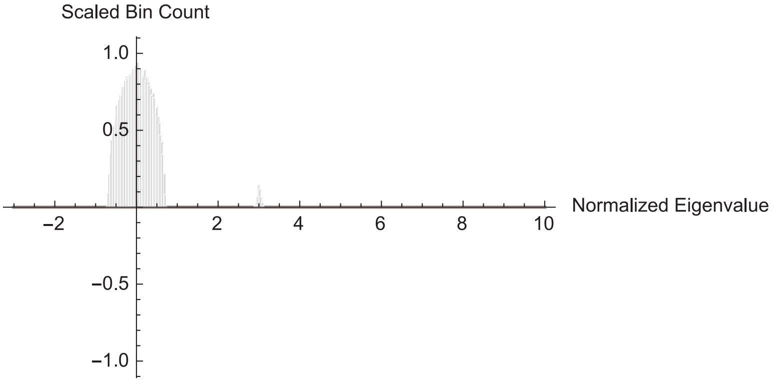

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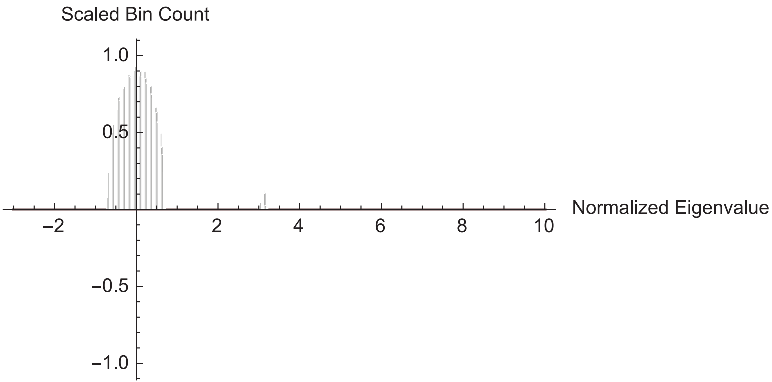

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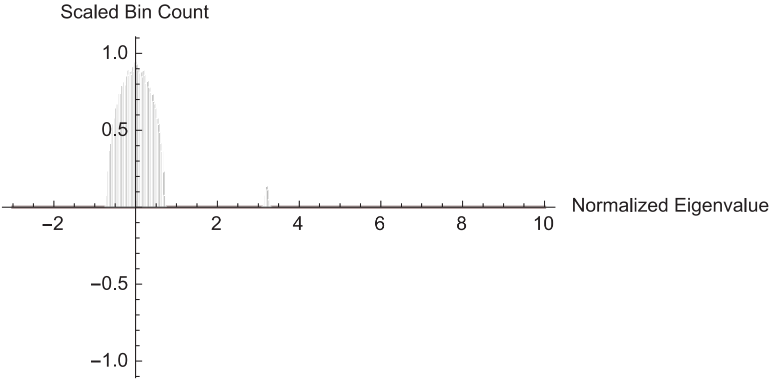

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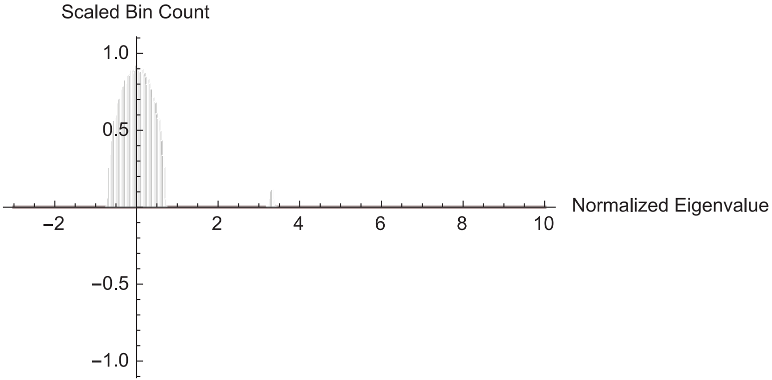

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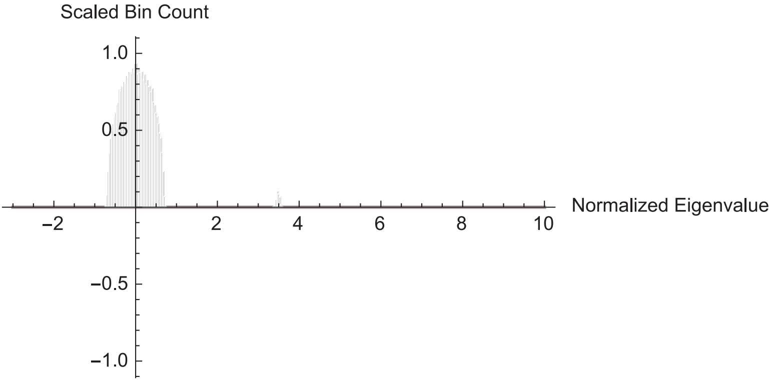

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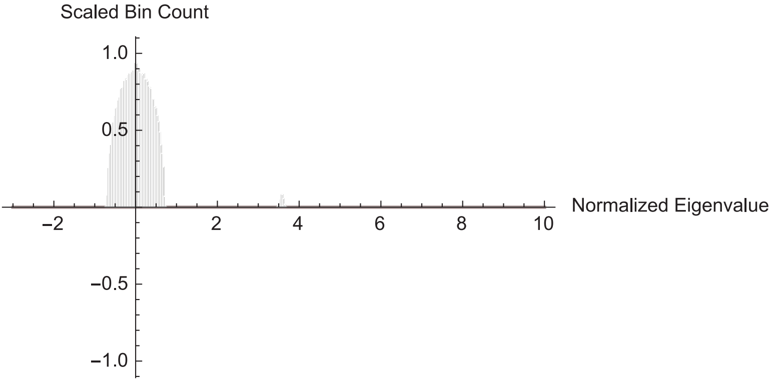

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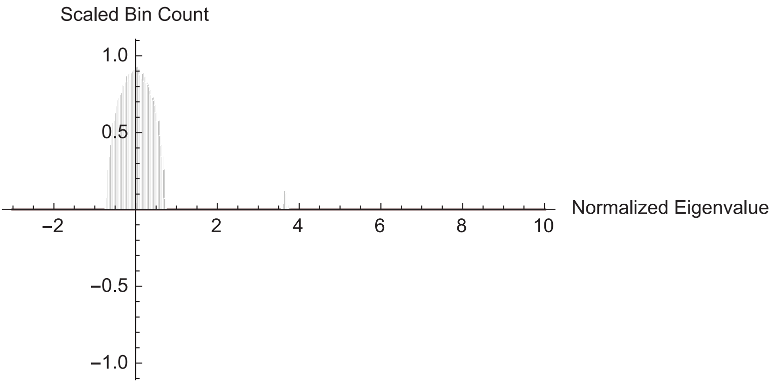

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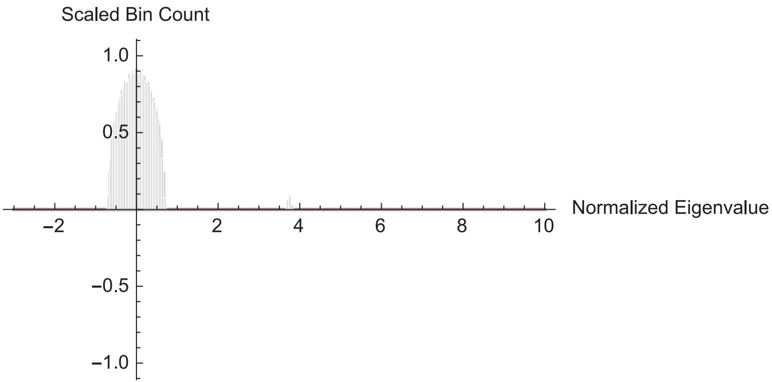

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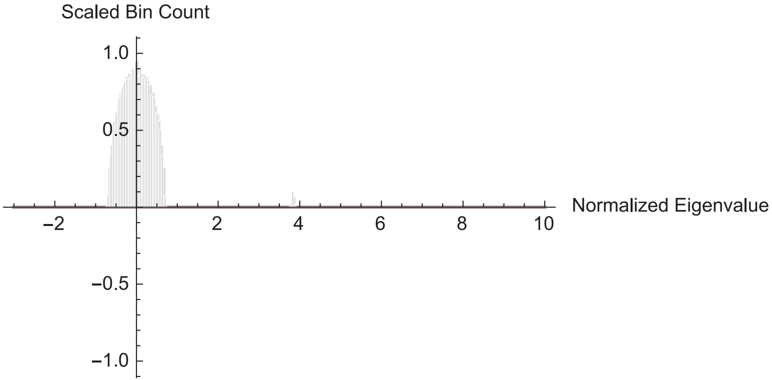

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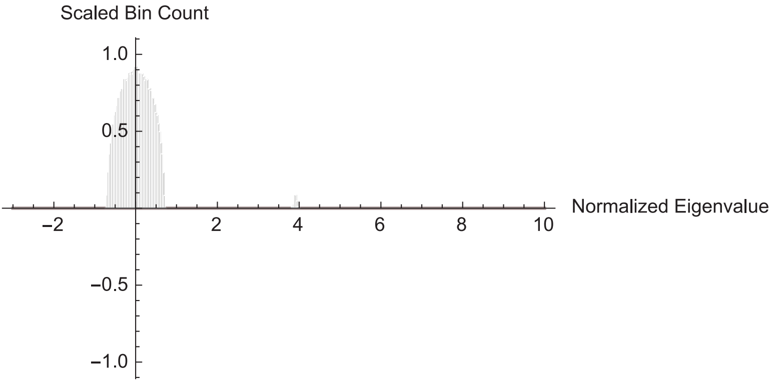

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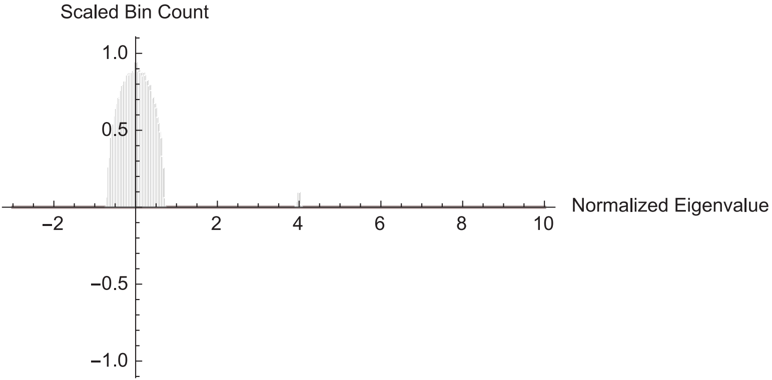

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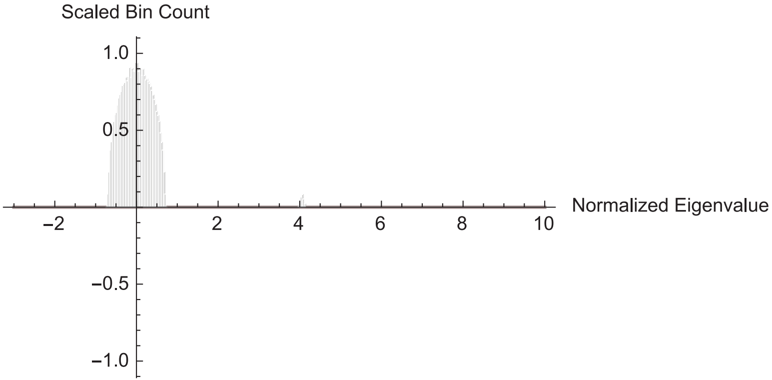

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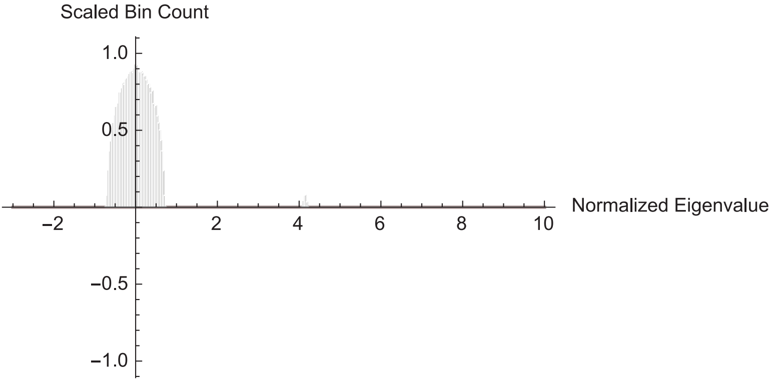

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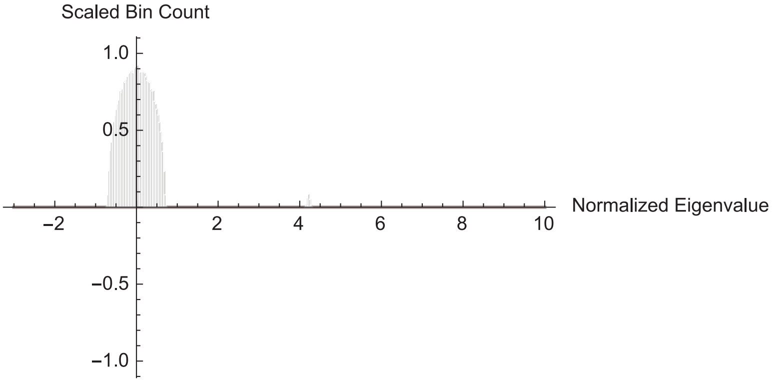

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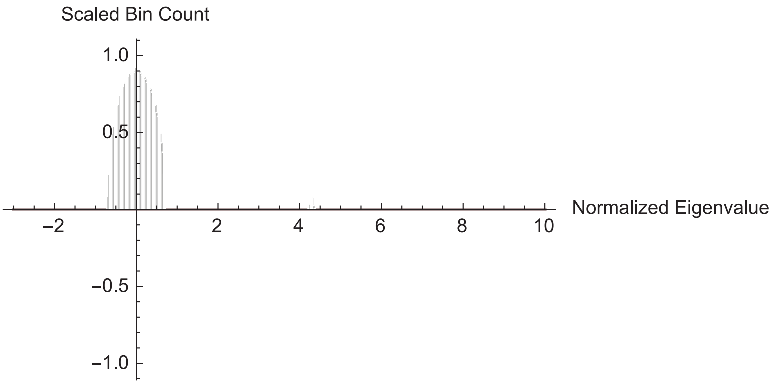

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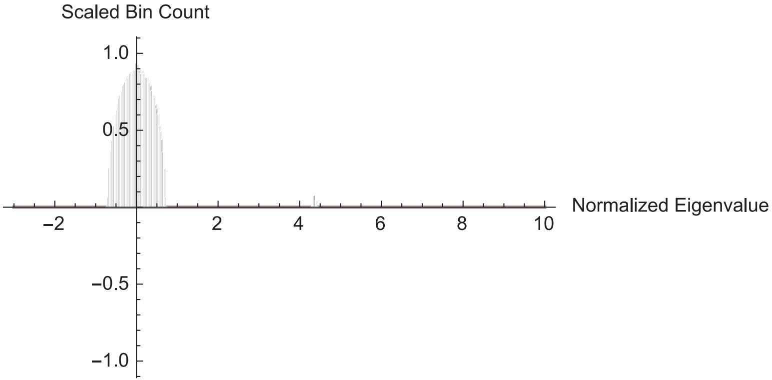

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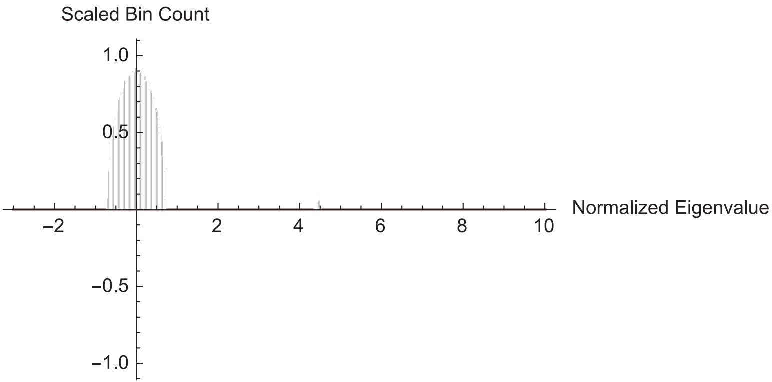

Evolving Results

Some key results:

- All but k eigenvalues in bulk and converge to semi-circle, remaining k in *blip* may be thought of as deviations from the “trivial eigenvalues”.

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- All but k eigenvalues in bulk and converge to semi-circle, remaining k in *blip* may be thought of as deviations from the “trivial eigenvalues”.
- Eigenvalues in blip are of size approximately N/k.
- Variance in blip is independent of N.

Key ideas

Weights: To concentrate in blip, let $f_n(x) = x^{2n}(x-2)^{2n}$, and set

$$\mu_{A,N}(x) \;=\; \frac{1}{k}\sum_{\lambda} f_{n(N)}\left(\frac{\lambda}{N/k}\right)\delta\left(x-\left(\lambda-\frac{N}{k}\right)\right),$$

where $n(N)$ is a function for which there is some $\epsilon > 0$ so that $N^{\epsilon} \ll n(N) \ll N^{1-\epsilon}$.

Combinatorics: Reduce to $k \times k$ GOE.

Combinatorics

Combinatorics overview:

- Get nice expression for expected moments in terms of expected traces.

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- Use degree-of-freedom arguments and combinatorial cancellation to reduce work.
- Analyze remaining terms from eigenvalue-trace lemma.

Combinatorics

Combinatorics result:

Theorem

Denote the m^{th} centered moment of the empirical blip spectral distribution of an $N \times N$ k-checkerboard matrix A by $\bar{\mu}_{A,N}^{(m)}$. Then

$$\lim_{N\to\infty} \mathbb{E}[\bar{\mu}_{A,N}^{(m)}] = \frac{1}{k}\mathbb{E}_k \operatorname{Tr} B^m,$$

where $\mathbb{E}_k \operatorname{Tr} B^m$ is expectation taken over the $k \times k$ hollow GOE ensemble.

Example and Generalizations

Example: The limiting expected centered moments of the 2-checkerboard are those of a standard Gaussian.

Generalizations: One can similarly define complex and quaternion k-checkerboard ensembles and reduce them to $k \times k$ hollow GUE and GSE ensembles.

Analysis

Analysis overview:

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- Here, there are only k significant eigenvalues, even as $N \to \infty$, so the above fails.
- Need to average over many matrices and look at high moments (not just second, fourth).

Analysis

Analysis result:

Theorem

Let $g : \mathbb{N} \to \mathbb{N}$ *be such that* $\exists \delta > 0$ *such that* $N^\delta \ll g(N)$. *Then as* $N \to \infty$, *the empirical spectral measures given by averaging over* $g(N)$ $N \times N$ k*-checkerboard matrices converge weakly almost-surely to the measure with moments* $\frac{1}{k}\mathbb{E}_k \operatorname{Tr} B^m$.

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