Rank and Bias in Families of Curves via Nagao's Conjecture

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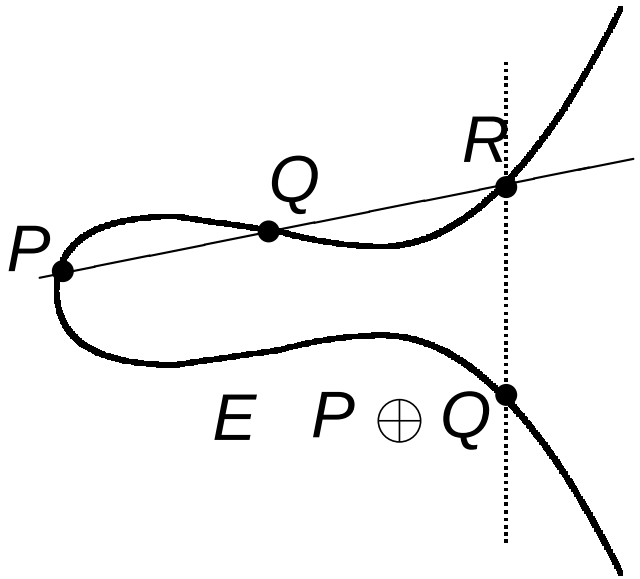
Outline

- Review Basics of Elliptic Curves.
- Describe Construction of Moderate Rank Families.
- Discuss Applications.
- Explore Generalizations.

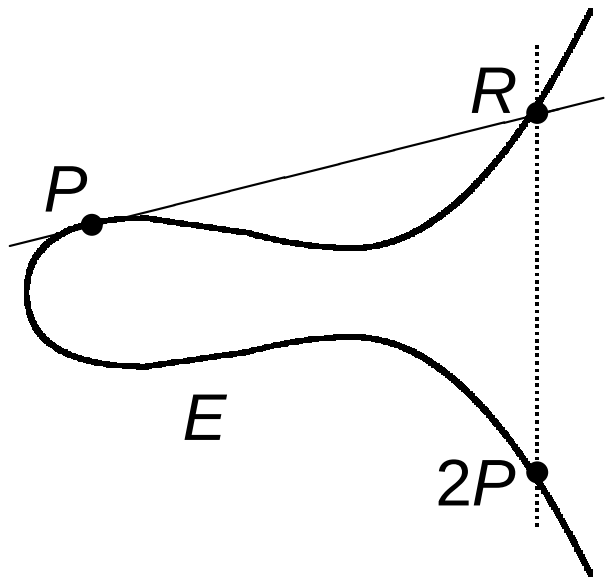
Introduction

Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

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Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.

Theorem: Preliminaries

Consider a one-parameter family

$$\mathcal{E} : y^2 + a_1(T)xy + a_3(T)y = x^3 + a_2(T)x^2 + a_4(T)x + a_6(T).$$

Let $a_t(p) = p + 1 - N_p$, where N_p is the number of solutions mod p (including ∞). Define

$$A_{\mathcal{E}}(p) := \frac{1}{p}\sum_{t(p)} a_t(p).$$

$A_{\mathcal{E}}(p)$ is bounded independent of p (Deligne).

Methods for Obtaining Explicit Formulas

For a family $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, we can write

$$a_{\mathcal{E}(t)}(p) = -\sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p} \right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol mod$p$ given by

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & \text{if } x \text{ is a non-zero square modulo } p \\ 0 & \text{if } x \equiv 0 \bmod p \\ -1 & \text{otherwise.} \end{cases}$$

Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

$$\sum_{x \bmod p} \left(\frac{ax+b}{p}\right) = 0 \quad \text{if } p \nmid a$$

$$\sum_{x \bmod p} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \nmid b^2-4ac \\ (p-1)\left(\frac{a}{p}\right) & \text{if } p \mid b^2-4ac \end{cases}$$

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Average Values of Legendre Symbols

The value of $\left(\frac{x}{p}\right)$ for $x \in \mathbb{Z}$, when averaged over all primes p, is 1 if x is a non-zero square, and 0 otherwise.

Tate's Conjecture

Tate's Conjecture for Elliptic Surfaces

Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\text{ét}}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$-\text{ord}_{s=2} L_2(\mathcal{E}, s) \;=\; \text{rank}\, NS(\mathcal{E}/\mathbb{Q}),$$

where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $\text{Re}(s) = 2$.

Theorem: Preliminaries

Theorem

Rosen-Silverman (Conjecture of Nagao): For an elliptic surface (a one-parameter family), assume Tate's conjecture. Then

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X}-A_{\mathcal{E}}(p)\log p \;=\; \operatorname{rank}\mathcal{E}(\mathbb{Q}(T)).$$

Tate's conjecture is known for rational surfaces: An elliptic surface $y^2 = x^3 + A(T)x + B(T)$ is rational iff one of the following is true:

- $0 < \max\{3\deg A, 2\deg B\} < 12$;
- $3\deg A = 2\deg B = 12$ and $\operatorname{ord}_{T=0} T^{12}\Delta(T^{-1}) = 0$.

Small Rank

Moderate Rank

Rank 6 Family

Rational Surface of Rank 6 over $\mathbb{Q}(T)$:

$$\begin{aligned} y^2 &= x^3 + (2aT - B)x^2 + (2bT - C)(T^2 + 2T - A + 1)x \\ &\quad + (2cT - D)(T^2 + 2T - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Constructing Rank 6 Family

Idea: can explicitly evaluate linear and quadratic Legendre sums.

Use: a and b are not both zero mod p and $p > 2$, then for $t \in \mathbb{Z}$

$$\sum_{t=0}^{p-1} \left(\frac{at^2 + bt + c}{p} \right) = \begin{cases} (p-1)\left(\frac{a}{p}\right) & \text{if } p|(b^2 - 4ac) \\ -\left(\frac{a}{p}\right) & \text{otherwise.} \end{cases}$$

Thus if $p|(b^2 - 4ac)$, the summands are $\left(\frac{a(t-t')^2}{p}\right) = \left(\frac{a}{p}\right)$, and the t-sum is large.

Constructing Rank 6 Family

$$\begin{aligned} y^2 = f(x,T) &= x^3T^2 + 2g(x)T - h(x) \\ g(x) &= x^3 + ax^2 + bx + c,\ c \neq 0 \\ h(x) &= (A-1)x^3 + Bx^2 + Cx + D \\ D_T(x) &= g(x)^2 + x^3h(x). \end{aligned}$$

$D_T(x)$ is one-fourth of the discriminant of the quadratic (in T) polynomial $f(x,T)$.

$\mathcal{E}$ not in standard form, as the coefficient of x^3 is T^2, harmless. As $y^2 = f(x,T)$, for the fiber at $T = t$:

$$a_t(p) = -\sum_{x(p)} \left(\frac{f(x,t)}{p}\right) = -\sum_{x(p)} \left(\frac{x^3t^2 + 2g(x)t - h(x)}{p}\right).$$

Constructing Rank 6 Family

We study $-pA_{\mathcal{E}}(p) = \sum_{x=0}^{p-1} \sum_{t=0}^{p-1} \left(\frac{f(x,t)}{p}\right)$.
When $x \equiv 0$ the t-sum vanishes if $c \not\equiv 0$, as it is just $\sum_{t=0}^{p-1} \left(\frac{2ct-D}{p}\right)$.

Assume now $x \not\equiv 0$. By the lemma on Quadratic Legendre Sums

$$\sum_{t=0}^{p-1} \left(\frac{x^3t^2 + 2g(x)t - h(x)}{p}\right) = \begin{cases} (p-1)\left(\frac{x^3}{p}\right) & \text{if } p \mid D_t(x) \\ -\left(\frac{x^3}{p}\right) & \text{otherwise.} \end{cases}$$

Goal:find coefficients a, b, c, A, B, C, D so that $D_t(x)$ has six distinct, non-zero roots that are squares.

Constructing Rank 6 Family

Assume we can find such coefficients. Then

$$\begin{aligned}
-pA_{\mathcal{E}}(p) &= \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right) = \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{x^3t^2+2g(x)t-h(x)}{p}\right) \\
&= \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right) + \sum_{x:D_t(x)\equiv 0}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right) \\
&\quad + \sum_{x:xD_t(x)\not\equiv 0}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right) \\
&= 0 + 6(p-1) - \sum_{x:xD_t(x)\not\equiv 0}\left(\frac{x^3}{p}\right) = 6p.
\end{aligned}$$

Constructing Rank 6 Family

We must find $a, \ldots, D$ such that $D_t(x)$ has six distinct, non-zero roots ρ_i^2:

$$\begin{aligned} D_t(x) &= g(x)^2 + x^3 h(x) \\ &= Ax^6 + (B+2a)x^5 + (C + a^2 + 2b)x^4 \\ &\quad + (D + 2ab + 2c)x^3 \\ &\quad + (2ac + b^2)x^2 + (2bc)x + c^2 \\ &= A(x^6 + R_5x^5 + R_4x^4 + R_3x^3 + R_2x^2 + R_1x + R_0) \\ &= A(x - \rho_1^2)(x - \rho_2^2)(x - \rho_3^2)(x - \rho_4^2)(x - \rho_5^2)(x - \rho_6^2). \end{aligned}$$

Constructing Rank 6 Family

Because of the freedom to choose B, C, D there is no problem matching coefficients for the x^5, x^4, x^3 terms. We must simultaneously solve in integers

$$\begin{aligned} 2ac + b^2 &= R_2 A \\ 2bc &= R_1 A \\ c^2 &= R_0 A. \end{aligned}$$

For simplicity, take $A = 64R_0^3$. Then

$$\begin{aligned} c^2 &= 64R_0^4 \longrightarrow c = 8R_0^2 \\ 2bc &= 64R_0^3 R_1 \longrightarrow b = 4R_0 R_1 \\ 2ac + b^2 &= 64R_0^3 R_2 \longrightarrow a = 4R_0 R_2 - R_1^2. \end{aligned}$$

Constructing Rank 6 Family

For an explicit example, take $r_i = \rho_i^2 = i^2$. For these choices of roots,

$$R_0 = 518400,\ R_1 = -773136,\ R_2 = 296296.$$

Solving for a through D yields

$$\begin{array}{rcrcr}
A & = & 64R_0^3 & = & 8916100448256000000 \\
c & = & 8R_0^2 & = & 2149908480000 \\
b & = & 4R_0R_1 & = & -1603174809600 \\
a & = & 4R_0R_2 - R_1^2 & = & 16660111104 \\
B & = & R_5A - 2a & = & -8113651408246162222208 \\
C & = & R_4A - a^2 - 2b & = & 26497490347321493520384 \\
D & = & R_3A - 2ab - 2c & = & -3431075943454881336320 0
\end{array}$$

Constructing Rank 6 Family

We convert $y^2 = f(x,t)$ to $y^2 = F(x,T)$, which is in Weierstrass normal form. We send $y \to \frac{y}{T^2+2T-A+1}$, $x \to \frac{x}{T^2+2T-A+1}$, and then multiply both sides by $(T^2+2T-A+1)^2$. For future reference, we note that

$$\begin{aligned} T^2+2T-A+1 &= (T+1-\sqrt{A})(T+1+\sqrt{A}) \\ &= (T-t_1)(T-t_2) \\ &= (T-2985983999)(T+2985984001). \end{aligned}$$

We have

$$\begin{aligned} f(x,T) &= T^2x^3 + (2x^3 + 2ax^2 + 2bx + 2c)T - (A-1)x^3 - Bx^2 - Cx - D \\ &= (T^2+2T-A+1)x^3 + (2aT-B)x^2 + (2bT-C)x + (2cT-D) \\ F(x,T) &= x^3 + (2aT-B)x^2 + (2bT-C)(T^2+2T-A+1)x \\ &\quad +(2cT-D)(T^2+2T-A+1)^2. \end{aligned}$$

Constructing Rank 6 Family

We now study the $-pA_{\mathcal{E}}(p)$ arising from $y^2 = F(x,T)$. It is enough to show this is $6p + O(1)$ for all p greater than some p_0. Note that t_1, t_2 are the unique roots of $t^2 + 2t - A + 1 \equiv 0 \bmod p$. We find

$$-pA_{\mathcal{E}}(p) = \sum_{t=0}^{p-1}\sum_{x=0}^{p-1}\left(\frac{F(x,t)}{p}\right) = \sum_{t\neq t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{F(x,t)}{p}\right) + \sum_{t=t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{F(x,t)}{p}\right).$$

For $t \neq t_1, t_2$, send $x \longrightarrow (t^2 + 2t - A + 1)x$. As $(t^2 + 2t - A + 1) \not\equiv 0$, $\left(\frac{(t^2+2t-A+1)^2}{p}\right) = 1$. Simple algebra yields

$$\begin{aligned} -pA_{\mathcal{E}}(p) &= 6p + O(1) + \sum_{t=t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{f_t(x)}{p}\right) + O(1) \\ &= 6p + O(1) + \sum_{t=t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{(2at-B)x^2 + (2bt-C)x + (2ct-D)}{p}\right). \end{aligned}$$

Constructing Rank 6 Family

The last sum above is negligible (i.e., is $O(1)$) if

$$D(t) = (2bt - C)^2 - 4(2at - B)(2ct - D) \not\equiv 0(p).$$

Calculating yields

$$\begin{aligned} D(t_1) &= 42912434802438365611230921435802099054018 56 \\ &= 2^{32} \cdot 3^{25} \cdot 7^5 \cdot 11^2 \cdot 13 \cdot 19 \cdot 29 \cdot 31 \cdot 47 \cdot 67 \cdot 83 \cdot 97 \cdot 103 \\ D(t_2) &= 42912438166624527518950932553917195154882 56 \\ &= 2^{33} \cdot 3^{12} \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 173 \cdot 17389 \cdot 805873 \cdot 9447850813. \end{aligned}$$

Constructing Rank 6 Family

Hence, except for finitely many primes (coming from factors of $D(t_i)$, $a, \ldots, D$, t_1 and t_2), $-A_{\mathcal{E}}(p) = 6p + O(1)$ as desired.

We have shown: There exist integers a, b, c, A, B, C, D so that the curve $\mathcal{E} : y^2 = x^3T^2 + 2g(x)T - h(x)$ over $\mathbb{Q}(T)$, with $g(x) = x^3 + ax^2 + bx + c$ and $h(x) = (A-1)x^3 + Bx^2 + Cx + D$, has rank 6 over $\mathbb{Q}(T)$. In particular, with the choices of a through D above, $\mathcal{E}$ is a rational elliptic surface and has Weierstrass form

$$\begin{aligned} y^2 &= x^3 + (2aT - B)x^2 + (2bT - C)(T^2 + 2T - A + 1)x \\ &\quad + (2cT - D)(T^2 + 2T - A + 1)^2 \end{aligned}$$

Constructing Rank 6 Family

We show $\mathcal{E}$ is a rational elliptic surface by translating $x \mapsto x - (2aT - B)/3$, which yields $y^2 = x^3 + A(T)x + B(T)$ with $\deg(A) = 3, \deg(B) = 5$.

The Rosen-Silverman theorem is applicable, and as we can compute $A_{\mathcal{E}}(p)$, we know the rank is exactly 6 (and we never need to calculate height matrices). $\square$

Applications

Biases in Lower Order Terms

Let $n_{3,2,p}$ equal the number of cube roots of 2 modulo p, and set $c_0(p) = \left[\left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)\right] p$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3-x}{p}\right)\right]^2$, $c_{3/2}(p) = p \sum_{x(p)} \left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p\equiv 2 \bmod 3 \\ 0 & p\equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p\equiv 1 \bmod 4 \\ 0 & p\equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and otherwise is 0.

Biases in Lower Order Terms

The first family is the family of all elliptic curves; it is a two parameter family and we expect the main term of its second moment to be p^3.

Note that except for our family $y^2 = x^3 + Tx^2 + 1$, all the families $\mathcal{E}$ have $A_{2,\mathcal{E}}(p) = p^2 - h(p)p + O(1)$, where $h(p)$ is non-negative. Further, many of the families have $h(p) = m_{\mathcal{E}} > 0$.

Note $c_1(p)$ is the square of the coefficients from an elliptic curve with complex multiplication. It is non-negative and of size p for $p \not\equiv 3 \bmod 4$, and zero for $p \equiv 1 \bmod 4$ (send $x \mapsto -x \bmod p$ and note $\left(\frac{-1}{p}\right) = -1$).

It is somewhat remarkable that all these families have a correction to the main term in Michel's theorem in the same direction, and we analyze the consequence this has on the average rank. For our family which has a $p^{3/2}$ term, note that on average this term is zero and the p term is negative.

Lower order terms and average rank

$$\frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t}\phi\left(\gamma_t\frac{\log R}{2\pi}\right) = \widehat{\phi}(0)+\phi(0)-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right)a_t(p)$$
$$-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)a_t(p)^2+O\left(\frac{\log\log R}{\log R}\right).$$

If ϕ is non-negative, we obtain a bound for the average rank in the family by restricting the sum to be only over zeros at the central point. The error $O\left(\frac{\log\log R}{\log R}\right)$ comes from trivial estimation and ignores probable cancellation, and we expect $O\left(\frac{1}{\log R}\right)$ or smaller to be the correct magnitude. For most families $\log R \sim \log N^a$ for some integer a.

Lower order terms and average rank (cont)

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

Assume the second moment of $a_t(p)^2$ is $p^2 - m_{\mathcal{E}}p + O(1)$, $m_{\mathcal{E}} > 0$.

We have already handled the contribution from p^2, and $-m_{\mathcal{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathcal{E}}p) \\ &= \frac{2m_{\mathcal{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned}$$

Thus there is a contribution of size $1/\log R$.

Lower order terms and average rank (cont)

A good choice of test functions (see Appendix A of [ILS]) is the Fourier pair

$$\phi(x) = \frac{\sin^2(2\pi\frac{\sigma}{2}x)}{(2\pi x)^2}, \quad \widehat{\phi}(u) = \begin{cases} \frac{\sigma-|u|}{4} & \text{if } |u| \le \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Note $\phi(0) = \frac{\sigma^2}{4}$, $\widehat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma}$, and evaluating the prime sum gives

$$S_2 \sim \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_{\mathcal{E}}}{\log R}\, \phi(0).$$

Lower order terms and average rank (cont)

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned} \frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\leq \frac{\phi(0)}{\sigma} + \left(r+\frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\phi(0) \\ \text{Ave Rank}_{[N,2N]}(\mathcal{E}) &\leq \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}. \end{aligned}$$

$\sigma = 1$, $m_{\mathcal{E}} = 1$: for conductors of size 10^{12}, the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

$\sigma = 2$: lower order correction contributes .02 for conductors of size 10^{12}, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier's bound (already there without the potential correction term!).

Hyperelliptic curves with moderately large rank over $\mathbb{Q}(T)$

Hyperelliptic Curves

Define a hyperelliptic curve of genus g over $\mathbb{Q}(T)$:

$$\mathcal{X} : y^2 = f(x,T) = x^{2g+1} + A_{2g}(T)x^{2g} + \cdots + A_1(T)x + A_0(T).$$

Let $a_{\mathcal{X}}(p) = p + 1 - \#\mathcal{X}(\mathbb{F}_p)$. Then

$$a_{\mathcal{X}}(p) \;=\; -\sum_{x(p)} \left(\frac{f(x,t)}{p}\right)$$

and its m^{th} power sum

$$A_{m,\mathcal{X}}(p) \;=\; \sum_{t(p)} a_{\mathcal{X}}(p)^m.$$

Generalized Nagao's conjecture

Generalized Nagao's Conjecture

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X}-\frac{1}{p}A_{1,\chi}(p)\log p=\operatorname{rank}\mathrm{J}_{\mathcal{X}}\left(\mathbb{Q}(\mathrm{T})\right).$$

Goal: Construct families of hyperelliptic curves with high rank.

Moderate-Rank Family

Theorem (HLKM, 2018)

Assume the Generalized Nagao Conjecture and trivial Chow trace Jacobian. For any $g \geq 1$, we can construct infinitely many genus g hyperelliptic curves $\mathcal{X}$ over $\mathbb{Q}(T)$ such that

$$\operatorname{rank} \mathrm{J}_{\mathcal{X}}\left(\mathbb{Q}(\mathrm{T})\right) = 4\mathrm{g} + 2.$$

- Close to current record of $4g + 7$.
- No height matrix or basis computation.

Generalizes construction of Arms, Lozano-Robledo, and Miller in the elliptic surface case.

Idea of Construction

Define a genus g curve

$$\mathcal{X} : y^2 = f(x,T) = x^{2g+1}T^2 + 2g(x)T - h(x)$$

$$g(x) = x^{2g+1} + \sum_{i=0}^{2g} a_i x^i$$

$$h(x) = (A-1)x^{2g+1} + \sum_{i=0}^{2g} A_i x^i.$$

The discriminant of the quadratic polynomial is

$$D_T(x) := g(x)^2 + x^{2g+1}h(x).$$

Idea of Construction

$$-A_{1,\mathcal{X}}(p) = \sum_{t(p)} \sum_{x(p)} \left(\frac{f(x,t)}{p}\right)$$

$$= \sum_{\substack{x(p) \\ D_t(x)\equiv 0}} (p-1)\left(\frac{x^{2g+1}}{p}\right) + \sum_{\substack{x(p) \\ D_t(x)\not\equiv 0}} (-1)\left(\frac{x^{2g+1}}{p}\right)$$

$$= \sum_{\substack{x(p) \\ D_t(x)\equiv 0}} p\left(\frac{x}{p}\right)$$

Therefore, $-A_{1,\mathcal{X}}(p)$ is $p\left(\frac{x}{p}\right)$ summed over the roots of $D_t(x)$. To maximize the sum, we make each x a perfect square.

Idea of Construction

Key Idea

Make the roots of $D_t(x)$ distinct nonzero perfect squares.

- Choose roots ρ_i^2 of $D_t(x)$ so that

$$D_t(x) = A \prod_{i=1}^{4g+2} \left(x - \rho_i^2\right).$$

- Equate coefficients in

$$D_t(x) = A \prod_{i=1}^{4g+2} \left(x - \rho_i^2\right) = g(x)^2 + x^{2g+1}h(x).$$

- Solve the nonlinear system for the coefficients of g, h.

Idea of the Construction

$$
\begin{aligned}
-A_{1,\chi}(p) &= p \sum_{\substack{x \bmod p \\ D_t(x)\equiv 0}} \left(\frac{x^{2g+1}}{p}\right) \\
&= p \cdot (\#\text{ of perfect-square roots of } D_t(x)) \\
&= p \cdot (4g+2)\,.
\end{aligned}
$$

Then by the Generalized Nagao Conjecture

$$
\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \frac{1}{p} \cdot p \cdot (4g+2) \log p = 4g+2 = \operatorname{rank} J_{\mathcal{X}}\left(\mathbb{Q}(T)\right).
$$

Bias Conjecture

Bias Conjecture

Michel's Theorem

For one-parameter families of elliptic curves $\mathcal{E}$, the second moment $A_{2,\mathcal{E}}(p)$ is

$$A_{2,\mathcal{E}}(p) = p^2 + O\left(p^{3/2}\right).$$

Bias Conjecture (Miller)

The largest lower order term in the second moment expansion that does not average to 0 is on average **negative**.

Goal: Find as many hyperelliptic families with as much bias as possible.

Bias Example

Theorem (HLKM 2018)

Consider $\mathcal{X} : y^2 = x^n + x^h T^k$. If h is odd assume $\nu_2(p-1) > \nu_2(n-h)$, where ν_2 is the 2-adic valuation. If $\gcd(k, n-h, p-1) = 1$, then

$$A_{2,\mathcal{X}}(p) = \begin{cases} (\gcd(n-h, p-1) - 1)(p^2 - p) & h \text{ even} \\ \gcd(n-h, p-1)(p^2 - p) & h \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Calculations Part 1: k-Periodicity

$$A_{2,\mathcal{X}}(p) = \sum_{t,x,y(p)} \left(\frac{x^n + x^h t^k}{p}\right)\left(\frac{y^n + y^h t^k}{p}\right)$$
$$= \sum_{t,x,y(p)} \left(\frac{(t^{-n}x^n) + (t^{-h}x^h)t^k}{p}\right)\left(\frac{(t^{-n}y^n) + (t^{-h}y^h)t^k}{p}\right)$$
$$= \sum_{t,x,y(p)} \left(\frac{x^n + x^h t^{(k+(n-h))}}{p}\right)\left(\frac{y^n + y^h t^{(k+(n-h))}}{p}\right)$$

The second moment is periodic in k with period $(n-h)$.

Calculations Part 2: Assume $\gcd(n-h,k,p-1)=1$

$$A_{2,\mathcal{X}}(p) = \sum_{t,x,y(p)} \left(\frac{x^n + x^h t^k}{p}\right)\left(\frac{y^n + y^h t^k}{p}\right)$$

$$= \sum_{t,x,y(p)} \left(\frac{x^n + x^h t^m}{p}\right)\left(\frac{y^n + y^h t^m}{p}\right) \qquad (m \equiv_{n-h} k)$$

$$= \sum_{t,x,y(p)} \left(\frac{x^n + x^h t}{p}\right)\left(\frac{y^n + y^h t}{p}\right) \qquad \text{(Frobenius)}$$

Thus, this reduces to calculating the second moment of $y^2 = x^n + x^h T$, which is straightforward.

Open Questions and References

Open Questions

- Higher rank.
- Bias conjecture.
- Higher moments.
- Higher degrees.

Pi Mu Epsilon Opportunities: sjm1@williams.edu

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to \$100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31. Congratulations to the Missouri State University Problem Solving Group and Columbus State University, which both qualified; the randomly selected winner was the Missouri State University Problem Solving Group.

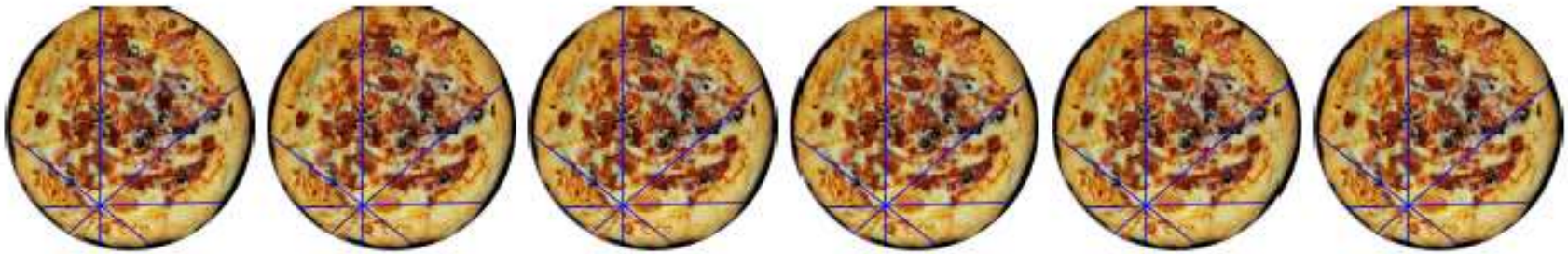

FIGURE 1. Pizza motivation; can you name the theorem that's represented here?

Publications: Elliptic Curves

1. 1- *and 2-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952–992. http://arxiv.org/pdf/math/0310159

2. *Variation in the number of points on elliptic curves and applications to excess rank*, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120. http://arxiv.org/abs/math/0506461

3. *Constructing one-parameter families of elliptic curves over* $\mathbb{Q}(T)$ *with moderate rank* (with Scott Arms and Álvaro Lozano-Robledo), Journal of Number Theory **123** (2007), no. 2, 388–402. http://arxiv.org/abs/math/0406579

4. *Towards an 'average' version of the Birch and Swinnerton-Dyer Conjecture* (with John Goes), Journal of Number Theory **130** (2010), no. 10, 2341–2358. http://arxiv.org/abs/0911.2871

5. *Moments of the rank of elliptic curves* (with Siman Wong), Canad. J. of Math. **64** (2012), no. 1, 151–182. http://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/mwMomentsRanksEC812final.pdf

6. *Lower-Order Biases in Elliptic Curve Fourier Coefficients in Families* (with B. Mackall, C. Rapti and K. Winsor), Frobenius Distributions: Lang-Trotter and Sato-Tate Conjectures (David Kohel and Igor Shparlinski, editors), Contemporary Mathematics **663**, AMS, Providence, RI 2016.

7. *Lower-Order Biases Second Moments of Fourier Coefficients in Families of L-Functions* (with Megumi Asada, Ryan Chen, Eva Fourakis, Yujin Kim, Andrew Kwon, Jared Lichtman, Blake Mackall, Eric Winsor, Karl Winsor, Jianing Yang, Kevin Yang), preprint.

8. *Rank and Bias in Families of Hyperelliptic Curves via Nagao's Conjecture* (with Trajan Hammonds, Seoyoung Kim, Benjamin Logsdon), 2019, in preparation.