Continued Fraction Digit Averages and Maclaurin's Inequalities

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Introduction

Plan of the talk

- Classical ergodic theory of continued fractions.
  - ◇ Almost surely geometric mean $\sqrt[n]{a_1 \cdots a_n} \to K_0$.
  - ◇ Almost surely arithmetic mean $(a_1 + \cdots + a_n)/n \to \infty$.
- Symmetric averages and Maclaurin's inequalities.
  - ◇ $S(x, n, k) := \binom{n}{k}^{-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$.
  - ◇ $\mathrm{AM} = S(x, n, 1)^{1/1} \ge S(x, n, 2)^{1/2} \ge \cdots \ge S(x, n, n)^{1/n} = \mathrm{GM}$.
- Results / conjectures on typical / periodic continued fraction averages.
- Elementary proofs of weak results, sketch of stronger results.

To appear in Exp. Math.:http://arxiv.org/abs/1402.0208.

Continued Fractions

- Every real number $\alpha \in (0,1)$ can be expressed as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}} = [a_1, a_2, a_3, \dots], \quad a_i \in \{1, 2, \dots\}.$$

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- The sequence $\{a_i\}_i$ is finite iff $\alpha \in \mathbb{Q}$.

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- $x = \frac{p}{q} \in \mathbb{Q}$ then a_i's the partial quotients of Euclidean Alg.

$$\frac{\mathbf{106}}{\mathbf{333}} = [3, 7, 15] \qquad \begin{aligned} \mathbf{333} &= 3 \cdot \mathbf{106} + 15 \\ 106 &= 7 \cdot 15 + 1 \\ 15 &= 15 \cdot 1 + 0. \end{aligned}$$

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- $\{a_i\}_i$ preperiodic iff α a quadratic irrational;
  ex: $\sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \dots]$.

Gauss Map: Definition

- The Gauss map $T : (0,1] \to (0,1]$, $T(x) = \{\frac{1}{x}\} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ generates the continued fraction digits

$$a_1 \;=\; \lfloor 1/T^0(\alpha) \rfloor, \quad a_{i+1} \;=\; \lfloor 1/T^i(\alpha) \rfloor, \quad \dots$$

  corresponding to the Markov partition

$$(0,1] \;=\; \bigsqcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k} \right].$$

- T preserves the measure $d\mu = \frac{1}{\log 2} \frac{1}{1+x} dx$ and it is mixing.

Gauss Map: Example: $\sqrt{3}-1 = [1,2,1,2,1,2,\dots]$

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$$\begin{aligned} T^1(\sqrt{3}-1) &= \frac{1}{\sqrt{3}-1}-\left\lfloor\frac{1}{\sqrt{3}-1}\right\rfloor = \frac{\sqrt{3}+1}{3-1}-1=\frac{\sqrt{3}-1}{2} \\ a_2 &= \left\lfloor\frac{2}{\sqrt{3}-1}\right\rfloor = 2. \end{aligned}$$

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Statistics of Continued Fraction Digits 1/3

- The digits a_i follow the Gauss-Kuzmin distribution:

$$\lim_{n\to\infty} \mathbb{P}(a_n = k) = \log_2\left(1 + \frac{1}{k(k+2)}\right)$$

(note the expectation is infinite).

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- Pointwise ergodic theorem (applied to f and $\log f$) reads

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty \quad \text{almost surely}$$

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = e^{\int \log f \, d\mu} \quad \text{almost surely.}$$

Statistics of Continued Fraction Digits 2/3

- Geometric mean converges a.s. to Khinchin's constant:

$$\lim_{n\to\infty}(a_1a_2\cdots a_n)^{1/n} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\log_2 k} = K_0 \approx 2.6854.$$

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- Hölder means: For $p < 1$, almost surely

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{1/p} = K_p = \left(\sum_{k=1}^{\infty} -k^p \log_2\left(1-\frac{1}{(k+1)^2}\right)\right)^{1/p}.$$

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- $\lim_{p\to 0} K_p = K_0$.

Statistics of Continued Fraction Digits 3/3

- Khinchin also proved: For $a'_m = a_m$ if $a_m < m(\log m)^{4/3}$ and 0 otherwise:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a'_i}{n \log n} = \frac{1}{\log 2} \quad \text{in measure.}$$

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- Diamond and Vaaler (1986) showed that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i - \max_{1\le i\le n} a_i}{n\log n} = \frac{1}{\log 2} \quad \text{almost surely.}$$

Maclaurin Inequalities

Definitions and Maclaurin's Inequalities

- Both $\frac{1}{n}\sum_{i=1}^{n} x_i$ and $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$ are defined in terms of elementary symmetric polynomials in $x_1, \dots, x_n$.
- Define k^{th} **elementary symmetric mean of** $x_1, \dots, x_n$ by

$$S(x,n,k) := \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

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Maclaurin's Inequalities

For positive $x_1, \dots, x_n$ we have

$$\mathrm{AM} := S(x,n,1)^{1/1} \geq S(x,n,2)^{1/2} \geq \cdots \geq S(x,n,n)^{1/n} =: \mathrm{GM}$$

(and equalities hold iff $x_1 = \cdots = x_n$).

Maclaurin's work

IV. *A ſecond Letter from* Mr. Colin M^c^ Laurin, *Profeſſor of Mathematicks in the Univerſity of* Edinburgh *and F. R. S. to* Martin Folkes, *Eſq;* *concerning the Roots of Equations, with the Demonſtration of other Rules in Algebra; being the Continuation of the Letter publiſhed in the* Philoſophical Tranſactions, N° 394.

Edinburgh, April 19th, 1729.

S I R,

IN the Year 1725, I wrote to you that I had a Method of demonſtrating Sir *Iſaac Newton*'s Rule concerning the impoſſible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities muſt always be poſitive; and ſome time after, I ſent you the firſt Principles of that Method, which were publiſhed in the *Philoſophical Tranſactions* for the Month of *May*, 1726. The

This laſt is the Theorem publiſhed by the learned Mr. *Bernouilli* in the *Acta Lipſiæ* 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Reſpect and Eſteem with which

I am,

S I R,

Your moſt Obedient,

Moſt Humble Servant,

Colin Mac Laurin.

Proof

Standard proof through Newton's inequalities.

Define the k^{th} **elementary symmetric function** by

$$s_k(x) \ = \ \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

and the k^{th} **elementary symmetric mean** by

$$E_k(x) \ = \ s_k(x) \Big/ \binom{n}{k}.$$

Newton's inequality: $E_k(x)^2 \geq E_{k-1}(x) E_{k+1}(x)$.

New proof by Iddo Ben-Ari and Keith Conrad:

http://homepages.uconn.edu/benari/pdf/maclaurinMathMagFinal.pdf.

Sketch of Ben-Ari and Conrad's Proof

Bernoulli's inequality: $t > -1$: $(1+t)^n \geq 1 + nt$ or $1 + \frac{1}{n}x \geq (1+x)^{1/n}$.

Generalized Bernoulli: $x > -1$:

$$1 + \frac{1}{n}x \geq \left(1 + \frac{2}{n}x\right)^{1/2} \geq \left(1 + \frac{3}{n}x\right)^{1/3} \geq \cdots \geq \left(1 + \frac{n}{n}x\right)^{1/n}.$$

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Proof: Equivalent to $\frac{1}{k}\log\left(1 + \frac{k}{n}x\right) \geq \frac{1}{k+1}\log\left(1 + \frac{k+1}{n}x\right)$, which follows by $\log t$ is strictly concave:

$$\lambda = \frac{1}{k+1},\ 1 + \frac{k}{n}x = \lambda \cdot 1 + (1-\lambda)\cdot\left(1 + \frac{k+1}{n}x\right).$$

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Proof of Maclaurin's Inequalities:

Trivial for $n \in \{1, 2\}$, wlog assume $x_1 \leq x_2 \leq \cdots \leq x_n$.

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Set $E_k := s_k(x)/\binom{n}{k}$, $\epsilon_k := E_k(x_1, \dots, x_{n-1})$.

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Have
$E_k(x_1, \dots, x_n) = \left(1 - \frac{k}{n}\right) E_k(x_1, \dots, x_{n-1}) + \frac{k}{n} E_k(x_1, \dots, x_{n-1}) x_n$.

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Proceed by induction in number of variables, use Generalized Bernoulli.

Main Results
(Elementary Techniques)

Symmetric Averages and Maclaurin's Inequalities

- Recall: $S(x,n,k) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$ and $S(x,n,1)^{1/1} \ge S(x,n,2)^{1/2} \ge \cdots \ge S(x,n,n)^{1/n}$.

Symmetric Averages and Maclaurin's Inequalities

- Recall: $S(x,n,k) = \frac{1}{\binom{n}{k}} \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}$ and $S(x,n,1)^{1/1} \ge S(x,n,2)^{1/2} \ge \cdots \ge S(x,n,n)^{1/n}$.

- Khinchin's results: almost surely as $n \to \infty$

$$S(\alpha,1,1)^{1/1} \to \infty \quad \text{and} \quad S(\alpha,n,n)^{1/n} \to K_0.$$

- We study the intermediate means $S(\alpha,n,k)^{1/k}$ as $n \to \infty$ when $k = k(n)$, with

$$S(\alpha,n,k(n))^{1/k(n)} = S(\alpha,n,\lceil k(n)\rceil)^{1/\lceil k(n)\rceil}.$$

Our results on typical continued fraction averages

Recall: $S(\alpha,n,k) = \frac{1}{\binom{n}{k}} \sum_{1\le i_1<\cdots<i_k\le n} a_{i_1}\cdots a_{i_k}$

and $S(\alpha,n,1)^{1/1} \ge S(\alpha,n,2)^{1/2} \ge \cdots \ge S(\alpha,n,n)^{1/n}$.

Theorem 1

Let $f(n) = o(\log\log n)$ as $n\to\infty$. Then, almost surely,

$$\lim_{n\to\infty} S(\alpha,n,f(n))^{1/f(n)} \;=\; \infty.$$

Theorem 2

Let $f(n) = o(n)$ as $n\to\infty$. Then, almost surely,

$$\lim_{n\to\infty} S(\alpha,n,n-f(n))^{1/(n-f(n))} \;=\; K_0.$$

Note: Theorems do not cover the case $f(n) = cn$ for $0 < c < 1$.

Sketch of Proofs of Theorems 1 and 2

Theorem 1: For $f(n) = o(\log\log n)$ as $n \to \infty$:

$$\text{Almost surely} \lim_{n\to\infty} S(\alpha, n, f(n))^{1/f(n)} \ = \ \infty.$$

Uses Niculescu's strengthening of Maclaurin (2000):

$$S(n, tj + (1-t)k) \ \geq \ S(n,j)^t \cdot S(n,k)^{1-t}.$$

Sketch of Proofs of Theorems 1 and 2

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Theorem 2: For $f(n) = o(n)$ as $n \to \infty$:

$$\text{Almost surely } \lim_{n\to\infty} S(\alpha, n, n - f(n))^{1/(n-f(n))} \;=\; K_0.$$

Use (a.s.) $K_0 \leq \limsup_{n\to\infty} S(\alpha, n, cn)^{1/cn} \leq K_0^{1/c} < \infty, 0 < c < 1.$

Proof of Theorem 1: Preliminaries

Lemma

Let X be a sequence of positive real numbers. Suppose $\lim_{n\to\infty} S(X,n,k(n))^{1/k(n)}$ exists. Then, for any $f(n) = o(k(n))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S(X,n,k(n)+f(n))^{1/(k(n)+f(n))} \;=\; \lim_{n\to\infty} S(n,k(n))^{1/k(n)}.$$

Proof: Assume $f(n) \geq 0$ for large enough n, and for display purposes write k and f for $k(n)$ and $f(n)$.

From Newton's inequalities and Maclaurin's inequalities, we get

$$\left(S(X,n,k)^{1/k}\right)^{\frac{k}{k+f}} \;=\; S(X,n,k)^{1/(k+f)} \;\leq\; S(X,n,k+f)^{1/(k+f)} \;\leq\; S(X,n,k)^{1/k}.$$

Proof of Theorem 1: $f(n) = o(\log\log n)$

Each entry of α is at least 1.
Let $f(n) = o(\log\log n)$. Set $t = 1/2$ and $(j, k) = (1, 2f(n) - 1)$, so that $tj + (1-t)k = f(n)$. Niculescu's result yields

$$S(\alpha, n, f(n)) \;\geq\; \sqrt{S(\alpha, n, 1) \cdot S(\alpha, n, 2f(n) - 1)} \;>\; \sqrt{S(\alpha, n, 1)}.$$

Square both sides, raise to the power $1/f(n)$:

$$S(\alpha, n, f(n))^{2/f(n)} \;\geq\; S(\alpha, n, 1)^{1/f(n)}.$$

From Khinchin almost surely if $g(n) = o(\log n)$

$$\lim_{n\to\infty} \frac{S(\alpha, n, 1)}{g(n)} \;=\; \infty.$$

Let $g(n) = \log n / \log\log n$. Taking logs:

$$\log\left(S(\alpha, n, 1)^{1/f(n)}\right) \;>\; \frac{\log g(n)}{f(n)} \;>\; \frac{\log\log n}{2f(n)}.$$

Proof of Theorem 2

Theorem 2: Let $f(n) = o(n)$ as $n \to \infty$. Then, almost surely,

$$\lim_{n\to\infty} S(\alpha, n, n - f(n))^{1/(n-f(n))} \;=\; K_0.$$

Proof: Follows immediately from:
For any constant $0 < c < 1$ and almost all α have

$$K_0 \;\le\; \limsup_{n\to\infty} S(\alpha, n, cn)^{1/cn} \;\le\; K_0^{1/c} \;<\; \infty.$$

To see this, note

$$S(\alpha, n, cn)^{1/cn} \;=\; \left(\prod_{i=1}^{n} a_i(\alpha)^{1/n}\right)^{n/cn} \left(\frac{\sum\limits_{i_1<\cdots<i_{(1-c)n}\le n} 1/(a_{i_1}(\alpha)\cdots a_{i_{(1-c)n}}(\alpha))}{\binom{n}{cn}}\right)^{1/cn}.$$

Limiting Behavior

Recall $S(\alpha, n, k) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} \cdots a_{i_k}$
and $S(\alpha, n, 1)^{1/1} \ge S(\alpha, n, 2)^{1/2} \ge \cdots \ge S(\alpha, n, n)^{1/n}$.

Proposition

For $0 < c < 1$ and for almost every α

$$K_0 \le \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn} \le K_0^{1/c} (K_{-1})^{1-1/c}.$$

Conjecture

Almost surely $F_+^\alpha(c) = F_-^\alpha(c) = F(c)$ for all $0 < c < 1$, with

$$F_+^\alpha(c) = \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn},$$
$$F_-^\alpha(c) = \liminf_{n \to \infty} S(\alpha, n, cn)^{1/cn}.$$

Limiting Behavior

Recall

$$F_{+}^{\alpha}(c) = \limsup_{n\to\infty} S(\alpha,n,cn)^{1/cn}$$
$$F_{-}^{\alpha}(c) = \liminf_{n\to\infty} S(\alpha,n,cn)^{1/cn},$$

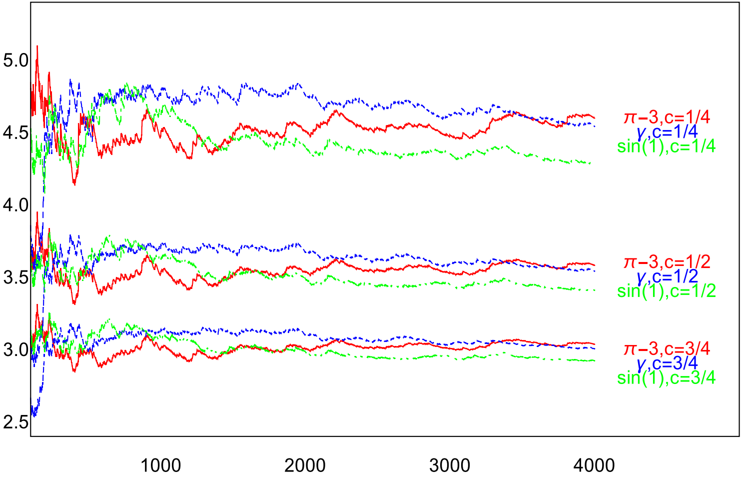
and we conjecture $F_{+}^{\alpha}(c) = F_{-}^{\alpha}(c) = F(c)$ a.s.

Assuming conjecture, can show that the function $c \mapsto F(c)$ is continuous.

Assuming conjecture is false, we can show that for every $0 < c < 1$ the set of limit points of the sequence $\{S(\alpha,n,cn)^{1/cn})\}_{n\in\mathbb{N}}$ is a non-empty interval inside $[K, K^{1/c}]$.

Evidence for Conjecture 1

- $n \mapsto S(\alpha, n, cn)^{1/cn}$ for $c = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and $\alpha = \pi - 3$, γ, $\sin(1)$.



Our results on periodic continued fraction averages 1/2

- For $\alpha = \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots]$,

$$\lim_{n\to\infty} S(\alpha, n, 1)^{1/1} = \frac{3}{2} \neq \infty$$
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- Let us look at $S(\alpha, n, cn)^{1/cn}$ for $c = 1/2$.

$$S(\alpha, n, \lceil \tfrac{n}{2} \rceil) = \begin{cases} S(\alpha, n, \frac{n}{2}) & \text{if } n \equiv 0 \bmod 2; \\ S(\alpha, n, \frac{n+1}{2}) & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

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- We find the limit $\lim_{n\to\infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil}$ in terms of x, y.

Our results on periodic continued fraction averages 2/2

Theorem 3

Let $\alpha = [\overline{x, y}]$. Then $S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil}$ converges as $n \to \infty$ to the $\frac{1}{2}$-Hölder mean of x and y:

$$\lim_{n\to\infty} S(\alpha, n, \lceil \tfrac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left(\frac{x^{1/2} + y^{1/2}}{2} \right)^2 .$$

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Suffices to show for $n \equiv 0 \bmod 2$, say $n = 2k$.

In this case we have that $S(\alpha, 2k, k)^{1/k} \to \left(\frac{x^{1/2}+y^{1/2}}{2} \right)^2$ monotonically as $k \to \infty$.

On the proof of Theorem 3, 1/2

$$\text{Goal}: \alpha = [\overline{x,y}] \Rightarrow \lim_{n\to\infty} S(\alpha, n, \lceil \tfrac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left(\frac{x^{1/2}+y^{1/2}}{2} \right)^2.$$

The proof uses an asymptotic formula for Legendre polynomials P_k (with $t = \frac{x}{y} < 1$ and $u = \frac{1+t}{1-t} > 1$):

$$P_k(u) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j}^2 (u-1)^{k-j}(u+1)^j$$

$$S(\alpha, 2k, k) = \frac{1}{\binom{2k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 x^j y^{k-j} = \frac{y^k}{\binom{2k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 t^j$$

$$= \frac{y^k}{\binom{2k}{k}} (1-t)^k P_k(u).$$

On the proof of Theorem 3, 2/2

$$\text{Goal}: \alpha = [\overline{x, y}] \Rightarrow \lim_{n\to\infty} S(\alpha, n, \lceil \tfrac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left(\frac{x^{1/2} + y^{1/2}}{2}\right)^2.$$

Using the *generalized Laplace-Heine asymptotic formula* for $P_k(u)$ for $u > 1$ and $t = \frac{x}{y} < 1$ and $u = \frac{1+t}{1-t} > 1$ gives

$$\begin{aligned} S(\alpha, 2k, k)^{1/k} &= y(1-t)\left(\frac{P_k(u)}{\binom{2k}{k}}\right)^{1/k} \\ &\longrightarrow y(1-t)\frac{u + \sqrt{u^2 - 1}}{4} = y\left(\frac{1+\sqrt{t}}{2}\right)^2 \\ &= \left(\frac{x^{1/2} + y^{1/2}}{2}\right)^2. \end{aligned}$$

A conjecture on periodic continued fraction averages 1/3

Expect the same result of Theorem 3 to hold for every quadratic irrational α and for every c.

Conjecture 2

For every $\alpha = [\overline{x_1, \dots, x_L}]$ and every $0 \leq c \leq 1$ the limit

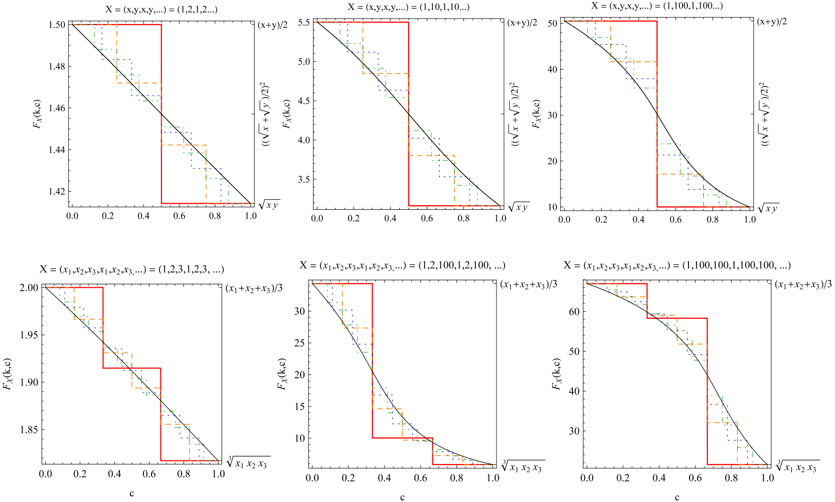
$$\lim_{n\to\infty} S(\alpha, n, \lceil cn \rceil)^{1/\lceil cn \rceil} =: F(\alpha, c)$$

exists and it is a continuous function of c.

Notice $c \mapsto F(\alpha, c)$ is automatically decreasing by Maclaurin's inequalities.

A conjecture on periodic continued fraction averages 2/3

Conjecture 2 for period 2 and period 3, $0 \leq c \leq 1$.



Main Results
(Sketch of More Technical Arguments)

Explicit Formula for $F(c)$

Result of Halász and Székely yields conjecture and $F(c)$.

Theorem 4

If $\lim_{n\to\infty} \frac{k}{n} = c \in (0,1]$, then for almost all $\alpha \in [0,1]$

$$\lim_{n\to\infty} S(\alpha, n, k)^{1/k} \;=:\; F(c)$$

exists, and $F(c)$ is continuous and given explicitly by

$$c(1-c)^{\frac{1-c}{c}} \exp\left\{\frac{1}{c}\left((c-1)\log r_c - \sum_{k=1}^{\infty} \log\left(r_c + k\right) \log_2\left(1 - \frac{1}{(k+1)^2}\right)\right)\right\},$$

where r_c is the unique nonnegative solution of the equation

$$\sum_{k=1}^{\infty} \frac{r}{r+k} \log_2\left(1 - \frac{1}{(k+1)^2}\right) \;=\; c - 1.$$

Proof: Work of Halász and Székely

- Halász and Székely calculate asymptotic properties of iidrv $\xi_1, \dots, \xi_n$ when
  - ⋄ $c = \lim_{n\to\infty} k/n \in [0,1]$.
  - ⋄ ξ_j non-negative.
  - ⋄ $\mathbb{E}[\log \xi_j] < \infty$ if $c = 1$.
  - ⋄ $\mathbb{E}[\log(1+\xi_j) < \infty$ if $0 < c < 1$.
  - ⋄ $\mathbb{E}[\xi_j] < \infty$ if $c = 0$.
- Prove $\lim_{n\to\infty} \sqrt[k]{S(\xi,n,k)/\binom{n}{k}}$ exists with probability 1 and determine it.

Proof: Work of Halász and Székely

Random variables $a_i(\alpha)$ not independent, but Halász and Székely only use independence to conclude sum of the form

$$\frac{1}{n}\sum_{k=1}^{n} f(T^k(\alpha))$$

(where T is the Gauss map and f is some function integrable with respect to the Gauss measure) converges a.e. to $\mathbb{E}f$ as $n \to \infty$.

Arrive at the same conclusion by appealing to the pointwise ergodic theorem.

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