Continued Fraction Digit Averages and Maclaurin’s Inequalities

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Introduction
Plan of the talk

- Classical ergodic theory of continued fractions.
  - Almost surely geometric mean $\sqrt[n]{a_1 \cdots a_n} \to K_0$.
  - Almost surely arithmetic mean $(a_1 + \cdots + a_n)/n \to \infty$.

- Symmetric averages and Maclaurin’s inequalities.
  - $S(x, n, k) := \binom{n}{k}^{-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$.
  - $AM = S(x, n, 1)^{1/1} \geq S(x, n, 2)^{1/2} \geq \cdots \geq S(x, n, n)^{1/n} = GM$.

- Results / conjectures on typical / periodic continued fraction averages.

- Elementary proofs of weak results, sketch of stronger results.

Continued Fractions

Every real number \( \alpha \in (0, 1) \) can be expressed as

\[
x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_1, a_2, a_3, \ldots], \quad a_i \in \{1, 2, \ldots\}.
\]
Every real number $\alpha \in (0, 1)$ can be expressed as

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_1, a_2, a_3, \ldots], \ a_i \in \{1, 2, \ldots\}.$$ 

The sequence $\{a_i\}_i$ is finite iff $\alpha \in \mathbb{Q}$. 

Continued Fractions
Continued Fractions

- Every real number $\alpha \in (0, 1)$ can be expressed as
  \[ x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \ldots], \quad a_i \in \{1, 2, \ldots\}. \]

- If $x = \frac{p}{q} \in \mathbb{Q}$ then $a_i$’s the partial quotients of Euclidean Alg.
  
  \[
  \frac{333}{106} = [3, 7, 15] \\
  333 = 3 \cdot 106 + 15 \\
  106 = 7 \cdot 15 + 1 \\
  15 = 15 \cdot 1 + 0.
  \]
Continued Fractions

- Every real number $\alpha \in (0, 1)$ can be expressed as
  \[
x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_1, a_2, a_3, \ldots], \quad a_i \in \{1, 2, \ldots\}.
  \]

- \(\{a_i\}_i\) preperiodic iff $\alpha$ a quadratic irrational;
  ex: $\sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots]$.  

\[
\begin{array}{c}
\text{Notes} \\
\text{References} \\
\text{Technical Results} \\
\text{Main Results (Elementary)} \\
\text{Maclaurin Inequalities} \\
\text{Intro}
\end{array}
\]
Gauss Map: Definition

The Gauss map $T : (0, 1] \to (0, 1]$, $T(x) = \{ \frac{1}{x} \} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$
generates the continued fraction digits

$$a_1 = \lfloor \frac{1}{T^0(\alpha)} \rfloor, \quad a_{i+1} = \lfloor \frac{1}{T^i(\alpha)} \rfloor, \quad \ldots$$
corresponding to the Markov partition

$$(0, 1] = \bigcup_{k=1}^{\infty} \left( \frac{1}{k+1}, \frac{1}{k} \right).$$

$T$ preserves the measure $d\mu = \frac{1}{\log 2} \frac{1}{1+x} dx$ and it is mixing.
Gauss Map: Example: \( \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots] \)

\[
T : (0, 1] \to (0, 1], \ T(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \ 	ext{generates digits}
\]

\[
a_1 = \lfloor 1/T^0(\alpha) \rfloor, \quad a_{i+1} = \lfloor 1/T^i(\alpha) \rfloor, \quad \ldots
\]

\[
\alpha = \sqrt{3} - 1 = [1, 2, 1, 2, \ldots]: \text{Note } a_1 = \lfloor \frac{1}{\sqrt{3} - 1} \rfloor = 1
\]
Gauss Map: Example: $\sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots]$

$T : (0, 1] \rightarrow (0, 1], \ T(x) = \{\frac{1}{x}\} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ generates digits

$a_1 = \lfloor 1/T^0(\alpha) \rfloor, \ a_{i+1} = \lfloor 1/T^i(\alpha) \rfloor, \ \ldots$

$\alpha = \sqrt{3} - 1 = [1, 2, 1, 2, \ldots]$: Note $a_1 = \lfloor \frac{1}{\sqrt{3}-1} \rfloor = 1$ and

$T^1(\sqrt{3} - 1) = \frac{1}{\sqrt{3} - 1} - \lfloor \frac{1}{\sqrt{3} - 1} \rfloor = \frac{\sqrt{3} + 1}{3 - 1} - 1 = \frac{\sqrt{3} - 1}{2}$

$a_2 = \lfloor \frac{2}{\sqrt{3} - 1} \rfloor = 2.$
Gauss Map: Example: \( \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots] \)

\( T : (0, 1] \rightarrow (0, 1], \ T(x) = \{\frac{1}{x}\} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor \) generates digits

\[ a_1 = \lfloor 1/T^0(\alpha) \rfloor, \quad a_{i+1} = \lfloor 1/T^i(\alpha) \rfloor, \quad \ldots \]

\( \alpha = \sqrt{3} - 1 = [1, 2, 1, 2, \ldots] \): Note \( a_1 = \lfloor \frac{1}{\sqrt{3}-1} \rfloor = 1 \) and

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\[ a_2 = \lfloor \frac{2}{\sqrt{3} - 1} \rfloor = 2. \]

\[ T^2(\sqrt{3} - 1) = \frac{2}{\sqrt{3} - 1} - \lfloor \frac{2}{\sqrt{3} - 1} \rfloor = \frac{2\sqrt{3} + 2}{2} - 2 = \sqrt{3} - 1 \]

\[ a_3 = \lfloor \frac{1}{\sqrt{3} - 1} \rfloor = 1. \]
Gauss Map: Example: \( \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots] \)

\( T : (0, 1] \rightarrow (0, 1], \ T(x) = \{ \frac{1}{x} \} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \) generates digits

\[
\begin{align*}
a_1 &= \left\lfloor \frac{1}{T^0(\alpha)} \right\rfloor, \quad a_{i+1} = \left\lfloor \frac{1}{T^i(\alpha)} \right\rfloor, \quad \ldots \\
\alpha &= \sqrt{3} - 1 = [1, 2, 1, 2, \ldots]: \text{Note } a_1 = \left\lfloor \frac{1}{\sqrt{3} - 1} \right\rfloor = 1 \text{ and} \\
T^1(\sqrt{3} - 1) &= \frac{1}{\sqrt{3} - 1} - \left\lfloor \frac{1}{\sqrt{3} - 1} \right\rfloor = \frac{\sqrt{3} + 1}{3 - 1} - 1 = \frac{\sqrt{3} - 1}{2} \\
a_2 &= \left\lfloor \frac{2}{\sqrt{3} - 1} \right\rfloor = 2. \\
T^2(\sqrt{3} - 1) &= \frac{2}{\sqrt{3} - 1} - \left\lfloor \frac{2}{\sqrt{3} - 1} \right\rfloor = \frac{2\sqrt{3} + 2}{2} - 2 = \sqrt{3} - 1 \\
a_3 &= \left\lfloor \frac{1}{\sqrt{3} - 1} \right\rfloor = 1.
\end{align*}
\]
The digits $a_i$ follow the Gauss-Kuzmin distribution:

$$\lim_{n \to \infty} P(a_n = k) = \log_2 \left( 1 + \frac{1}{k(k + 2)} \right)$$

(note the expectation is infinite).
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The function $x \mapsto f(x) = \lfloor 1/T(x) \rfloor$ on $(0, 1]$ is not integrable wrt $\mu$. However, $\log f \in L^1(\mu)$. 
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Pointwise ergodic theorem (applied to $f$ and $\log f$) reads

$$
\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty \quad \text{almost surely}
$$

$$
\lim_{n \to \infty} (a_1 a_2 \cdots a_n)^{1/n} = e^{\int \log f \, d\mu} \quad \text{almost surely}.
$$
Geometric mean converges a.s. to Khinchin’s constant:

\[
\lim_{n \to \infty} \left( a_1 a_2 \cdots a_n \right)^{1/n} = \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k(k+2)} \right)^{\log_2 k} = K_0 \approx 2.6854.
\]
Statistics of Continued Fraction Digits 2/3

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- Hölder means: For \( p < 1 \), almost surely

\[
\lim_{n \to \infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p\right)^{1/p} = K_p = \left(\sum_{k=1}^{\infty} -k^p \log_2 \left(1 - \frac{1}{(k+1)^2}\right)\right)^{1/p}.
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Statistics of Continued Fraction Digits 2/3

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\]

- Example: The harmonic mean \( K_{-1} = 1.74540566 \ldots \).
Statistics of Continued Fraction Digits 2/3

- Geometric mean converges a.s. to Khinchin’s constant:

$$\lim_{n \to \infty} (a_1 a_2 \cdots a_n)^{1/n} = \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k(k+2)} \right)^{\log_2 k} = K_0 \approx 2.6854.$$  

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- Example: The harmonic mean $K_{-1} = 1.74540566 \ldots$

- $\lim_{p \to 0} K_p = K_0.$
Khinchin also proved: For $a'_m = a_m$ if $a_m < m(\log m)^{4/3}$ and 0 otherwise:

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} a'_i}{n \log n} = \frac{1}{\log 2} \quad \text{in measure.}$$
Khinchin also proved: For $a_m' = a_m$ if $a_m < m(\log m)^{4/3}$ and 0 otherwise:

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} a_i'}{n \log n} = \frac{1}{\log 2} \quad \text{in measure.}$$

Diamond and Vaaler (1986) showed that

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} a_i - \max_{1 \leq i \leq n} a_i}{n \log n} = \frac{1}{\log 2} \quad \text{almost surely.}$$
Maclaurin Inequalities
Definitions and Maclaurin’s Inequalities

- Both $\frac{1}{n} \sum_{i=1}^{n} x_i$ and $\left( \prod_{i=1}^{n} x_i \right)^{1/n}$ are defined in terms of elementary symmetric polynomials in $x_1, \ldots, x_n$.

- Define the $k^{th}$ elementary symmetric mean of $x_1, \ldots, x_n$ by

$$S(x, n, k) := \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$
Definitions and Maclaurin’s Inequalities

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Maclaurin’s Inequalities

For positive $x_1, \ldots, x_n$ we have

$$AM := S(x, n, 1)^{1/1} \geq S(x, n, 2)^{1/2} \geq \cdots \geq S(x, n, n)^{1/n} =: GM$$

(and equalities hold iff $x_1 = \cdots = x_n$).
Maclaurin’s work

IV. A second Letter from Mr. Colin Mac Laurin, Professor of Mathematicks in the University of Edinburgh and F. R. S. to Martin Folkes, Esq; concerning the Roots of Equations, with the Demonstration of other Rules in Algebra, being the Continuation of the Letter published in the Philosophical Transactions, No. 394.

Edinburgh, April 19th, 1729.

S I R,

In the Year 1725, I wrote to you that I had a Method of demonstrating Sir Isaac Newton’s Rule concerning the Imposible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities must always be positive; and some time after, I sent you the first Principles of that Method, which were published in the Philosophical Transactions for the Month of May, 1726. The

This last is the Theorem published by the learned Mr. Bernouilli in the Acta Lipsia 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Respect and Esteem with which

I am,

S I R,

Your most Obedient,

Most Humble Servant,

Colin Mac Laurin.
Proof

Standard proof through Newton’s inequalities.

Define the \( k^{\text{th}} \) elementary symmetric function by

\[
S_k(x) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k},
\]

and the \( k^{\text{th}} \) elementary symmetric mean by

\[
E_k(x) = \frac{S_k(x)}{\binom{n}{k}}.
\]

Newton’s inequality: \( E_k(x)^2 \geq E_{k-1}(x)E_{k+1}(x) \).

New proof by Iddo Ben-Ari and Keith Conrad:

Sketch of Ben-Ari and Conrad’s Proof

Bernoulli’s inequality: \( t > -1 \): \((1 + t)^n \geq 1 + nt\) or 
\[ 1 + \frac{1}{n}x \geq (1 + x)^{1/n}. \]

Generalized Bernoulli: \( x > -1 \):

\[ 1 + \frac{1}{n}x \geq \left(1 + \frac{2}{n}x\right)^{1/2} \geq \left(1 + \frac{3}{n}x\right)^{1/3} \geq \cdots \geq \left(1 + \frac{n}{n}x\right)^{1/n}. \]
Sketch of Ben-Ari and Conrad’s Proof

Bernoulli’s inequality: $t > -1$: $(1 + t)^n \geq 1 + nt$ or $1 + \frac{1}{n}x \geq (1 + x)^{1/n}$.

Generalized Bernoulli: $x > -1$:

$$1 + \frac{1}{n}x \geq \left(1 + \frac{2}{n}x\right)^{1/2} \geq \left(1 + \frac{3}{n}x\right)^{1/3} \geq \cdots \geq \left(1 + \frac{n}{n}x\right)^{1/n}.$$ 

**Proof:** Equivalent to $\frac{1}{k} \log \left(1 + \frac{k}{n}x\right) \geq \frac{1}{k+1} \log \left(1 + \frac{k+1}{n}x\right)$, which follows by $\log t$ is strictly concave:

$$\lambda = \frac{1}{k+1}, \quad 1 + \frac{k}{n}x = \lambda \cdot 1 + \left(1 - \lambda\right) \cdot \left(1 + \frac{k+1}{n}x\right).$$
Sketch of Ben-Ari and Conrad’s Proof

Proof of Maclaurin’s Inequalities:

Trivial for $n \in \{1, 2\}$, wlog assume $x_1 \leq x_2 \leq \cdots \leq x_n$. 
Sketch of Ben-Ari and Conrad’s Proof

Proof of Maclaurin’s Inequalities:

Trivial for \( n \in \{1, 2\} \), wlog assume \( x_1 \leq x_2 \leq \cdots \leq x_n \).

Set \( E_k := s_k(x)/(\binom{n}{k}), \epsilon_k := E_k(x_1, \ldots, x_{n-1}) \).
Sketch of Ben-Ari and Conrad’s Proof

Proof of Maclaurin’s Inequalities:

Trivial for \( n \in \{1, 2\} \), wlog assume \( x_1 \leq x_2 \leq \cdots \leq x_n \).

Set \( E_k := s_k(x)/(n\choose k) \), \( \epsilon_k := E_k(x_1, \ldots, x_{n-1}) \).

Have
\[
E_k(x_1, \ldots, x_n) = \left(1 - \frac{k}{n}\right) E_k(x_1, \ldots, x_{n-1}) + \frac{k}{n} E_k(x_1, \ldots, x_{n-1}) x_n.
\]
Proof of Maclaurin’s Inequalities:

Trivial for \( n \in \{1, 2\} \), wlog assume \( x_1 \leq x_2 \leq \cdots \leq x_n \).

Set \( E_k := s_k(x)/\binom{n}{k} \), \( \epsilon_k := E_k(x_1, \ldots, x_{n-1}) \).

Have
\[
E_k(x_1, \ldots, x_n) = (1 - \frac{k}{n}) E_k(x_1, \ldots, x_{n-1}) + \frac{k}{n} E_k(x_1, \ldots, x_{n-1}) x_n.
\]

Proceed by induction in number of variables, use Generalized Bernoulli.
Main Results
(Elementary Techniques)
Symmetric Averages and Maclaurin’s Inequalities

Recall: \( S(x, n, k) = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1} \cdots x_{i_k} \)

and \( S(x, n, 1)^{1/1} \geq S(x, n, 2)^{1/2} \geq \cdots \geq S(x, n, n)^{1/n} \).
Symmetric Averages and Maclaurin’s Inequalities

Recall: \( S(x, n, k) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \)
and \( S(x, n, 1)^{1/1} \geq S(x, n, 2)^{1/2} \geq \cdots \geq S(x, n, n)^{1/n} \).

Khinchin’s results: almost surely as \( n \to \infty \)

\[ S(\alpha, 1, 1)^{1/1} \to \infty \quad \text{and} \quad S(\alpha, n, n)^{1/n} \to K_0. \]

We study the intermediate means \( S(\alpha, n, k)^{1/k} \) as \( n \to \infty \)
when \( k = k(n) \), with

\[ S(\alpha, n, k(n))^{1/k(n)} = S(\alpha, n, \lceil k(n) \rceil)^{1/\lceil k(n) \rceil}. \]
Our results on typical continued fraction averages

Recall: \( S(\alpha, n, k) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k} \)

and \( S(\alpha, n, 1)^{1/1} \geq S(\alpha, n, 2)^{1/2} \geq \cdots \geq S(\alpha, n, n)^{1/n} \).

**Theorem 1**

Let \( f(n) = o(\log \log n) \) as \( n \to \infty \). Then, almost surely,

\[
\lim_{n \to \infty} S(\alpha, n, f(n))^{1/f(n)} = \infty.
\]

**Theorem 2**

Let \( f(n) = o(n) \) as \( n \to \infty \). Then, almost surely,

\[
\lim_{n \to \infty} S(\alpha, n, n - f(n))^{1/(n - f(n))} = K_0.
\]

Note: Theorems do not cover the case \( f(n) = cn \) for \( 0 < c < 1 \).
Theorem 1: For $f(n) = o(\log \log n)$ as $n \to \infty$:

Almost surely $\lim_{n \to \infty} S(\alpha, n, f(n))^{1/f(n)} = \infty$.

Uses Niculescu’s strengthening of Maclaurin (2000):

$S(n, tj + (1 - t)k) \geq S(n, j)^t \cdot S(n, k)^{1-t}$. 
Sketch of Proofs of Theorems 1 and 2

**Theorem 1:** For \( f(n) = o(\log \log n) \) as \( n \to \infty \):

Almost surely \( \lim_{n \to \infty} S(\alpha, n, f(n))^{1/f(n)} = \infty. \)

Uses Niculescu’s strengthening of Maclaurin (2000):

\[
S(n, tj + (1 - t)k) \geq S(n, j)^t \cdot S(n, k)^{1-t}.
\]

**Theorem 2:** For \( f(n) = o(n) \) as \( n \to \infty \):

Almost surely \( \lim_{n \to \infty} S(\alpha, n, n - f(n))^{1/(n - f(n))} = K_0. \)

Use (a.s.) \( K_0 \leq \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn} \leq K_0^{1/c} < \infty, 0 < c < 1. \)
Proof of Theorem 1: Preliminaries

Lemma

Let $X$ be a sequence of positive real numbers. Suppose $\lim_{n \to \infty} S(X, n, k(n))^{1/k(n)}$ exists. Then, for any $f(n) = o(k(n))$ as $n \to \infty$, we have

$$\lim_{n \to \infty} S(X, n, k(n) + f(n))^{1/(k(n)+f(n))} = \lim_{n \to \infty} S(n, k(n))^{1/k(n)}.$$ 

Proof: Assume $f(n) \geq 0$ for large enough $n$, and for display purposes write $k$ and $f$ for $k(n)$ and $f(n)$.

From Newton’s inequalities and MacLaurin’s inequalities, we get

$$\left( S(X, n, k)^{1/k} \right)^{k/(k+f)} = S(X, n, k)^{1/(k+f)} \leq S(X, n, k+f)^{1/(k+f)} \leq S(X, n, k)^{1/k}.$$
Proof of Theorem 1: \( f(n) = o(\log \log n) \)

Each entry of \( \alpha \) is at least 1.
Let \( f(n) = o(\log \log n) \). Set \( t = 1/2 \) and \((j, k) = (1, 2f(n) - 1)\), so that \( tj + (1 - t)k = f(n) \). Niculescu’s result yields

\[
S(\alpha, n, f(n)) \geq \sqrt{S(\alpha, n, 1) \cdot S(\alpha, n, 2f(n) - 1)} > \sqrt{S(\alpha, n, 1)}.
\]

Square both sides, raise to the power \( 1/f(n) \):

\[
S(\alpha, n, f(n))^{2/f(n)} \geq S(\alpha, n, 1)^{1/f(n)}.
\]

From Khinchin almost surely if \( g(n) = o(\log n) \)

\[
\lim_{n \to \infty} \frac{S(\alpha, n, 1)}{g(n)} = \infty.
\]

Let \( g(n) = \log n/\log \log n \). Taking logs:

\[
\log \left( S(\alpha, n, 1)^{1/f(n)} \right) > \frac{\log g(n)}{f(n)} > \frac{\log \log n}{2f(n)}.
\]
Proof of Theorem 2

**Theorem 2:** Let $f(n) = o(n)$ as $n \to \infty$. Then, almost surely,

$$\lim_{n \to \infty} S(\alpha, n, n - f(n))^{1/(n-f(n))} = K_0.$$

**Proof:** Follows immediately from:

For any constant $0 < c < 1$ and almost all $\alpha$ have

$$K_0 \leq \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn} \leq K_0^{1/c} < \infty.$$

To see this, note

$$S(\alpha, n, cn)^{1/cn} = \left(\prod_{i=1}^{n} a_i(\alpha)^{1/n}\right)^{n/cn} \left(\sum_{i_1 < \cdots < i_{(1-c)n} \leq n} 1/(a_{i_1}(\alpha) \cdots a_{i_{(1-c)n}}(\alpha))\right)^{1/cn}.$$
Limiting Behavior

Recall $S(\alpha, n, k) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k}$

and $S(\alpha, n, 1)^{1/1} \geq S(\alpha, n, 2)^{1/2} \geq \cdots \geq S(\alpha, n, n)^{1/n}$.

**Proposition**

For $0 < c < 1$ and for almost every $\alpha$

$$K_0 \leq \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn} \leq K_0^{1/c}(K_{-1})^{1-1/c}.$$

**Conjecture**

Almost surely $F_{+}^{\alpha}(c) = F_{-}^{\alpha}(c) = F(c)$ for all $0 < c < 1$, with

$$F_{+}^{\alpha}(c) = \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn},$$

$$F_{-}^{\alpha}(c) = \liminf_{n \to \infty} S(\alpha, n, cn)^{1/cn}.$$
Limiting Behavior

Recall

\[ F_+^\alpha(c) = \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn} \]
\[ F_-^\alpha(c) = \liminf_{n \to \infty} S(\alpha, n, cn)^{1/cn}, \]

and we conjecture \( F_+^\alpha(c) = F_-^\alpha(c) = F(c) \) a.s.

Assuming conjecture, can show that the function \( c \mapsto F(c) \) is continuous.

Assuming conjecture is false, we can show that for every \( 0 < c < 1 \) the set of limit points of the sequence \( \{ S(\alpha, n, cn)^{1/cn} \}_{n \in \mathbb{N}} \) is a non-empty interval inside \([K, K^{1/c}]\).
Evidence for Conjecture 1

\[ n \mapsto S(\alpha, n, cn)^{1/cn} \text{ for } c = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \text{ and } \alpha = \pi - 3, \gamma, \sin(1). \]
Our results on periodic continued fraction averages 1/2

For \( \alpha = \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots] \),

\[
\lim_{n \to \infty} S(\alpha, n, 1)^{1/1} = \frac{3}{2} \neq \infty \\
\lim_{n \to \infty} S(\alpha, n, n)^{1/n} = \sqrt{2} \neq K_0
\]
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- What can we say about \( \lim_{n \to \infty} S(\alpha, n, cn)^{1/cn} \)?
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- What can we say about \( \lim_{n \to \infty} S(\alpha, n, cn)^{1/cn} \)?
- Consider the quadratic irrational \( \alpha = [x, y, x, y, x, y, \ldots] \).
Our results on periodic continued fraction averages 1/2

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- What can we say about $\lim_{n \to \infty} S(\alpha, n, cn)^{1/cn}$?
- Consider the quadratic irrational $\alpha = [x, y, x, y, x, y, \ldots]$. 
- Let us look at $S(\alpha, n, cn)^{1/cn}$ for $c = 1/2$.

$$S(\alpha, n, \lceil \frac{n}{2} \rceil) = \begin{cases} S(\alpha, n, \frac{n}{2}) & \text{if } n \equiv 0 \mod 2; \\ S(\alpha, n, \frac{n+1}{2}) & \text{if } n \equiv 1 \mod 2. \end{cases}$$
Our results on periodic continued fraction averages 1/2

- For $\alpha = \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots]$,
  
  \[
  \lim_{n \to \infty} S(\alpha, n, 1)^{1/1} = \frac{3}{2} \neq \infty
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S(\alpha, n, \lceil \frac{n}{2} \rceil) = \begin{cases} 
S(\alpha, n, \frac{n}{2}) & \text{if } n \equiv 0 \mod 2; \\
S(\alpha, n, \frac{n+1}{2}) & \text{if } n \equiv 1 \mod 2.
\end{cases}
\]

- We find the limit $\lim_{n \to \infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil}$ in terms of $x, y$. 
Our results on periodic continued fraction averages 2/2

**Theorem 3**

Let \( \alpha = [x, y] \). Then \( S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} \) converges as \( n \to \infty \) to the \( \frac{1}{2} \)-Hölder mean of \( x \) and \( y \):

\[
\lim_{n \to \infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left( \frac{x^{1/2} + y^{1/2}}{2} \right)^2.
\]
Theorem 3

Let $\alpha = [x, y]$. Then $S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil}$ converges as $n \to \infty$ to the $\frac{1}{2}$-Hölder mean of $x$ and $y$:

$$\lim_{n \to \infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left( \frac{x^{1/2} + y^{1/2}}{2} \right)^2.$$

Suffices to show for $n \equiv 0 \mod 2$, say $n = 2k$. In this case we have that $S(\alpha, 2k, k)^{1/k} \to \left( \frac{x^{1/2} + y^{1/2}}{2} \right)^2$ monotonically as $k \to \infty$. 
On the proof of Theorem 3, 1/2

Goal: \( \alpha = [x, y] \Rightarrow \lim_{n \to \infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left( \frac{x^{1/2} + y^{1/2}}{2} \right)^2. \)

The proof uses an asymptotic formula for Legendre polynomials \( P_k \) (with \( t = \frac{x}{y} < 1 \) and \( u = \frac{1+t}{1-t} > 1 \)):

\[
P_k(u) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j}^2 (u-1)^{k-j}(u+1)^j
\]

\[
S(\alpha, 2k, k) = \frac{1}{\binom{2k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 x^j y^{k-j} = \frac{y^k}{\binom{2k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 t^j
\]

\[
= \frac{y^k}{\binom{2k}{k}} (1 - t)^k P_k(u).
\]
On the proof of Theorem 3, 2/2

Goal: \( \alpha = [x, y] \Rightarrow \lim_{n \to \infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left( \frac{x^{1/2} + y^{1/2}}{2} \right)^2. \)

Using the generalized Laplace-Heine asymptotic formula for \( P_k(u) \) for \( u > 1 \) and \( t = \frac{x}{y} < 1 \) and \( u = \frac{1+t}{1-t} > 1 \) gives

\[
S(\alpha, 2k, k)^{1/k} = y(1-t) \left( \frac{P_k(u)}{\binom{2k}{k}} \right)^{1/k}
\]

\[
\rightarrow y(1-t) \frac{u + \sqrt{u^2 - 1}}{4} = y \left( \frac{1 + \sqrt{t}}{2} \right)^2
\]

\[
= \left( \frac{x^{1/2} + y^{1/2}}{2} \right)^2.
\]
A conjecture on periodic continued fraction averages 1/3

Expect the same result of Theorem 3 to hold for every quadratic irrational $\alpha$ and for every $c$.

**Conjecture 2**

For every $\alpha = [x_1, \ldots, x_L]$ and every $0 \leq c \leq 1$ the limit

$$\lim_{n \to \infty} S(\alpha, n, \lceil cn \rceil)^{1/\lceil cn \rceil} =: F(\alpha, c)$$

exists and it is a continuous function of $c$.

Notice $c \mapsto F(\alpha, c)$ is automatically decreasing by Maclaurin’s inequalities.
A conjecture on periodic continued fraction averages 2/3

Conjecture 2 for period 2 and period 3, $0 \leq c \leq 1$. 
Main Results
(Sketch of More Technical Arguments)
Explicit Formula for $F(c)$

Result of Halász and Székely yields conjecture and $F(c)$.

**Theorem 4**

If $\lim_{n \to \infty} \frac{k}{n} = c \in (0, 1]$, then for almost all $\alpha \in [0, 1]$

$$\lim_{n \to \infty} S(\alpha, n, k)^{1/k} =: F(c)$$

exists, and $F(c)$ is continuous and given explicitly by

$$c(1-c)^{1-c} \exp \left\{ \frac{1}{c} \left( (c-1) \log r_c - \sum_{k=1}^{\infty} \log (r_c + k) \log_2 \left( 1 - \frac{1}{(k+1)^2} \right) \right) \right\},$$

where $r_c$ is the unique nonnegative solution of the equation

$$\sum_{k=1}^{\infty} \frac{r}{r + k} \log_2 \left( 1 - \frac{1}{(k+1)^2} \right) = c - 1.$$
Halász and Székely calculate asymptotic properties of iid random variables $\xi_1, \ldots, \xi_n$ when

- $c = \lim_{n \to \infty} k/n \in [0, 1]$.
- $\xi_j$ non-negative.
- $E[\log \xi_j] < \infty$ if $c = 1$.
- $E[\log(1 + \xi_j)] < \infty$ if $0 < c < 1$.
- $E[\xi_j] < \infty$ if $c = 0$.

Prove $\lim_{n \to \infty} \sqrt[k]{S(\xi, n, k)/\binom{n}{k}}$ exists with probability 1 and determine it.
Proof: Work of Halász and Székely

Random variables $a_i(\alpha)$ not independent, but Halász and Székely only use independence to conclude sum of the form

$$\frac{1}{n} \sum_{k=1}^{n} f(T^k(\alpha))$$

(where $T$ is the Gauss map and $f$ is some function integrable with respect to the Gauss measure) converges a.e. to $\mathbb{E}f$ as $n \to \infty$.

Arrive at the same conclusion by appealing to the pointwise ergodic theorem.
References


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