Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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Summary / Acknowledgements

- Previous results: Zeckendorf and Lekkerkerker.
- New approach: Joint with Carlos Dominguez, Gene Kopp, Murat Kolğlu and Yinghui Wang.
- Thanks: Ed Burger and his SMALL REU students (David Clyde, Cory Colbert, Gea Shin and Nancy

Previous Results

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$F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5 \ldots$.

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Lekkerkerker's Theorem

The average number of non-consecutive Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Main Results

Lemma: Application of Cookie Counting

The 'probability' (ie, percentage of the time) an integer in $[F_n, F_{n+1})$ has exactly $k+1$ non-consecutive Fibonacci summands is $\binom{n-1-k}{k}/F_{n-1}$.

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The above lemma yields Zeckendorf's Theorem, Lekkerkerker's Theorem, and

An Erdos-Kac Type Theorem: SMALL 2010

As $n \to \infty$, the distribution of the number of non-consecutive Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

Properties of Fibonacci Numbers
and needed Combinatorial Results

Binet's Formula

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$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

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Alternate proof via generating functions useful for

Combinatorial Review

The Cookie Problem

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Example: 10 cookies and 5 people:

$$\odot \; \odot \; \otimes \; \otimes \; \odot \; \odot \; \otimes \; \odot \; \odot \; \odot \; \odot \; \odot \; \otimes \; \odot$$

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Reinterpreting the Cookie Problem

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This follows by setting $x_i = y_i + c_i$ with y_i a non-negative integer.

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Must have F_n one of the summands, must not have F_{n-1}.

For each Fibonacci number from F_1 to F_{n-1} we either include or not, cannot have two consecutive, must end with a non-taken number.

Proof of Zeckendorf's Theorem (continued)

Consider all subsets of $k+1$ non-consecutive Fibonaccis from $\{F_1, \dots, F_n\}$ where F_n is taken. Let y_0 be number of Fibonaccis not taken until first one taken, and then y_i $(1 \le i \le k)$ be the number not taken between two taken.

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Example: 2010 = 1597+377+34+2 = $F_{16} + F_{13} + F_8 + F_2$, so $n = 16$, $k+1 = 4$, $y_0 = 1$, $y_1 = 5$, $y_2 = 4$, $y_3 = 2$.

Equivalently: $y_0 + y_1 + \cdots + y_k + k = n - 1$, $y_i \ge 1$ if $i \ge 1$.

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Obtain $\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}$ integers in $[F_n, F_{n+1})$; as all distinct and this many integers in interval, done. $\square$

Lekkerkerker’s Theorem

Preliminaries

$$\mathcal{E}(n) \ := \ \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k}.$$

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Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n-2) \ = \ (n-2)F_{n-3}.$$

Recurrence Relation

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$$\mathcal{E}(n)+\mathcal{E}(n-2) \;=\; (n-2)F_{n-3}.$$

Proof by algebra (details in appendix):

$$\begin{aligned}
\mathcal{E}(n) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k\binom{n-1-k}{k} \\
&= (n-2)\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell\binom{n-3-\ell}{\ell} \\
&= (n-2)F_{n-3} - \mathcal{E}(n-2).
\end{aligned}$$

Solving Recurrence Relation

Formula for $\mathcal{E}(n)$ (i.e., Lekkerkerker's Theorem)

$$\mathcal{E}(n) \ = \ \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).$$

$$\begin{aligned} & \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} \left(\mathcal{E}(n-2\ell) + \mathcal{E}(n-2(\ell+1))\right) \\ = \ & \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} (n-2-2\ell) F_{n-3-2\ell}. \end{aligned}$$

Result follows from Binet's formula, the geometric series formula, and differentiating identities: $\sum_{j=0}^{m} jx^j = x\frac{(m+1)x^m(x-1)-(x^{m+1}-1)}{(x-1)^2}$. Details in appendix.

An Erdos-Kac Type Theorem

Generalizing Lekkerkerker

Theorem (SMALL 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

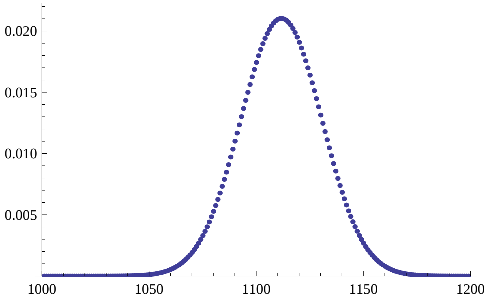

Figure: Number of summands in $[F_{2010}, F_{2011})$

Generalizing Lekkerkerker: Erdos-Kac type result

Theorem (SMALL 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Numerics: At $F_{100,000}$: Ratio of $2m^{\text{th}}$ moment σ_{2m} to $(2m-1)!!\sigma_2^m$ is between .999955 and 1 for $2m \leq 10$.

Sketch of proof: Use Stirling's formula,

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

to approximates binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.

Additional Generalizations

Further Generalization

Generalized Fibonacci Numbers

Let $H_n = c_1 H_{n-1} + \cdots + c_L H_{n-L}$ with $c_1 \geq \cdots \geq c_L \geq 1$. Then every positive integer can be written as a unique sum of the H_i's such that cannot use the recurrence relation to remove any summands.

Key ingredients in proof: generating functions, matching coefficients of polynomials.

Further Generalization (cont)

In 2009 Hannah Alpert proved every positive integer can be written uniquely as a sum and difference of Fibonacci numbers, such that all terms of the same sign are at least 4 apart and those of different sign at least 3. We can show

Signed Representations

The number of positive and negative summands are Gaussianly distributed as $n \to \infty$. They are not independent, and have a negative correlation coefficient.

Conclusion

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- Re-derive Zeckendorf and Lekkerkerker’s results through combinatorics.
- Method yields an Erdos-Kac type result on Gaussian behavior of the number of summands.
- Method applicable to other, related questions.

NOTE: These and similar questions are being studied by the students at the 2010 SMALL REU at Williams College; we expect to be able to provide papers and proofs by the end of the summer.

Appendix:
Details of Computations

Needed Binomial Identity

Binomial identity involving Fibonacci Numbers

Let F_m denote the m^{th} Fibonacci number, with $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$ and so on. Then

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}.$$

Proof by induction: The base case is trivially verified. Assume our claim holds for n and show that it holds for $n+1$. We may extend the sum to $n-1$, as $\binom{n-1-k}{k} = 0$ whenever $k > \lfloor \frac{n-1}{2} \rfloor$. Using the standard identity that

$$\binom{m}{\ell} + \binom{m}{\ell+1} = \binom{m+1}{\ell+1},$$

and the convention that $\binom{m}{\ell} = 0$ if ℓ is a negative integer, we find

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n-k}{k} &= \sum_{k=0}^{n} \left[\binom{n-1-k}{k-1} + \binom{n-1-k}{k} \right] \\
&= \sum_{k=1}^{n} \binom{n-1-k}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} \\
&= \sum_{k=1}^{n} \binom{n-2-(k-1)}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-2} + F_{n-1}
\end{aligned}$$

by the inductive assumption; noting $F_{n-2} + F_{n-1} = F_n$ completes the proof. □

Derivation of Recurrence Relation for $\mathcal{E}(n)$

$$\begin{aligned}
\mathcal{E}(n) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} k \frac{(n-1-k)!}{k!(n-1-2k)!} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-k) \frac{(n-2-k)!}{(k-1)!(n-1-2k)!} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2-(k-1)) \frac{(n-3-(k-1)!}{(k-1)!(n-3-2(k-1))!} \\
&= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (n-2-\ell) \binom{n-3-\ell}{\ell} \\
&= (n-2) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell \binom{n-3-\ell}{\ell} \\
&= (n-2)F_{n-3} - \mathcal{E}(n-2),
\end{aligned}$$

which proves the claim (note we used the binomial identity to replace the sum of binomial coefficients with a Fibonacci number).

Formula for $\mathcal{E}(n)$

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$$\mathcal{E}(n) \ = \ \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).$$

Proof: The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by inputting Binet's formula and differentiating identities. Explicitly, consider

$$\begin{aligned}\sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^\ell\left(\mathcal{E}(n-2\ell)+\mathcal{E}(n-2(\ell+1))\right) \ &= \ \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^\ell(n-2-2\ell)F_{n-3-2\ell} \\ &= \ \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^\ell(n-3-2\ell)F_{n-3-2\ell} + \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^\ell(2\ell)F_{n-3-2\ell} \\ &= \ \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^\ell(n-3-2\ell)F_{n-3-2\ell} + O(F_{n-2});\end{aligned}$$

while we could evaluate the last sum exactly, trivially estimating it suffices to obtain the main term (as we have a sum of every other Fibonacci number, the sum is at most the next Fibonacci number after the largest one in our sum).

Formula for $\mathcal{E}(n)$ (continued)

We now use Binet's formula to convert the sum into a geometric series. Letting $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden mean, we have

$$F_n = \frac{\varphi}{\sqrt{5}} \cdot \varphi^n - \frac{1-\varphi}{\sqrt{5}} \cdot (1-\varphi)^n$$

(our constants are because our counting has $F_1 = 1$, $F_2 = 2$ and so on). As $|1-\varphi| < 1$, the error from dropping the $(1-\varphi)^n$ term is $O(\sum_{\ell \le n} n) = O(n^2) = o(F_{n-2})$, and may thus safely be absorbed in our error term. We thus find

$$\begin{aligned}
\mathcal{E}(n) &= \frac{\varphi}{\sqrt{5}} \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (n-3-2\ell)(-1)^\ell \varphi^{n-3-2\ell} + O(F_{n-2}) \\
&= \frac{\varphi^{n-2}}{\sqrt{5}} \left[(n-3) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-\varphi^{-2})^\ell - 2 \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell(-\varphi^{-2})^\ell \right] + O(F_{n-2}).
\end{aligned}$$

Formula for $\mathcal{E}(n)$ (continued)

We use the geometric series formula to evaluate the first term. We drop the upper boundary term of $(-\varphi^{-1})^{\lfloor \frac{n-3}{2} \rfloor}$, as this term is negligible since $\varphi > 1$. We may also move the 3 from the $n - 3$ into the error term, and are left with

$$\begin{aligned}
\mathcal{E}(n) &= \frac{\varphi^{n-2}}{\sqrt{5}} \left[\frac{n}{1+\varphi^{-2}} - 2 \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell(-\varphi^{-2})^{\ell} \right] + O(F_{n-2}) \\
&= \frac{\varphi^{n-2}}{\sqrt{5}} \left[\frac{n}{1+\varphi^{-2}} - 2S\left(\left\lfloor \frac{n-3}{2} \right\rfloor, -\varphi^{-2} \right) \right] + O(F_{n-2}),
\end{aligned}$$

where

$$\mathcal{S}(m,x) = \sum_{j=0}^{m} jx^j.$$

There is a simple formula for $\mathcal{S}(m,x)$. As

$$\sum_{j=0}^{m} x^j = \frac{x^{m+1}-1}{x-1},$$

applying the operator $x\frac{d}{dx}$ gives

$$\mathcal{S}(m,x) = \sum_{j=0}^{m} jx^j = x\frac{(m+1)x^m(x-1)-(x^{m+1}-1)}{(x-1)^2} = \frac{mx^{m+2}-(m+1)x^{m+1}+x}{(x-1)^2}.$$

Formula for $\mathcal{E}(n)$ (continued)

Taking $x = -\varphi^{-2}$, we see that the contribution from this piece may safely be absorbed into the error term $O(F_{n-2})$, leaving us with

$$\mathcal{E}(n) \;=\; \frac{n\varphi^{n-2}}{\sqrt{5}(1+\varphi^{-2})} + O(F_{n-2}) \;=\; \frac{n\varphi^{n}}{\sqrt{5}(\varphi^{2}+1)} + O(F_{n-2}).$$

Noting that for large n we have $F_{n-1} = \frac{\varphi^n}{\sqrt{5}} + O(1)$, we finally obtain

$$\mathcal{E}(n) \;=\; \frac{nF_{n-1}}{\varphi^{2}+1} + O(F_{n-2}).\Box$$

(Sketch of the) Proof of Gaussianity

The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is $f_n(k) = \binom{n-1-k}{k}/F_{n-1}$. Consider the density for the $n+1$ case. Then we have, by Stirling

$$\begin{aligned} f_{n+1}(k) &= \binom{n-k}{k}\frac{1}{F_n} \\ &= \frac{(n-k)!}{(n-2k)!k!}\frac{1}{F_n} = \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{(n-k+\frac{1}{2})}}{k^{(k+\frac{1}{2})}(n-2k)^{(n-2k+\frac{1}{2})}}\frac{1}{F_n} \end{aligned}$$

plus a lower order correction term.

Also we can write $F_n = \frac{1}{\sqrt{5}}\phi^{n+1} = \frac{\phi}{\sqrt{5}}\phi^n$ for large n, where ϕ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on n.

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}\right)\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right)$$

Define

$$N_n = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}.$$

Thus, write the density function as

$$f_{n+1}(k) = N_n S_n$$

where N_n is the first term that is of order $n^{-1/2}$ and S_n is the second term with exponential dependence on n.

(Sketch of the) Proof of Gaussianity (cont)

Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where μ and σ are the mean and the standard deviation, and depend on n. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.$$

Using the change of variable, we can write N_n as

$$\begin{aligned}
N_n &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\phi}{\sqrt{5}} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\phi} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}}\frac{\sqrt{5}}{\phi} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}}\frac{\sqrt{5}}{\phi}
\end{aligned}$$

where $C = \mu/n \approx 1/(\phi+2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large n, the y term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C}{C(1-2C)}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{(\phi+1)(\phi+2)}{\phi}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{5(\phi+2)}{\phi}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$.

(Sketch of the) Proof of Gaussianity (cont)

For the second term S_n, take the logarithm and once again change variable $k = \mu + x\sigma$,

$$\begin{aligned}
\log(S_n) &= \log\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right) \\
&= -n\log(\phi) + (n-k)\log(n-k) - (k)\log(k) \\
&\quad - (n-2k)\log(n-2k) \\
&= -n\log(\phi) + (n-(\mu+x\sigma))\log(n-(\mu+x\sigma)) \\
&\quad - (\mu+x\sigma)\log(\mu+x\sigma) \\
&\quad - (n-2(\mu+x\sigma))\log(n-2(\mu+x\sigma)) \\
&= -n\log(\phi) \\
&\quad + (n-(\mu+x\sigma))\left(\log(n-\mu)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right) \\
&\quad - (\mu+x\sigma)\left(\log(\mu)+\log\left(1+\frac{x\sigma}{\mu}\right)\right) \\
&\quad - (n-2(\mu+x\sigma))\left(\log(n-2\mu)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right) \\
&= -n\log(\phi) \\
&\quad + (n-(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-1\right)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right) \\
&\quad - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) \\
&\quad - (n-2(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-2\right)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity (cont)

Note that, since $n/\mu = \phi + 2$ for large n, the constant terms vanish. We have $\log(S_n)$

$$
\begin{aligned}
=\; & -n\log(\phi) + (n-k)\log\left(\frac{n}{\mu} - 1\right) - (n-2k)\log\left(\frac{n}{\mu} - 2\right) + (n - (\mu + x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu + x\sigma))\log\left(1 - \frac{x\sigma}{n - 2\mu}\right) \\
=\; & -n\log(\phi) + (n-k)\log(\phi + 1) - (n-2k)\log(\phi) + (n - (\mu + x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu + x\sigma))\log\left(1 - \frac{x\sigma}{n - 2\mu}\right) \\
=\; & n(-\log(\phi) + \log\left(\phi^2\right) - \log(\phi)) + k(\log(\phi^2) + 2\log(\phi)) + (n - (\mu + x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu + x\sigma))\log\left(1 - 2\frac{x\sigma}{n - 2\mu}\right) \\
=\; & (n - (\mu + x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) \\
& - (n - 2(\mu + x\sigma))\log\left(1 - 2\frac{x\sigma}{n - 2\mu}\right).
\end{aligned}
$$

(Sketch of the) Proof of Gaussianity (cont)

Finally, we expand the logarithms and collect powers of $x\sigma/n$. $\log(S_n)$

$$\begin{aligned}
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n-\mu}-\frac{1}{2}\left(\frac{x\sigma}{n-\mu}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\mu}-\frac{1}{2}\left(\frac{x\sigma}{\mu}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-2\frac{x\sigma}{n-2\mu}-\frac{1}{2}\left(2\frac{x\sigma}{n-2\mu}\right)^2+\dots\right)\\
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}-\frac{1}{2}\left(\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\frac{n}{\phi+2}}-\frac{1}{2}\left(\frac{x\sigma}{\frac{n}{\phi+2}}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-\frac{2x\sigma}{n\frac{\phi}{\phi+2}}-\frac{1}{2}\left(\frac{2x\sigma}{n\frac{\phi}{\phi+2}}\right)^2+\dots\right)\\
&= \frac{x\sigma}{n}n\left(-\left(1-\frac{1}{\phi+2}\right)\frac{(\phi+2)}{(\phi+1)}-1+2\left(1-\frac{2}{\phi+2}\right)\frac{\phi+2}{\phi}\right)\\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n\left(-2\frac{\phi+2}{\phi+1}+\frac{\phi+2}{\phi+1}+2(\phi+2)-(\phi+2)+4\frac{\phi+2}{\phi}\right)\\
&\quad +O\left(n\left(x\sigma/n\right)^3\right)
\end{aligned}$$

(Sketch of the) Proof of Gaussianity (cont)

$$
\begin{aligned}
= \quad & \frac{x\sigma}{n} n \left(-\frac{\phi+1}{\phi+2} \frac{\phi+2}{\phi+1} - 1 + 2\frac{\phi}{\phi+2} \frac{\phi+2}{\phi} \right) \\
& -\frac{1}{2} \left(\frac{x\sigma}{n} \right)^2 n(\phi+2) \left(-\frac{1}{\phi+1} + 1 + \frac{4}{\phi} \right) \\
& + O\left(n \left(\frac{x\sigma}{n} \right)^3 \right) \\
= \quad & -\frac{1}{2} \frac{(x\sigma)^2}{n} (\phi+2) \left(\frac{3\phi+4}{\phi(\phi+1)} + 1 \right) + O\left(n \left(\frac{x\sigma}{n} \right)^3 \right) \\
= \quad & -\frac{1}{2} \frac{(x\sigma)^2}{n} (\phi+2) \left(\frac{3\phi+4+2\phi+1}{\phi(\phi+1)} \right) + O\left(n \left(\frac{x\sigma}{n} \right)^3 \right) \\
= \quad & -\frac{1}{2} x^2 \sigma^2 \left(\frac{5(\phi+2)}{\phi n} \right) + O\left(n \left(x\sigma/n \right)^3 \right)
\end{aligned}
$$

(Sketch of the) Proof of Gaussianity (cont)

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi+2)}$$

Also, since $\sigma \sim n^{-1/2}$, $n\left(\frac{x\sigma}{n}\right)^3 \sim n^{-1/2}$. So for large n, the $O\left(n\left(\frac{x\sigma}{n}\right)^3\right)$ term vanishes. Thus we are left with

$$\begin{aligned} \log S_n &= -\frac{1}{2}x^2 \\ S_n &= e^{-\frac{1}{2}x^2} \end{aligned}$$

Hence, as n gets large, the density converges to the normal distribution.

$$\begin{aligned} f_n(k)dk &= N_n S_n dk \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2} \sigma dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \end{aligned}$$

□

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