Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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CANT 2010, CUNY, May 29, 2010

Summary / Acknowledgements

- Previous results: Zeckendorf's and Lekkerkerker's theorems.
- New approach: View as combinatorial problem.
- Thanks: Ed Burger and SMALL REU students.

Special Thanks

Special thanks go to Cameron and Kayla Miller
for playing quietly while key details were worked out
and for suggesting which colors to use and where!
We're ready for yellow now – hope you are too!

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5 \dots$.

Zeckendorf's Theorem

Every positive integer can be written in a unique way as a sum of non-consecutive Fibonacci numbers.

Example: 2010 = 1597+377+34+2 = $F_{16} + F_{13} + F_8 + F_2$.

Lekkerkerker's Theorem

The average number of non-consecutive Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Main Results

Lemma: Application of Cookie Counting

The 'probability' (ie, percentage of the time) an integer in $[F_n, F_{n+1})$ has exactly $k+1$ non-consecutive Fibonacci summands is $\binom{n-1-k}{k}/F_{n-1}$.

The above lemma yields Zeckendorf's Theorem, Lekkerkerker's Theorem, and *will* (hopefully) yield

An Erdos-Kac Type Theorem: SMALL 2010

As $n \to \infty$, the distribution of the number of non-consecutive Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

Properties of Fibonacci Numbers
and needed Combinatorial Results

Binet's Formula

Binet's Formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Proof: $F_{n+1} = F_n + F_{n-1}$.

Guess $F_n = n^r$: $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$.

Roots $r = (1 \pm \sqrt{5})/2$.

General solution: $F_n = c_1 r_1^n + c_2 r_2^n$, solve for c_i's. □

Combinatorial Review

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line.

Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.

Divides the cookies into P sets. $\square$

Example: 10 cookies and 5 people:

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Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with x_i a non-negative integer is $\binom{C+P-1}{P-1}$.

Generalization: If have constraints $x_i \geq c_i$, then number of solutions is $\binom{C-\sum_i c_i+P-1}{P-1}$.

This follows by setting $x_i = y_i + c_i$ with y_i a non-negative integer.

Zeckendorf's Theorem

Proof of Zeckendorf's Theorem

Uniqueness: Same standard argument (induction).

Existence: Consider all sums of non-consecutive Fibonacci numbers equaling an $m \in [F_n, F_{n+1})$; note there are $F_{n+1} - F_n = F_{n-1}$ such integers.

Must have F_n one of the summands, must not have F_{n-1}.

For each Fibonacci number from F_1 to F_{n-1} we either include or not, cannot have two consecutive, must end with a non-taken number.

Proof of Zeckendorf's Theorem (continued)

Consider all subsets of $k+1$ non-consecutive Fibonaccis from $\{F_1, \dots, F_n\}$ where F_n is taken. Let y_0 be number of Fibonaccis not taken until first one taken, and then y_i $(1 \le i \le k)$ be the number not taken between two taken.

Example: 2010 = 1597+377+34+2 = $F_{16} + F_{13} + F_8 + F_2$, so $n = 16$, $k+1 = 4$, $y_0 = 1$, $y_1 = 5$, $y_2 = 4$, $y_3 = 2$.

Equivalently: $y_0 + y_1 + \cdots + y_k + k = n - 1$, $y_i \ge 1$ if $i \ge 1$.

Equivalently: $x_0 + \cdots + x_k + 2k = n - 1$, $x_i \ge 0$. Number of solutions is $\binom{n-1-k}{k}$.

Obtain $\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}$ integers in $[F_n, F_{n+1})$; as all distinct and this many integers in interval, done. $\square$

Lekkerkerker's Theorem

Preliminaries

$$\mathcal{E}(n) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k}.$$

Average number of summands in $[F_n, F_{n+1})$ is

$$\frac{\mathcal{E}(n)}{F_{n-1}} + 1.$$

Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n-2) = (n-2)F_{n-3}.$$

Recurrence Relation

Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n-2) \;=\; (n-2)F_{n-3}.$$

Proof by algebra (details in appendix):

$$\begin{aligned}
\mathcal{E}(n) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k} \\
&= (n-2) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell \binom{n-3-\ell}{\ell} \\
&= (n-2)F_{n-3} - \mathcal{E}(n-2).
\end{aligned}$$

Solving Recurrence Relation

Formula for $\mathcal{E}(n)$ (i.e., Lekkerkerker's Theorem)

$$\mathcal{E}(n) \;=\; \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).$$

$$\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^\ell \left(\mathcal{E}(n-2\ell) + \mathcal{E}(n-2(\ell+1))\right)$$

$$= \quad \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^\ell (n-2-2\ell) F_{n-3-2\ell}.$$

Result follows from Binet's formula, the geometric series formula, and differentiating identities: $\sum_{j=0}^{m} jx^j = x\frac{(m+1)x^m(x-1)-(x^{m+1}-1)}{(x-1)^2}$. Details in appendix.

An Erdos-Kac Type Theorem

Generalizing Lekkerkerker

Theorem (SMALL 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Proof should follow from Markov's Method of Moments, Binet's formula for the Fibonacci numbers, and then differentiating identities to evaluate sums of the form $\sum_j j^m x^m$.

Generalizing Lekkerkerker

Theorem (SMALL 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

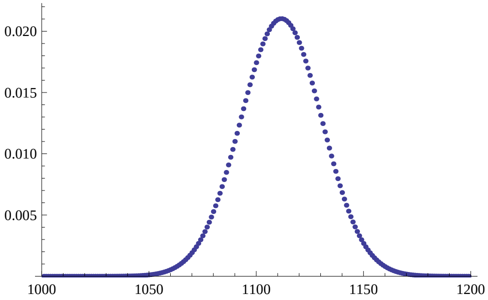

Figure: Number of summands in $[F_{2010}, F_{2011})$

Generalizing Lekkerkerker

Theorem (SMALL 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Numerics: At $F_{100,000}$: Ratio of $2m^{\text{th}}$ moment σ_{2m} to $(2m-1)!!\sigma_2^m$ is between .999955 and 1 for $2m \leq 10$.

Conclusion

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- Re-derive Zeckendorf and Lekkerkerker's results through combinatorics.
- Method should yield an Erdos-Kac type result on Gaussian behavior of the number of summands.
- Method should be applicable to other, related questions.

NOTE: These and similar questions will be studied by the students at the 2010 SMALL REU at Williams College; we expect to be able to provide papers and proofs by the end of the summer.

Appendix:
Details of Computations

Needed Binomial Identity

Binomial identity involving Fibonacci Numbers

Let F_m denote the m^{th} Fibonacci number, with $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$ and so on. Then

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}.$$

Proof by induction: The base case is trivially verified. Assume our claim holds for n and show that it holds for $n+1$. We may extend the sum to $n-1$, as $\binom{n-1-k}{k} = 0$ whenever $k > \lfloor \frac{n-1}{2} \rfloor$. Using the standard identity that

$$\binom{m}{\ell} + \binom{m}{\ell+1} = \binom{m+1}{\ell+1},$$

and the convention that $\binom{m}{\ell} = 0$ if ℓ is a negative integer, we find

$$\begin{aligned} \sum_{k=0}^{n} \binom{n-k}{k} &= \sum_{k=0}^{n} \left[\binom{n-1-k}{k-1} + \binom{n-1-k}{k} \right] \\ &= \sum_{k=1}^{n} \binom{n-1-k}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} \\ &= \sum_{k=1}^{n} \binom{n-2-(k-1)}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-2} + F_{n-1} \end{aligned}$$

by the inductive assumption; noting $F_{n-2} + F_{n-1} = F_n$ completes the proof. □

Derivation of Recurrence Relation for $\mathcal{E}(n)$

$$\begin{aligned}
\mathcal{E}(n) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} k \frac{(n-1-k)!}{k!(n-1-2k)!} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-k) \frac{(n-2-k)!}{(k-1)!(n-1-2k)!} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2-(k-1)) \frac{(n-3-(k-1)!}{(k-1)!(n-3-2(k-1))!} \\
&= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (n-2-\ell) \binom{n-3-\ell}{\ell} \\
&= (n-2) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell \binom{n-3-\ell}{\ell} \\
&= (n-2)F_{n-3} - \mathcal{E}(n-2),
\end{aligned}$$

which proves the claim (note we used the binomial identity to replace the sum of binomial coefficients with a Fibonacci number).

Formula for $\mathcal{E}(n)$

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$$\mathcal{E}(n) \;=\; \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).$$

Proof: The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by inputting Binet's formula and differentiating identities. Explicitly, consider

$$\begin{aligned}\sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^{\ell}\left(\mathcal{E}(n-2\ell)+\mathcal{E}(n-2(\ell+1))\right) &= \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^{\ell}(n-2-2\ell)F_{n-3-2\ell}\\ &= \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^{\ell}(n-3-2\ell)F_{n-3-2\ell}+\sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^{\ell}(2\ell)F_{n-3-2\ell}\\ &= \sum_{\ell=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^{\ell}(n-3-2\ell)F_{n-3-2\ell}+O(F_{n-2});\end{aligned}$$

while we could evaluate the last sum exactly, trivially estimating it suffices to obtain the main term (as we have a sum of every other Fibonacci number, the sum is at most the next Fibonacci number after the largest one in our sum).

Formula for $\mathcal{E}(n)$ (continued)

We now use Binet's formula to convert the sum into a geometric series. Letting $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden mean, we have

$$F_n \;=\; \frac{\varphi}{\sqrt{5}} \cdot \varphi^n - \frac{1-\varphi}{\sqrt{5}} \cdot (1-\varphi)^n$$

(our constants are because our counting has $F_1 = 1$, $F_2 = 2$ and so on). As $|1-\varphi| < 1$, the error from dropping the $(1-\varphi)^n$ term is $O(\sum_{\ell \le n} n) = O(n^2) = o(F_{n-2})$, and may thus safely be absorbed in our error term. We thus find

$$\begin{aligned} \mathcal{E}(n) &= \frac{\varphi}{\sqrt{5}} \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (n-3-2\ell)(-1)^\ell \varphi^{n-3-2\ell} + O(F_{n-2}) \\ &= \frac{\varphi^{n-2}}{\sqrt{5}} \left[(n-3) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-\varphi^{-2})^\ell - 2 \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell(-\varphi^{-2})^\ell \right] + O(F_{n-2}). \end{aligned}$$

Formula for $\mathcal{E}(n)$ (continued)

We use the geometric series formula to evaluate the first term. We drop the upper boundary term of $(-\varphi^{-1})^{\lfloor \frac{n-3}{2} \rfloor}$, as this term is negligible since $\varphi > 1$. We may also move the 3 from the $n-3$ into the error term, and are left with

$$\begin{aligned}
\mathcal{E}(n) &= \frac{\varphi^{n-2}}{\sqrt{5}} \left[\frac{n}{1+\varphi^{-2}} - 2 \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell(-\varphi^{-2})^{\ell} \right] + O(F_{n-2}) \\
&= \frac{\varphi^{n-2}}{\sqrt{5}} \left[\frac{n}{1+\varphi^{-2}} - 2S\left(\left\lfloor \frac{n-3}{2} \right\rfloor, -\varphi^{-2} \right) \right] + O(F_{n-2}),
\end{aligned}$$

where

$$\mathcal{S}(m,x) = \sum_{j=0}^{m} j x^j.$$

There is a simple formula for $\mathcal{S}(m,x)$. As

$$\sum_{j=0}^{m} x^j = \frac{x^{m+1}-1}{x-1},$$

applying the operator $x\frac{d}{dx}$ gives

$$\mathcal{S}(m,x) = \sum_{j=0}^{m} j x^j = x\frac{(m+1)x^m(x-1)-(x^{m+1}-1)}{(x-1)^2} = \frac{mx^{m+2}-(m+1)x^{m+1}+x}{(x-1)^2}.$$

Formula for $\mathcal{E}(n)$ (continued)

Taking $x = -\varphi^{-2}$, we see that the contribution from this piece may safely be absorbed into the error term $O(F_{n-2})$, leaving us with

$$\mathcal{E}(n) \;=\; \frac{n\varphi^{n-2}}{\sqrt{5}(1+\varphi^{-2})} + O(F_{n-2}) \;=\; \frac{n\varphi^{n}}{\sqrt{5}(\varphi^{2}+1)} + O(F_{n-2}).$$

Noting that for large n we have $F_{n-1} = \frac{\varphi^n}{\sqrt{5}} + O(1)$, we finally obtain

$$\mathcal{E}(n) \;=\; \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).\Box$$

A similar calculation should yield the higher moments; this will be done by the students of the 2010 SMALL REU at Williams College.

References

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