Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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Summary / Acknowledgements

- **Previous results:** Zeckendorf’s and Lekkerkerker’s theorems.

- **New approach:** View as combinatorial problem.

- **Thanks:** Ed Burger and SMALL REU students.
Special Thanks

Special thanks go to Cameron and Kayla Miller for playing quietly while key details were worked out and for suggesting which colors to use and where! We’re ready for yellow now – hope you are too!
Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$; $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$.

Zeckendorf’s Theorem
Every positive integer can be written in a unique way as a sum of non-consecutive Fibonacci numbers.

Example: $2010 = 1597 + 377 + 34 + 2 = F_{16} + F_{13} + F_8 + F_2$.

Lekkerkerker’s Theorem
The average number of non-consecutive Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2 + 1} \approx 0.276n$, where $\varphi = \frac{1 + \sqrt{5}}{2}$ is the golden mean.
Lemma: Application of Cookie Counting

The ‘probability’ (ie, percentage of the time) an integer in $[F_n, F_{n+1})$ has exactly $k + 1$ non-consecutive Fibonacci summands is $(\binom{n-1-k}{k})/F_{n-1}$.

The above lemma yields Zeckendorf’s Theorem, Lekkerkerker’s Theorem, and will (hopefully) yield an Erdos-Kac type theorem: SMALL 2010.

As $n \to \infty$, the distribution of the number of non-consecutive Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.
Properties of Fibonacci Numbers and needed Combinatorial Results
Binet’s Formula

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}.
\]

Proof: \( F_{n+1} = F_n + F_{n-1} \).

Guess \( F_n = nr^2 \): \( r^{n+1} = r^n + r^{n-1} \) or \( r^2 = r + 1 \).

Roots \( r = (1 \pm \sqrt{5})/2 \).

General solution: \( F_n = c_1 r_1^n + c_2 r_2^n \), solve for \( c_i \)'s.
The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line.

Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.

Divides the cookies into $P$ sets.

Example: 10 cookies and 5 people:
Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i$ a non-negative integer is $\binom{C + P - 1}{P - 1}$.

Generalization: If have constraints $x_i \geq c_i$, then number of solutions is $\binom{C - \sum_i c_i + P - 1}{P - 1}$.

This follows by setting $x_i = y_i + c_i$ with $y_i$ a non-negative integer.
Zeckendorf’s Theorem
Proof of Zeckendorf’s Theorem

**Uniqueness:** Same standard argument (induction).

**Existence:** Consider all sums of non-consecutive Fibonacci numbers equaling an $m \in [F_n, F_{n+1})$; note there are $F_{n+1} - F_n = F_{n-1}$ such integers.

Must have $F_n$ one of the summands, must not have $F_{n-1}$.

For each Fibonacci number from $F_1$ to $F_{n-1}$ we either include or not, cannot have two consecutive, must end with a non-taken number.
Proof of Zeckendorf’s Theorem (continued)

Consider all subsets of $k + 1$ non-consecutive Fibonaccis from $\{F_1, \ldots, F_n\}$ where $F_n$ is taken. Let $y_0$ be number of Fibonaccis not taken until first one taken, and then $y_i$ $(1 \leq i \leq k)$ be the number not taken between two taken.

Example: $2010 = 1597+377+34+2 = F_{16} + F_{13} + F_8 + F_2$, so $n = 16$, $k + 1 = 4$, $y_0 = 1$, $y_1 = 5$, $y_2 = 4$, $y_3 = 2$.

Equivalently: $y_0 + y_1 + \cdots + y_k + k = n - 1$, $y_i \geq 1$ if $i \geq 1$.

Equivalently: $x_0 + \cdots + x_k + 2k = n - 1$, $x_i \geq 0$. Number of solutions is $\binom{n-1-k}{k}$.

Obtain $\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-k}{k} = F_{n-1}$ integers in $[F_n, F_{n+1})$; as all distinct and this many integers in interval, done. $\square$
Lekkerkerkerker’s Theorem
**Preliminaries**

\[ \mathcal{E}(n) := \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \binom{n - 1 - k}{k}. \]

Average number of summands in \([F_n, F_{n+1})\) is

\[ \frac{\mathcal{E}(n)}{F_{n-1}} + 1. \]

**Recurrence Relation for \( \mathcal{E}(n) \)**

\[ \mathcal{E}(n) + \mathcal{E}(n - 2) = (n - 2)F_{n-3}. \]
Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n-2) = (n-2)F_{n-3}.$$ 

Proof by algebra (details in appendix):

$$\mathcal{E}(n) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k\binom{n-1-k}{k}$$

$$= (n-2) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell \binom{n-3-\ell}{\ell}$$

$$= (n-2)F_{n-3} - \mathcal{E}(n-2).$$
Solving Recurrence Relation

**Formula for $\mathcal{E}(n)$ (i.e., Lekkerkerker’s Theorem)**

$$\mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^2 + 1} + O(F_{n-2}).$$

\[
\sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell \left( \mathcal{E}(n - 2\ell) + \mathcal{E}(n - 2(\ell + 1)) \right)
\]

\[
= \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (n - 2 - 2\ell) F_{n-3-2\ell}.
\]

Result follows from Binet’s formula, the geometric series formula, and differentiating identities: $\sum_{j=0}^{m} jx^j = x \frac{(m+1)x^m(x-1) - (x^{m+1} - 1)}{(x-1)^2}$. Details in appendix.
An Erdos-Kac Type Theorem
Generalizing Lekkerkerkerker

**Theorem (SMALL 2010)**

As \( n \to \infty \), the distribution of the number of summands in Zeckendorf’s Theorem is a Gaussian.

Proof should follow from Markov’s Method of Moments, Binet’s formula for the Fibonacci numbers, and then differentiating identities to evaluate sums of the form \( \sum_j j^m x^m \).
Generalizing Lekkerkerker

**Theorem (SMALL 2010)**

As $n \to \infty$, the distribution of the number of summands in Zeckendorf’s Theorem is a Gaussian.

**Figure:** Number of summands in $[F_{2010}, F_{2011})$
Generalizing Lekkerkerkerker

Theorem (SMALL 2010)
As $n \to \infty$, the distribution of the number of summands in Zeckendorf’s Theorem is a Gaussian.

Numerics: At $F_{100,000}$: Ratio of $2m^{th}$ moment $\sigma_{2m}$ to $(2m - 1)!!\sigma_m^2$ is between .999955 and 1 for $2m \leq 10$. 
Conclusion
Conclusion

- Re-derive Zeckendorf and Lekkerkerker’s results through combinatorics.

- Method should yield an Erdos-Kac type result on Gaussian behavior of the number of summands.

- Method should be applicable to other, related questions.

NOTE: These and similar questions will be studied by the students at the 2010 SMALL REU at Williams College; we expect to be able to provide papers and proofs by the end of the summer.
Appendix:
Details of Computations
Needed Binomial Identity

Binomial identity involving Fibonacci Numbers

Let $F_m$ denote the $m^{th}$ Fibonacci number, with $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$ and so on. Then

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}.$$

Proof by induction: The base case is trivially verified. Assume our claim holds for $n$ and show that it holds for $n + 1$. We may extend the sum to $n - 1$, as $\binom{n-1-k}{k} = 0$ whenever $k > \lfloor \frac{n-1}{2} \rfloor$. Using the standard identity that

$$\binom{m}{\ell} + \binom{m}{\ell + 1} = \binom{m + 1}{\ell + 1},$$

and the convention that $\binom{m}{\ell} = 0$ if $\ell$ is a negative integer, we find

$$\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=0}^{n} \left[ \binom{n-1-k}{k-1} + \binom{n-1-k}{k} \right]$$

$$= \sum_{k=1}^{n} \binom{n-1-k}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k}$$

$$= \sum_{k=1}^{n} \binom{n-2-(k-1)}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-2} + F_{n-1}$$

by the inductive assumption; noting $F_{n-2} + F_{n-1} = F_n$ completes the proof. □
Derivation of Recurrence Relation for $\mathcal{E}(n)$

\[
\mathcal{E}(n) = \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \binom{n-1-k}{k}
\]

\[
= \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \frac{(n-1-k)!}{k!(n-1-2k)!}
\]

\[
= \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (n-1-k) \frac{(n-2-k)!}{(k-1)!(n-1-2k)!}
\]

\[
= \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (n-2-(k-1)) \frac{(n-3-(k-1))}{(k-1)!(n-3-2(k-1))!}
\]

\[
= \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (n-2-\ell) \binom{n-3-\ell}{\ell}
\]

\[
= (n-2) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell \binom{n-3-\ell}{\ell}
\]

\[
= (n-2)F_{n-3} - \mathcal{E}(n-2),
\]

which proves the claim (note we used the binomial identity to replace the sum of binomial coefficients with a Fibonacci number).
Formula for $\mathcal{E}(n)$

The formula for $\mathcal{E}(n)$ is given by:

$$\mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^2 + 1} + O(F_{n-2}).$$

**Proof:** The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by inputting Binet’s formula and differentiating identities. Explicitly, consider

$$\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} (\mathcal{E}(n - 2\ell) + \mathcal{E}(n - 2(\ell + 1))) = \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} (n - 2 - 2\ell)F_{n-3-2\ell}$$

$$= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} (n - 3 - 2\ell)F_{n-3-2\ell} + \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} (2\ell)F_{n-3-2\ell}$$

$$= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} (n - 3 - 2\ell)F_{n-3-2\ell} + O(F_{n-2});$$

while we could evaluate the last sum exactly, trivially estimating it suffices to obtain the main term (as we have a sum of every other Fibonacci number, the sum is at most the next Fibonacci number after the largest one in our sum).
We now use Binet’s formula to convert the sum into a geometric series. Letting \( \varphi = \frac{1+\sqrt{5}}{2} \) be the golden mean, we have

\[
F_n = \frac{\varphi}{\sqrt{5}} \cdot \varphi^n - \frac{1-\varphi}{\sqrt{5}} \cdot (1-\varphi)^n
\]

(our constants are because our counting has \( F_1 = 1, F_2 = 2 \) and so on). As \( |1-\varphi| < 1 \), the error from dropping the \( (1-\varphi)^n \) term is \( O(\sum_{\ell \leq n} n) = O(n^2) = o(F_{n-2}) \), and may thus safely be absorbed in our error term. We thus find

\[
\mathcal{E}(n) = \frac{\varphi}{\sqrt{5}} \left[ \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (n-3-2\ell)(-1)^\ell \varphi^{n-3-2\ell} + O(F_{n-2}) \right]
\]

\[
= \frac{\varphi^{n-2}}{\sqrt{5}} \left[ (n-3) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-\varphi^{-2})^\ell - 2 \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell(-\varphi^{-2})^\ell \right] + O(F_{n-2}).
\]
Formula for $\mathcal{E}(n)$ (continued)

We use the geometric series formula to evaluate the first term. We drop the upper boundary term of 

$$(-\varphi^{-1}) \lfloor \frac{n-3}{2} \rfloor,$$

as this term is negligible since $\varphi > 1$. We may also move the 3 from the $n - 3$ into the error term, and are left with

$$
\mathcal{E}(n) = \frac{\varphi^{n-2}}{\sqrt{5}} \left[ \frac{n}{1 + \varphi^{-2}} - 2 \sum_{\ell=0}^{\lfloor n/2 \rfloor} \ell(-\varphi^{-2})^{\ell} \right] + O(F_{n-2})
= \frac{\varphi^{n-2}}{\sqrt{5}} \left[ \frac{n}{1 + \varphi^{-2}} - 2S \left( \left\lfloor \frac{n-3}{2} \right\rfloor, -\varphi^{-2} \right) \right] + O(F_{n-2}),
$$

where

$$S(m, x) = \sum_{j=0}^{m} jx^j.$$

There is a simple formula for $S(m, x)$. As

$$\sum_{j=0}^{m} x^j = \frac{x^{m+1} - 1}{x - 1},$$

applying the operator $x \frac{d}{dx}$ gives

$$S(m, x) = \sum_{j=0}^{m} jx^j = x \frac{(m+1)x^m(x-1) - (x^{m+1} - 1)}{(x-1)^2} = \frac{mx^{m+2} - (m+1)x^{m+1} + x}{(x-1)^2}.$$
Taking $x = -\varphi^{-2}$, we see that the contribution from this piece may safely be absorbed into the error term $O(F_{n-2})$, leaving us with

$$\mathcal{E}(n) = \frac{n\varphi^{n-2}}{\sqrt{5}(1 + \varphi^{-2})} + O(F_{n-2}) = \frac{n\varphi^{n}}{\sqrt{5}(\varphi^{2} + 1)} + O(F_{n-2}).$$

Noting that for large $n$ we have $F_{n-1} = \frac{\varphi^{n}}{\sqrt{5}} + O(1)$, we finally obtain

$$\mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^{2} + 1} + O(F_{n-2}). \Box$$

A similar calculation should yield the higher moments; this will be done by the students of the 2010 SMALL REU at Williams College.
References


SMALL REU (2010, Williams College), preprint.