Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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Difference Equations and Applications Session
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Introduction
Goals of the Talk

- Difference equations: special linear recursions.
- Chat about ‘fun’ properties of Fibonacci numbers.
- Answer questions on number of summands and gaps.
- Methods: Combinatorial vantage, generating functions.
- Some open problems.

Thanks to colleagues from the Williams College 2010 and 2011 SMALL REU programs, and Louis Gaudet.
Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$
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Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.
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**Example:**
\[ 2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1. \]
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Lekkerkerker’s Theorem (1952)
The average number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) tends to \( \frac{n}{\varphi^2+1} \approx .276n, \)
where \( \varphi = \frac{1+\sqrt{5}}{2} \) is the golden mean.
Previous Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$. 
Theorem (Zeckendorf Gap Distribution (BM))

For Zeckendorf decompositions, \( P(k) = \frac{\phi(\phi - 1)}{\phi^k} \) for \( k \geq 2 \), with \( \phi = \frac{1 + \sqrt{5}}{2} \) the golden mean.

Figure: Distribution of gaps in \([F_{1000}, F_{1001})\); \( F_{2010} \approx 10^{208} \).
The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.
Preliminaries: The Cookie Problem

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Proof: Consider $C + P - 1$ cookies in a line. **Cookie Monster** eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do. Divides the cookies into $P$ sets.
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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is
$\binom{C+P-1}{P-1}$. 
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Reinterpreting the Cookie Problem

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Let $p_{n,k} = \# \{ N \in [F_n, F_{n+1}): \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \}$. 
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For $N \in [F_n, F_{n+1})$, the largest summand is $F_n$.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \ i_j - i_{j-1} \geq 2.$$
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$$d_1 := i_1 - 1, \ d_j := i_j - i_{j-1} - 2 \ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \ d_j \geq 0.$$
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Let \( p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of} \ N \text{ has exactly} \ k \text{ summands} \} \).

For \( N \in [F_n, F_{n+1}) \), the largest summand is \( F_n \).

\[
N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,
\]

\[
1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \ i_j - i_{j-1} \geq 2.
\]

\[
d_1 := i_1 - 1, \ d_j := i_j - i_{j-1} - 2 \ (j > 1).
\]

\[
d_1 + d_2 + \cdots + d_k = n - 2k + 1, \ d_j \geq 0.
\]

Cookie counting \( \Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1} \).
An Erdos-Kac Type Theorem
(note slightly different notation)
Generalizing Lekkerkerker: Erdos-Kac type result

**Theorem (KKMW 2010)**

As $n \to \infty$, the distribution of the number of summands in Zeckendorf’s Theorem is a Gaussian.

**Sketch of proof:** Use Stirling’s formula,

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

...to approximate binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.
The probability density for the number of Fibonacci numbers that add up to an integer in \([F_n, F_{n+1}]\) is 
\[ f_n(k) = \frac{(n-k)^{n-1-k}}{k^{n-1-k}} / F_{n-1} \]. Consider the density for the \(n+1\) case. Then we have, by Stirling

\[
 f_{n+1}(k) = \left( \frac{n-k}{k} \right) \frac{1}{F_n} 
 = \frac{(n-k)!}{(n-2k)!k!} \frac{1}{F_n} = \frac{1}{\sqrt{2\pi}} \frac{(n-k)^{n-k+\frac{1}{2}}}{k^{k+\frac{1}{2}}(n-2k)^{n-2k+\frac{1}{2}}} \frac{1}{F_n}
\]

plus a lower order correction term.
Also we can write \(F_n = \frac{1}{\sqrt{5}} \phi^{n+1} = \frac{\phi}{\sqrt{5}} \phi^n\) for large \(n\), where \(\phi\) is the golden ratio (we are using relabeled Fibonacci numbers where \(1 = F_1\) occurs once to help dealing with uniqueness and \(F_2 = 2\)). We can now split the terms that exponentially depend on \(n\).

\[
f_{n+1}(k) = \left( \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(n-k)}{k(n-2k)}} \frac{\sqrt{5}}{\phi} \right) \left( \phi^{-n} \frac{(n-k)^{n-k}}{k^{k}(n-2k)^{n-2k}} \right).
\]

Define

\[
 N_n = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(n-k)}{k(n-2k)}} \frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n} \frac{(n-k)^{n-k}}{k^{k}(n-2k)^{n-2k}}.
\]

Thus, write the density function as

\[
 f_{n+1}(k) = N_n S_n
\]

where \(N_n\) is the first term that is of order \(n^{-1/2}\) and \(S_n\) is the second term with exponential dependence on \(n\).
(Sketch of the) Proof of Gaussianity (cont)

Model the distribution as centered around the mean by the change of variable \( k = \mu + x\sigma \) where \( \mu \) and \( \sigma \) are the mean and the standard deviation, and depend on \( n \). The discrete weights of \( f_n(k) \) will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

\[
f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.
\]

Using the change of variable, we can write \( N_n \) as

\[
N_n = \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{n-k}{k(n-2k)}} \frac{\phi}{\sqrt{5}} \]

\[
= \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}} \frac{\sqrt{5}}{\phi} \]

\[
= \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}} \frac{\sqrt{5}}{\phi} \]

\[
= \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}} \frac{\sqrt{5}}{\phi} \]

where \( C = \mu/n \approx 1/(\phi + 2) \) (note that \( \phi^2 = \phi + 1 \)) and \( y = \sigma x/n \). But for large \( n \), the \( y \) term vanishes since \( \sigma \sim \sqrt{n} \) and thus \( y \sim n^{-1/2} \). Thus

\[
N_n \approx \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{1-C}{C(1-2C)}} \frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{(\phi + 1)(\phi + 2)}{\phi}} \frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{5(\phi + 2)}{\phi}} = \frac{1}{\sqrt{2\pi \sigma^2}}
\]

since \( \sigma^2 = n\frac{\phi}{5(\phi+2)} \).
(Sketch of the) Proof of Gaussianity (cont)

For the second term $S_n$, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\log(S_n) = \log \left( \frac{(n-k)^{n-k}}{k^n (n-2k)^{n-2k}} \right)$$

$$= -n \log(\phi) + (n-k) \log(n-k) - k \log(k)$$
$$- (n-2k) \log(n-2k)$$
$$= -n \log(\phi) + (n-(\mu+x\sigma)) \log(n-(\mu+x\sigma))$$
$$- (\mu+x\sigma) \log(\mu+x\sigma)$$
$$- (n-2(\mu+x\sigma)) \log(n-2(\mu+x\sigma))$$
$$= -n \log(\phi)$$
$$+ (n-(\mu+x\sigma)) \left( \log(n-\mu) + \log \left( 1 - \frac{x\sigma}{n-\mu} \right) \right)$$
$$- (\mu+x\sigma) \left( \log(\mu) + \log \left( 1 + \frac{x\sigma}{\mu} \right) \right)$$
$$- (n-2(\mu+x\sigma)) \left( \log(n-2\mu) + \log \left( 1 - \frac{x\sigma}{n-2\mu} \right) \right)$$
$$= -n \log(\phi)$$
$$+ (n-(\mu+x\sigma)) \left( \log \left( \frac{n}{\mu} - 1 \right) + \log \left( 1 - \frac{x\sigma}{n-\mu} \right) \right)$$
$$- (\mu+x\sigma) \log \left( 1 + \frac{x\sigma}{\mu} \right)$$
$$- (n-2(\mu+x\sigma)) \left( \log \left( \frac{n}{\mu} - 2 \right) + \log \left( 1 - \frac{x\sigma}{n-2\mu} \right) \right) \right).
(Sketch of the) Proof of Gaussianity (cont)

Note that, since \( n/\mu = \phi + 2 \) for large \( n \), the constant terms vanish. We have \( \log(S_n) \)

\[
\begin{align*}
\log(S_n) & = -n \log(\phi) + (n - k) \log\left(\frac{n}{\mu} - 1\right) - (n - 2k) \log\left(\frac{n}{\mu} - 2\right) + (n - (\mu + x\sigma)) \log\left(1 - \frac{x\sigma}{n - \mu}\right) \\
& \quad - (\mu + x\sigma) \log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu + x\sigma)) \log\left(1 - \frac{x\sigma}{n - 2\mu}\right) \\
& = -n \log(\phi) + (n - k) \log(\phi + 1) - (n - 2k) \log(\phi) + (n - (\mu + x\sigma)) \log\left(1 - \frac{x\sigma}{n - \mu}\right) \\
& \quad - (\mu + x\sigma) \log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu + x\sigma)) \log\left(1 - \frac{x\sigma}{n - 2\mu}\right) \\
& = n(- \log(\phi) + \log(\phi^2) - \log(\phi)) + k(\log(\phi^2) + 2 \log(\phi)) + (n - (\mu + x\sigma)) \log\left(1 - \frac{x\sigma}{n - \mu}\right) \\
& \quad - (\mu + x\sigma) \log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu + x\sigma)) \log\left(1 - 2 \frac{x\sigma}{n - 2\mu}\right) \\
& = (n - (\mu + x\sigma)) \log\left(1 - \frac{x\sigma}{n - \mu}\right) - (\mu + x\sigma) \log\left(1 + \frac{x\sigma}{\mu}\right) \\
& \quad - (n - 2(\mu + x\sigma)) \log\left(1 - 2 \frac{x\sigma}{n - 2\mu}\right).
\end{align*}
\]
Finally, we expand the logarithms and collect powers of $x\sigma/n$.

\[
\log(S_n) = (n - (\mu + x\sigma)) \left( -\frac{x\sigma}{n-\mu} - \frac{1}{2} \left( \frac{x\sigma}{n-\mu} \right)^2 + \ldots \right) \\
- (\mu + x\sigma) \left( \frac{x\sigma}{\mu} - \frac{1}{2} \left( \frac{x\sigma}{\mu} \right)^2 + \ldots \right) \\
- (n - 2(\mu + x\sigma)) \left( -2\frac{x\sigma}{n-2\mu} - \frac{1}{2} \left( 2\frac{x\sigma}{n-2\mu} \right)^2 + \ldots \right)
\]

\[
= (n - (\mu + x\sigma)) \left( -\frac{x\sigma}{n(\phi+1)} - \frac{1}{2} \left( \frac{x\sigma}{n(\phi+1)} \right)^2 + \ldots \right) \\
- (\mu + x\sigma) \left( \frac{x\sigma}{n\phi+2} - \frac{1}{2} \left( \frac{x\sigma}{n\phi+2} \right)^2 + \ldots \right) \\
- (n - 2(\mu + x\sigma)) \left( -\frac{2x\sigma}{n\phi+2} - \frac{1}{2} \left( \frac{2x\sigma}{n\phi+2} \right)^2 + \ldots \right)
\]

\[
= \frac{x\sigma}{n} \left( -\left(1 - \frac{1}{\phi+2}\right) \frac{(\phi+2)}{(\phi+1)} - 1 + 2 \left( 1 - \frac{2}{\phi+2} \right) \frac{\phi+2}{\phi} \right) \\
- \frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 \left( -2\frac{\phi+2}{\phi+1} + \frac{\phi+2}{\phi+1} + 2(\phi+2) - (\phi+2) + 4 \frac{\phi+2}{\phi} \right) \\
+ O \left( n(x\sigma/n)^3 \right)
\]
(Sketch of the) Proof of Gaussianity (cont)

\[
\log(S_n) = \frac{x\sigma}{n} \left( -\frac{\phi + 1}{\phi + 2} \frac{\phi + 2}{\phi + 1} - 1 + 2 \frac{\phi}{\phi + 2} \frac{\phi + 2}{\phi} \right) \\
- \frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 n(\phi + 2) \left( -\frac{1}{\phi + 1} + 1 + \frac{4}{\phi} \right) \\
+ O \left( n \left( \frac{x\sigma}{n} \right)^3 \right) \\
= -\frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 (\phi + 2) \left( \frac{3\phi + 4}{\phi(\phi + 1)} + 1 \right) + O \left( n \left( \frac{x\sigma}{n} \right)^3 \right) \\
= -\frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 (\phi + 2) \left( \frac{3\phi + 4 + 2\phi + 1}{\phi(\phi + 1)} \right) + O \left( n \left( \frac{x\sigma}{n} \right)^3 \right) \\
= -\frac{1}{2} x^2 \sigma^2 \left( \frac{5(\phi + 2)}{\phi n} \right) + O \left( n (x\sigma/n)^3 \right).
\]
(Sketch of the) Proof of Gaussianity (cont)

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi + 2)}.$$ 

Also, since $\sigma \sim n^{-1/2}$, $n \left( \frac{x\sigma}{n} \right)^3 \sim n^{-1/2}$. So for large $n$, the $O \left( n \left( \frac{x\sigma}{n} \right)^3 \right)$ term vanishes. Thus we are left with

$$\log S_n = -\frac{1}{2} x^2$$
$$S_n = e^{-\frac{1}{2} x^2}.$$ 

Hence, as $n$ gets large, the density converges to the normal distribution:

$$f_n(k) dk = N_n S_n dk$$
$$= \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2} x^2} \sigma dx$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx.$$ 

□
Generalizations
Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

\[ H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \quad n \geq L \]

with \( H_1 = 1, \quad H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \quad n < L, \)
coefficients \( c_i \geq 0; \quad c_1, c_L > 0 \) if \( L \geq 2; \quad c_1 > 1 \) if \( L = 1. \)

- **Zeckendorf**: Every positive integer can be written uniquely as \( \sum a_i H_i \) with natural constraints on the \( a_i \)'s (e.g. cannot use the recurrence relation to remove any summand).
- **Lekkerkerker**
- **Central Limit Type Theorem**
Generalizing Lekkerkerkerker

**Generalized Lekkerkerkerker’s Theorem**

The average number of summands in the generalized Zeckendorf decomposition for integers in \([H_n, H_{n+1})\) tends to \(Cn + d\) as \(n \to \infty\), where \(C > 0\) and \(d\) are computable constants determined by the \(c_i\)'s.

\[
C = -\frac{y'(1)}{y(1)} = \frac{\sum_{m=0}^{L-1} (s_m + s_{m+1} - 1)(s_{m+1} - s_m)y^m(1)}{2 \sum_{m=0}^{L-1} (m + 1)(s_{m+1} - s_m)y^m(1)}.
\]

\(s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m.\)

\(y(x)\) is the root of \(1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}.\)

\(y(1)\) is the root of \(1 - c_1 y - c_2 y^2 - \cdots - c_L y^L.\)
Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.
Example: the Special Case of \( L = 1, c_1 = 10 \)

\[ H_{n+1} = 10H_n, \quad H_1 = 1, \quad H_n = 10^{n-1}. \]

- Legal decomposition is decimal expansion: \( \sum_{i=1}^{m} a_i H_i: \)
  \[ a_i \in \{0,1,\ldots,9\} \quad (1 \leq i < m), \quad a_m \in \{1,\ldots,9\} . \]

- For \( N \in [H_n, H_{n+1}) \), \( m = n \), i.e., first term is \( a_n H_n = a_n 10^{n-1} \).

- \( A_i \): the corresponding random variable of \( a_i \).
  The \( A_i \)'s are independent.

- For large \( n \), the contribution of \( A_n \) is immaterial.
  \( A_i \) \((1 \leq i < n)\) are identically distributed random variables
  with mean 4.5 and variance 8.25.

- Central Limit Theorem: \( A_2 + A_3 + \cdots + A_n \to \) Gaussian
  with mean \( 4.5n + O(1) \)
  and variance \( 8.25n + O(1) \).
Far-difference Representation

**Theorem (Alpert, 2009) (Analogue to Zeckendorf)**

Every integer can be written uniquely as a sum of the $\pm F_n$’s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3).

**Example:** $1900 = F_{17} - F_{14} - F_{10} + F_6 + F_2$.

$K$: # of positive terms, $L$: # of negative terms.

**Generalized Lekkerkerkerker’s Theorem**

As $n \to \infty$, $E[K]$ and $E[L] \to n/10$. $E[K] - E[L] = \varphi/2 \approx .809$.

**Central Limit Type Theorem**

As $n \to \infty$, $K$ and $L$ converges to a bivariate Gaussian.

- $\text{corr}(K, L) = -(21 - 2\varphi)/(29 + 2\varphi) \approx -.551$, $\varphi = \frac{\sqrt{5}+1}{2}$.
- $K + L$ and $K - L$ are independent.
Gaps Between Summands
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$. 
Distribution of Gaps

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Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.
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Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$. 
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Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$.

What is $P(k) = \lim_{n \to \infty} P_n(k)$?
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$.

What is $P(k) = \lim_{n \to \infty} P_n(k)$?

Can ask similar questions about binary or other expansions: $2011 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0$. 
Main Results (Beckwith-Miller 2011)

**Theorem (Base B Gap Distribution)**

For base $B$ decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

**Theorem (Zeckendorf Gap Distribution)**

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.
Lekkerkerker ⇒ total number of gaps $\sim F_{n-1} \frac{n}{\phi^2 + 1}$. 
Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1} \frac{n}{\phi^2 + 1}$.

Let $X_{i,j} = \# \{ m \in [F_n, F_{n+1}) : \text{decomposition of } m \text{ includes } F_i, F_j, \text{but not } F_q \text{ for } i < q < j \}$. 
Lekkerkerker ⇒ total number of gaps \( \sim F_{n-1} \frac{n}{\phi^2 + 1} \).

Let \( X_{i,j} = \# \{ m \in [F_n, F_{n+1}) : \text{decomposition of } m \text{ includes } F_i, F_j, \text{ but not } F_q \text{ for } i < q < j \} \).

\[
P(k) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1} \frac{n}{\phi^2 + 1}}.
\]
Calculating $X_{i,j+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?
Calculating $X_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

$1 \leq i \leq n - k - 2$: 
Calculating $X_{i, i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

$1 \leq i \leq n - k - 2$:

For the indices less than $i$: $F_{i-1}$ choices.

For the indices greater than $i + k$: $F_{n-k-2-i}$ choices.
Calculating $X_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

1 ≤ $i$ ≤ $n − k − 2$:

For the indices less than $i$: $F_{i−1}$ choices.

For the indices greater than $i + k$: $F_{n−k−2−i}$ choices.

So total choices number of choices is $F_{n−k−2−i}F_{i−1}$. 
Determining $P(k)$

\[ \sum_{i=1}^{n-k} X_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2} \]

\[ \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3} \text{ is the } x^{n-k-3} \text{ coefficient of } (g(x))^2, \text{ where } g(x) \text{ is the generating function of the Fibonaccis.} \]
Determining $P(k)$

\[
\sum_{i=1}^{n-k} X_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1} F_{n-k-i-2}
\]

\[
\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3} \text{ is the } x^{n-k-3} \text{ coefficient of } (g(x))^2, \text{ where } g(x) \text{ is the generating function of the Fibonaccis.}
\]

\[
P(k) = \frac{C}{\phi^k} \text{ for some constant } C, \text{ so } P(k) = \frac{\phi(\phi-1)}{\phi^k}.
\]
Tribonacci Numbers: \( T_{n+1} = T_n + T_{n-1} + T_{n-2}; \)
\( F_1 = 1, \ F_2 = 2, \ F_3 = 4, \ F_4 = 7, \ldots \)
Tribonacci Gaps

Tribonacci Numbers: \( T_{n+1} = T_n + T_{n-1} + T_{n-2} \);
\( F_1 = 1, \ F_2 = 2, \ F_3 = 4, \ F_4 = 7, \ldots \).

Interval: \([T_n, T_{n+1})\), size \( Cn(T_{n-1} + T_{n-2}) + \text{smaller} \).
Tribonacci Gaps

Tribonacci Numbers: \( T_{n+1} = T_n + T_{n-1} + T_{n-2}; \)
\( F_1 = 1, F_2 = 2, F_3 = 4, F_4 = 7, \ldots. \)

Interval: \([T_n, T_{n+1})\), size \( C_n(T_{n-1} + T_{n-2}) + \) smaller.

Counting:

\[
X_{i,i+k}(n) = \begin{cases} 
    T_{i-1}(T_{n-i-3} + T_{n-i-4}) & \text{if } k = 1 \\
    (T_{i-1} + T_{i-2})(T_{n-k-i-1} + T_{n-k-i-3}) & \text{if } k \geq 2 
\end{cases}
\]
Tribonacci Gaps

Tribonacci Numbers: \( T_{n+1} = T_n + T_{n-1} + T_{n-2}; \)
\( F_1 = 1, \ F_2 = 2, \ F_3 = 4, \ F_4 = 7, \ldots. \)

Interval: \([T_n, T_{n+1}), \) size \( Cn(T_{n-1} + T_{n-2}) + \text{smaller}. \)

Counting:

\[
X_{i,i+k}(n) = \begin{cases} 
T_{i-1}(T_{n-i-3} + T_{n-i-4}) & \text{if } k = 1 \\
(T_{i-1} + T_{i-2})(T_{n-k-i-1} + T_{n-k-i-3}) & \text{if } k \geq 2 
\end{cases}
\]

Constants: \( P(1) = \frac{c_1}{C\lambda_1^3}, \ P(k) = \frac{2c_1}{C(1 + \lambda_1)}\lambda_1^{-k} \) (for \( k \geq 2 \)).
Other gaps?

- Gaps longer than recurrence – geometric decay.
- Interesting behavior with “short” gaps.
- “Skiponaccis”: $S_{n+1} = S_n + S_{n-2}$.
- “Doublanaccis”: $H_{n+1} = 2H_n + H_{n-1}$.
- Hope: Generalize to all positive linear recurrences.
Method of General Proof
Generating Function (Example: Binet’s Formula)

**Binet’s Formula**

\[
F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{-1 + \sqrt{5}}{2} \right)^n \right].
\]

- **Recurrence relation:** \( F_{n+1} = F_n + F_{n-1} \) \hspace{1cm} (1)

- **Generating function:** \( g(x) = \sum_{n>0} F_n x^n \).

\[
\begin{align*}
\sum_{n \geq 2} F_{n+1} x^{n+1} &= \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1} \\
\Rightarrow &\quad \sum_{n \geq 3} F_n x^n = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 1} F_n x^{n+2} \\
\Rightarrow &\quad \sum_{n \geq 3} F_n x^n = x \sum_{n \geq 2} F_n x^n + x^2 \sum_{n \geq 1} F_n x^n \\
\Rightarrow &\quad g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
\Rightarrow &\quad g(x) = x/(1 - x - x^2).
\end{align*}
\]
Partial Fraction Expansion (Example: Binet’s Formula)

- Generating function: \( g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}. \)

- Partial fraction expansion:

\[
\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left( \frac{\frac{1+\sqrt{5}}{2} x}{1 - \frac{1+\sqrt{5}}{2} x} - \frac{-\frac{1+\sqrt{5}}{2} x}{1 - -\frac{1+\sqrt{5}}{2} x} \right).
\]

**Coefficient of** \( x^n \) (power series expansion):

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( -\frac{1+\sqrt{5}}{2} \right)^n \right] - \text{Binet's Formula!}
\]

(Using geometric series: \( \frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots \).)
Differentiating Identities and Method of Moments

- **Differentiating identities**
  Example: Given a random variable $X$ such that
  \[ \text{Prob}(X = 1) = \frac{1}{2}, \text{Prob}(X = 2) = \frac{1}{4}, \text{Prob}(X = 3) = \frac{1}{8}, \ldots, \]
  then what’s the mean of $X$ (i.e., $E[X]$)?

  **Solution:** Let $f(x) = \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots = \frac{1}{1-x/2} - 1$.
  \[
  f'(x) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}x + 3 \cdot \frac{1}{8}x^2 + \cdots.
  \]
  \[
  f'(1) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots = E[X].
  \]

- **Method of moments:** Random variables $X_1, X_2, \ldots$.
  If the $\ell^{\text{th}}$ moment $E[X_n^\ell]$ converges to that of the standard normal distribution ($\forall \ell$), then $X_n$ converges to a Gaussian.

**Standard normal distribution:**
- $2m^{\text{th}}$ moment: $(2m - 1)!! = (2m - 1)(2m - 3)\cdots1$,
- $(2m - 1)^{\text{th}}$ moment: 0.
New Approach: Case of Fibonacci Numbers

\[ \rho_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} . \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}) : N = F_{n+1} + F_t + \cdots , \ t \leq n - 1. \]
  \[ \rho_{n+1,k+1} = \rho_{n-1,k} + \rho_{n-2,k} + \cdots \]
New Approach: Case of Fibonacci Numbers

\[ \rho_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \}. \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}) : N = F_{n+1} + F_t + \cdots, t \leq n - 1. \]
  \[ \rho_{n+1,k+1} = \rho_{n-1,k} + \rho_{n-2,k} + \cdots \]
  \[ \rho_{n,k+1} = \rho_{n-2,k} + \rho_{n-3,k} + \cdots \]
New Approach: Case of Fibonacci Numbers

\[ p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}): N = F_{n+1} + F_t + \cdots, \ t \leq n - 1. \]
  \[ \begin{align*}
  p_{n+1,k+1} &= p_{n-1,k} + p_{n-2,k} + \cdots \\
  p_{n,k+1} &= p_{n-2,k} + p_{n-3,k} + \cdots \\
  \Rightarrow p_{n+1,k+1} &= p_{n,k+1} + p_{n-1,k}.
  \end{align*} \]
New Approach: Case of Fibonacci Numbers

\[ p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \}. \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}): N = F_{n+1} + F_t + \cdots, t \leq n - 1. \]
  \[ p_{n+1,k+1} = p_{n-1,k} + p_{n-2,k} + \cdots \]
  \[ p_{n,k+1} = p_{n-2,k} + p_{n-3,k} + \cdots \]
  \[ \Rightarrow p_{n+1,k+1} = p_{n,k+1} + p_{n-1,k}. \]

- **Generating function:**
  \[ \sum_{n,k>0} p_{n,k} x^k y^n = \frac{y}{1 - y - xy^2}. \]

- **Partial fraction expansion:**
  \[ \frac{y}{1 - y - xy^2} = -\frac{y}{y_1(x) - y_2(x)} \left( \frac{1}{y - y_1(x)} - \frac{1}{y - y_2(x)} \right) \]

where \( y_1(x) \) and \( y_2(x) \) are the roots of \( 1 - y - xy^2 = 0. \)

**Coefficient of \( y^n \):**
\[ g(x) = \sum_{k>0} p_{n,k} x^k. \]
New Approach: Case of Fibonacci Numbers (Continued)

\( K_n \): the corresponding random variable associated with \( k \).

\[ g(x) = \sum_{k>0} p_{n,k} x^k. \]

- **Differentiating identities:**
  \[ g(1) = \sum_{k>0} p_{n,k} = F_{n+1} - F_n, \]
  \[ g'(x) = \sum_{k>0} kp_{n,k} x^{k-1}, \quad g'(1) = g(1) E[K_n], \]
  \[ (xg'(x))' = \sum_{k>0} k^2 p_{n,k} x^{k-1}, \]
  \[ (xg'(x))' \big|_{x=1} = g(1) E[K_n^2], \quad (x (xg'(x)))' \big|_{x=1} = g(1) E[K_n^3], \ldots \]

Similar results hold for the centralized \( K_n \): \( K'_n = K_n - E[K_n] \).

- **Method of moments** (for normalized \( K'_n \)):
  \[ E[(K'_n)^{2m}]/(SD(K'_n))^{2m} \rightarrow (2m - 1)!!, \]
  \[ E[(K'_n)^{2m-1}]/(SD(K'_n))^{2m-1} \rightarrow 0. \]

\( \Rightarrow K_n \rightarrow \text{Gaussian}. \)
Let $p_{n,k} = \# \{ N \in [H_n, H_{n+1}) : \text{the generalized Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\}$.

- **Recurrence relation:**
  - **Fibonacci:** $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$.
  - **General:**
    $$p_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} p_{n-m,k-j}.$$  
    where $s_0 = 0$, $s_m = c_1 + c_2 + \cdots + c_m$.

- **Generating function:**
  - **Fibonacci:** $\frac{y}{1-y-xy^2}$.
  - **General:**
    $$\sum_{n \leq L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n < L-m} p_{n,k} x^k y^n.$$  
    $$\frac{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}.$$  

New Approach: General Case (Continued)

- Partial fraction expansion:
  
  Fibonacci: \[-\frac{y}{y_1(x) - y_2(x)} \left( \frac{1}{y - y_1(x)} - \frac{1}{y - y_2(x)} \right) \cdot \]
  
  General:
  \[
  - \frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j} \sum_{i=1}^{L} \frac{B(x, y)}{(y - y_i(x)) \prod_{j \neq i} (y_j(x) - y_i(x))}.
  \]

  \[
  B(x, y) = \sum_{n \leq L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} p_{n,k} x^k y^n,
  \]

  \[
  y_i(x): \text{root of } 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} = 0.
  \]

  Coefficient of \(y^n\): \(g(x) = \sum_{n,k>0} p_{n,k} x^k\).

- Differentiating identities

- Method of moments \(\Rightarrow K_n \rightarrow \text{Gaussian}\)
Appendix:
Combinatorial Identities and Lekkerkerker’s Theorem
Needed Binomial Identity

Binomial identity involving Fibonacci Numbers

Let $F_m$ denote the $m^{th}$ Fibonacci number, with $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$ and so on. Then

$$\left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-1}.$$

Proof by induction: The base case is trivially verified. Assume our claim holds for $n$ and show that it holds for $n + 1$. We may extend the sum to $n − 1$, as $\binom{n-1-k}{k} = 0$ whenever $k > \left\lfloor \frac{n-1}{2} \right\rfloor$. Using the standard identity that

$$\binom{m}{\ell} + \binom{m}{\ell + 1} = \binom{m+1}{\ell + 1},$$

and the convention that $\binom{m}{\ell} = 0$ if $\ell$ is a negative integer, we find

$$\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=0}^{n} \left[ \binom{n-1-k}{k-1} + \binom{n-1-k}{k} \right]$$

$$= \sum_{k=1}^{n} \binom{n-1-k}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k}$$

$$= \sum_{k=1}^{n} \binom{n-2-(k-1)}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-2} + F_{n-1}$$

by the inductive assumption; noting $F_{n-2} + F_{n-1} = F_n$ completes the proof. \qed
Preliminaries for Lekkerkerker’s Theorem

\[ \mathcal{E}(n) := \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \binom{n-1}{k}. \]

Average number of summands in \([F_n, F_{n+1})\) is

\[ \frac{\mathcal{E}(n)}{F_{n-1}} + 1. \]

Recurrence Relation for \(\mathcal{E}(n)\)

\[ \mathcal{E}(n) + \mathcal{E}(n-2) = (n-2)F_{n-3}. \]
Recurrence Relation

Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n - 2) = (n - 2)F_{n-3}. $$

Proof by algebra (details later):

$$\mathcal{E}(n) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \binom{n-1-k}{k}$$

$$= (n-2) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell \binom{n-3-\ell}{\ell}$$

$$= (n-2)F_{n-3} - \mathcal{E}(n-2).$$
Solving Recurrence Relation

Formula for $\mathcal{E}(n)$ (i.e., Lekkerkerkerker’s Theorem)

$$\mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^2 + 1} + O(F_{n-2}).$$

$$\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^\ell (\mathcal{E}(n - 2\ell) + \mathcal{E}(n - 2(\ell + 1)))$$

$$= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^\ell (n - 2 - 2\ell) F_{n - 3 - 2\ell}.$$

Result follows from Binet’s formula, the geometric series formula, and differentiating identities: $$\sum_{j=0}^{m} jx^j = x \frac{(m+1)x^m(x-1)-(x^{m+1}-1)}{(x-1)^2}$$. Details later in the appendix.
Derivation of Recurrence Relation for $\mathcal{E}(n)$

\[
\mathcal{E}(n) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} k \binom{n - 1 - k}{k}
\]

\[
= \sum_{k=1}^{\left\lfloor n/2 \right\rfloor} k \frac{(n - 1 - k)!}{k!(n - 1 - 2k)!}
\]

\[
= \sum_{k=1}^{\left\lfloor n/2 \right\rfloor} (n - 1 - k) \frac{(n - 2 - k)!}{(k - 1)!(n - 1 - 2k)!}
\]

\[
= \sum_{k=1}^{\left\lfloor n/2 \right\rfloor} (n - 2 - (k - 1)) \frac{(n - 3 - (k - 1))!}{(k - 1)!(n - 3 - 2(k - 1))!}
\]

\[
\mathcal{E}(n) = (n - 2) \sum_{\ell=0}^{\left\lfloor (n - 3)/2 \right\rfloor} \binom{n - 3 - \ell}{\ell}
\]

\[
= (n - 2)F_{n-3} - \mathcal{E}(n - 2),
\]

which proves the claim (note we used the binomial identity to replace the sum of binomial coefficients with a Fibonacci number).
Formula for $\mathcal{E}(n)$

Formula for $\mathcal{E}(n)$

$$\mathcal{E}(n) = \frac{n F_{n-1}}{\varphi^2 + 1} + O(F_{n-2}).$$

Proof: The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by inputting Binet’s formula and differentiating identities. Explicitly, consider

$$\left\lfloor \frac{n-3}{2} \right\rfloor \sum_{\ell=0} \left( -1 \right)^{\ell} \left( \mathcal{E}(n - 2\ell) + \mathcal{E}(n - 2(\ell + 1)) \right) = \left\lfloor \frac{n-3}{2} \right\rfloor \sum_{\ell=0} \left( -1 \right)^{\ell} (n - 2 - 2\ell) F_{n-3-2\ell}$$

$$= \left\lfloor \frac{n-3}{2} \right\rfloor \sum_{\ell=0} \left( -1 \right)^{\ell} (n - 3 - 2\ell) F_{n-3-2\ell} + \sum_{\ell=0} \left( -1 \right)^{\ell} (2\ell) F_{n-3-2\ell} + O(F_{n-2});$$

while we could evaluate the last sum exactly, trivially estimating it suffices to obtain the main term (as we have a sum of every other Fibonacci number, the sum is at most the next Fibonacci number after the largest one in our sum).
Formula for $\mathcal{E}(n)$ (continued)

We now use Binet’s formula to convert the sum into a geometric series. Letting $\varphi = \frac{1 + \sqrt{5}}{2}$ be the golden mean, we have

$$F_n = \frac{\varphi}{\sqrt{5}} \cdot \varphi^n - \frac{1 - \varphi}{\sqrt{5}} \cdot (1 - \varphi)^n$$

(our constants are because our counting has $F_1 = 1, F_2 = 2$ and so on). As $|1 - \varphi| < 1$, the error from dropping the $(1 - \varphi)^n$ term is $O(\sum_{\ell \leq n} \ell^2) = O(n^2) = o(F_{n-2})$, and may thus safely be absorbed in our error term. We thus find

$$\mathcal{E}(n) = \frac{\varphi}{\sqrt{5}} \left[ \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (n - 3 - 2\ell)(-1)^\ell \varphi^{n-3-2\ell} + O(F_{n-2}) \right]$$

$$= \frac{\varphi^{n-2}}{\sqrt{5}} \left[ (n - 3) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-\varphi^{-2})^\ell - 2 \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell(-\varphi^{-2})^\ell \right] + O(F_{n-2}).$$
We use the geometric series formula to evaluate the first term. We drop the upper boundary term of 
\((-\varphi^{-1})^\left\lfloor \frac{n-3}{2} \right\rfloor\), as this term is negligible since \(\varphi > 1\). We may also move the 3 from the \(n - 3\) into the error term, and are left with

\[
\mathcal{E}(n) = \frac{\varphi^{n-2}}{\sqrt{5}} \left[ \frac{n}{1 + \varphi^{-2}} - 2 \sum_{\ell=0}^\left\lfloor \frac{n-3}{2} \right\rfloor \ell(-\varphi^{-2})^\ell \right] + O(F_{n-2})
\]

\[
= \frac{\varphi^{n-2}}{\sqrt{5}} \left[ \frac{n}{1 + \varphi^{-2}} - 2S\left(\left\lfloor \frac{n-3}{2} \right\rfloor, -\varphi^{-2}\right) \right] + O(F_{n-2}),
\]

where

\[
S(m, x) = \sum_{j=0}^{m} jx^j.
\]

There is a simple formula for \(S(m, x)\). As

\[
\sum_{j=0}^{m} x^j = \frac{x^{m+1} - 1}{x - 1},
\]

applying the operator \(x \frac{d}{dx}\) gives

\[
S(m, x) = \sum_{j=0}^{m} jx^j = x \frac{(m+1)x^m(x-1) - (x^{m+1} - 1)}{(x - 1)^2} = \frac{mx^{m+2} - (m + 1)x^{m+1} + x}{(x - 1)^2}.
\]
Formula for $\mathcal{E}(n)$ (continued)

Taking $x = -\varphi^{-2}$, we see that the contribution from this piece may safely be absorbed into the error term $O(F_{n-2})$, leaving us with

$$\mathcal{E}(n) = \frac{n\varphi^{n-2}}{\sqrt{5}(1 + \varphi^{-2})} + O(F_{n-2}) = \frac{n\varphi^{n}}{\sqrt{5}(\varphi^{2} + 1)} + O(F_{n-2}).$$

Noting that for large $n$ we have $F_{n-1} = \frac{\varphi^{n}}{\sqrt{5}} + O(1)$, we finally obtain

$$\mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^{2} + 1} + O(F_{n-2}).$$