Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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http://www.williams.edu/Mathematics/sjmiller/public_html

Colby College, March 28, 2012
Introduction
## Goals of the Talk

- Chat about ‘fun’ properties of Fibonacci numbers.
- Often enough to ask *any* question, not just right one.
- Look for the ‘right’ perspective.
- Bring techniques to bear: generating fns, partial fractions.
- You can join in – lots of other problems to study.

Thanks to colleagues from the Williams College 2010 and 2011 SMALL REU programs (especially Ed Burger, David Clyde, Cory Colbert, Carlos Dominguez, Gea Shin and Nancy Wang).
Let $X$ be random variable with density $p(x)$:

- $p(x) \geq 0; \int_{-\infty}^{\infty} p(x)dx = 1$;
- $\text{Prob}(a \leq X \leq b) = \int_{a}^{b} p(x)dx$. 
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Mean: $\mu = \int_{-\infty}^{\infty} xp(x)dx$.

Variance: $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx$. 

![Graph of a normal distribution]
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**Variance:** $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x) \, dx$.

**Gaussian:** Density $(2\pi\sigma^2)^{-1/2} \exp(- (x - \mu)^2 / 2\sigma^2)$.
Pre-requisites: Combinatorics Review

- $n!$: number of ways to order $n$ people, order matters.

- $\frac{n!}{k!(n-k)!} = nCk = \binom{n}{k}$: number of ways to choose $k$ from $n$, order doesn’t matter.

- Stirling’s Formula: $n! \approx n^n e^{-n} \sqrt{2\pi n}$. 
Previous Results

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**Zeckendorf’s Theorem**

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.
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Example:
$2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1$. 
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**Example:**
\[ 2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1. \]

**Lekkerkerkerker’s Theorem (1952)**

The average number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) tends to 
\[ \frac{n}{\varphi^2 + 1} \approx 0.276n, \]
where \(\varphi = \frac{1 + \sqrt{5}}{2}\) is the golden mean.
Central Limit Type Theorem (KKMW, MW)

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

![Graph showing distribution of summands]

**Figure:** Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$. 
Theorem (Zeckendorf Gap Distribution (BM))

For Zeckendorf decompositions, \( P(k) = \frac{\phi(\phi - 1)}{\phi^k} \) for \( k \geq 2 \), with \( \phi = \frac{1 + \sqrt{5}}{2} \) the golden mean.

Figure: Distribution of gaps in \([F_{1000}, F_{1001})\); \( F_{2010} \approx 10^{208} \).
The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$. 
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Example: 8 cookies and 5 people ($C = 8$, $P = 5$):
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Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is

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Preliminaries: The Cookie Problem: Reinterpretation

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The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is \( \binom{C+P-1}{P-1} \).

Let $p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \}$. 
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For $N \in [F_n, F_{n+1})$, the largest summand is $F_n$.

\[ N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n, \]
\[ 1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, i_j - i_{j-1} \geq 2. \]
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\]
\[
1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \ i_j - i_{j-1} \geq 2.
\]
\[
d_1 := i_1 - 1, \ d_j := i_j - i_{j-1} - 2 \ (j > 1).
\]
\[
d_1 + d_2 + \cdots + d_k = n - 2k + 1, \ d_j \geq 0.
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N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,
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1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \quad i_j - i_{j-1} \geq 2.
\]

\[
d_1 := i_1 - 1, \quad d_j := i_j - i_{j-1} - 2 \quad (j > 1).
\]

\[
d_1 + d_2 + \cdots + d_k = n - 2k + 1, \quad d_j \geq 0.
\]

Cookie counting \( \Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1} \).
An Erdos-Kac Type Theorem
(note slightly different notation)
Generalizing Lekkerkerker: Erdos-Kac type result

**Theorem (KKMW 2010)**

As $n \to \infty$, the distribution of the number of summands in Zeckendorf’s Theorem is a Gaussian.

**Sketch of proof:** Use Stirling’s formula,

$$ n! \approx n^n e^{-n} \sqrt{2\pi n} $$

to approximate binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.
The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is
\[f_n(k) = \binom{n-1-k}{k} / F_{n-1}.\]
Consider the density for the $n+1$ case. Then we have, by Stirling
\[
f_{n+1}(k) = \binom{n-k}{k} \frac{1}{F_n} = \frac{(n-k)!}{(n-2k)!k!} \frac{1}{F_n} = \frac{1}{\sqrt{2\pi}} \frac{(n-k)^{n-k+\frac{1}{2}}}{k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}} \frac{1}{F_n}
\]
plus a lower order correction term.
Also we can write $F_n = \frac{1}{\sqrt{5}} \phi^{n+1} = \frac{\phi}{\sqrt{5}} \phi^n$ for large $n$, where $\phi$ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on $n$.
\[
f_{n+1}(k) = \left(1 \over \sqrt{2\pi} \sqrt{{n-k \over k(n-2k)}} {\sqrt{5} \over \phi}\right) \left(\phi^{-n} {n-k \over k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}} \right).
\]
Define
\[N_n = \frac{1}{\sqrt{2\pi}} \sqrt{{n-k \over k(n-2k)}} {\sqrt{5} \over \phi}, \quad S_n = \phi^{-n} {n-k \over k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}}.
\]
Thus, write the density function as
\[f_{n+1}(k) = N_n S_n\]
where $N_n$ is the first term that is of order $n^{-1/2}$ and $S_n$ is the second term with exponential dependence on $n$. 
Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where $\mu$ and $\sigma$ are the mean and the standard deviation, and depend on $n$. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)\,dk = f_n(\mu + \sigma x)\sigma \,dx.$$ 

Using the change of variable, we can write $N_n$ as

$$N_n = \frac{1}{\sqrt{2\pi}} \int \frac{n-k}{\sqrt{k(n-2k)}} \frac{\phi}{\sqrt{5}} \,dk = \frac{1}{\sqrt{2\pi n}} \int \frac{1-k/n}{\sqrt{(k/n)(1-2k/n)}} \frac{\sqrt{5}}{\phi} \,dk = \frac{1}{\sqrt{2\pi n}} \int \frac{1-(\mu+\sigma x)/n}{\sqrt{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}} \frac{\sqrt{5}}{\phi} \,dk = \frac{1}{\sqrt{2\pi n}} \int \frac{1-C-y}{\sqrt{(C+y)(1-2C-2y)}} \frac{\sqrt{5}}{\phi} \,dk$$

where $C = \mu/n \approx 1/(\phi + 2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large $n$, the $y$ term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{1-C}{C(1-2C)}} \frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{(\phi + 1)(\phi + 2)}{\phi}} \frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{5(\phi + 2)}{\phi}} = \frac{1}{\sqrt{2\pi \sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$. 

**Sketch of the** Proof of Gaussianity (cont)
(Sketch of the) Proof of Gaussianity (cont)

For the second term $S_n$, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\log(S_n) = \log \left( \phi^{-n} \frac{(n-k)^{(n-k)}}{k^k(n-2k)(n-2k)} \right)$$

$$= -n \log(\phi) + (n-k) \log(n-k) - (k) \log(k) - (n-2k) \log(n-2k)$$

$$= -n \log(\phi) + (n-(\mu+x\sigma)) \log(n-(\mu+x\sigma)) - (\mu+x\sigma) \log(\mu+x\sigma) - (n-2(\mu+x\sigma)) \log(n-2(\mu+x\sigma))$$

$$= -n \log(\phi) + (n-(\mu+x\sigma)) \left( \log(n-\mu) + \log \left( 1 - \frac{x\sigma}{n-\mu} \right) \right)$$

$$- (\mu+x\sigma) \left( \log(\mu) + \log \left( 1 + \frac{x\sigma}{\mu} \right) \right)$$

$$- (n-2(\mu+x\sigma)) \left( \log(n-2\mu) + \log \left( 1 - \frac{x\sigma}{n-2\mu} \right) \right)$$

$$= -n \log(\phi) + (n-(\mu+x\sigma)) \left( \log \left( \frac{n}{\mu} - 1 \right) + \log \left( 1 - \frac{x\sigma}{n-\mu} \right) \right)$$

$$- (\mu+x\sigma) \log \left( 1 + \frac{x\sigma}{\mu} \right)$$

$$- (n-2(\mu+x\sigma)) \left( \log \left( \frac{n}{\mu} - 2 \right) + \log \left( 1 - \frac{x\sigma}{n-2\mu} \right) \right).$$
(Sketch of the) Proof of Gaussianity (cont)

Note that, since \( n/\mu = \phi + 2 \) for large \( n \), the constant terms vanish. We have \( \log(S_n) \)

\[
\begin{align*}
&= -n \log(\phi) + (n - k) \log \left( \frac{n}{\mu} - 1 \right) - (n - 2k) \log \left( \frac{n}{\mu} - 2 \right) + (n - (\mu + x\sigma)) \log \left( 1 - \frac{x\sigma}{n - \mu} \right) \\
&\quad - (\mu + x\sigma) \log \left( 1 + \frac{x\sigma}{\mu} \right) - (n - 2(\mu + x\sigma)) \log \left( 1 - \frac{x\sigma}{n - 2\mu} \right) \\
&= -n \log(\phi) + (n - k) \log (\phi + 1) - (n - 2k) \log (\phi) + (n - (\mu + x\sigma)) \log \left( 1 - \frac{x\sigma}{n - \mu} \right) \\
&\quad - (\mu + x\sigma) \log \left( 1 + \frac{x\sigma}{\mu} \right) - (n - 2(\mu + x\sigma)) \log \left( 1 - \frac{x\sigma}{n - 2\mu} \right) \\
&= n(- \log(\phi) + \log \left( \frac{\phi^2}{\mu} \right) - \log(\phi)) + k(\log(\phi^2) + 2 \log(\phi)) + (n - (\mu + x\sigma)) \log \left( 1 - \frac{x\sigma}{n - \mu} \right) \\
&\quad - (\mu + x\sigma) \log \left( 1 + \frac{x\sigma}{\mu} \right) - (n - 2(\mu + x\sigma)) \log \left( 1 - 2 \frac{x\sigma}{n - 2\mu} \right) \\
&= (n - (\mu + x\sigma)) \log \left( 1 - \frac{x\sigma}{n - \mu} \right) - (\mu + x\sigma) \log \left( 1 + \frac{x\sigma}{\mu} \right) \\
&\quad - (n - 2(\mu + x\sigma)) \log \left( 1 - 2 \frac{x\sigma}{n - 2\mu} \right).
\end{align*}
\]
(Sketch of the) Proof of Gaussianity (cont)

Finally, we expand the logarithms and collect powers of $x\sigma/n$.

$$
\log(S_n) = (n - (\mu + x\sigma)) \left( - \frac{x\sigma}{n - \mu} - \frac{1}{2} \left( \frac{x\sigma}{n - \mu} \right)^2 + \ldots \right) \\
- (\mu + x\sigma) \left( \frac{x\sigma}{\mu} - \frac{1}{2} \left( \frac{x\sigma}{\mu} \right)^2 + \ldots \right) \\
- (n - 2(\mu + x\sigma)) \left( -2 \frac{x\sigma}{n - 2\mu} - \frac{1}{2} \left( \frac{2x\sigma}{n - 2\mu} \right)^2 + \ldots \right)
$$

$$
= (n - (\mu + x\sigma)) \left( - \frac{x\sigma}{n(\phi + 1)} - \frac{1}{2} \left( \frac{x\sigma}{n(\phi + 1)} \right)^2 + \ldots \right) \\
- (\mu + x\sigma) \left( \frac{x\sigma}{n(\phi + 2)} - \frac{1}{2} \left( \frac{x\sigma}{n(\phi + 2)} \right)^2 + \ldots \right) \\
- (n - 2(\mu + x\sigma)) \left( - \frac{2x\sigma}{n(\phi + 2)} - \frac{1}{2} \left( \frac{2x\sigma}{n(\phi + 2)} \right)^2 + \ldots \right)
$$

$$
= \frac{x\sigma}{n} \left[ - \left( 1 - \frac{1}{\phi + 2} \right) \frac{\phi + 2}{(\phi + 1)} - 1 + 2 \left( 1 - \frac{2}{\phi + 2} \right) \frac{\phi + 2}{\phi} \right] \\
- \frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 \left[ -2 \frac{\phi + 2}{\phi + 1} + \frac{\phi + 2}{\phi + 1} + 2(\phi + 2) - (\phi + 2) + 4 \frac{\phi + 2}{\phi} \right] + O\left(n(x\sigma/n)^3\right)
$$
(Sketch of the) Proof of Gaussianity (cont)

$$\log(S_n) = \frac{x\sigma}{n} \left( - \frac{\phi + 1}{\phi + 2} + \frac{\phi}{\phi + 1} - 1 + 2 \frac{\phi}{\phi + 2} \right)$$

$$- \frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 n(\phi + 2) \left( - \frac{1}{\phi + 1} + 1 + \frac{4}{\phi} \right)$$

$$+ O \left( n \left( \frac{x\sigma}{n} \right)^3 \right)$$

$$= - \frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 (\phi + 2) \left( \frac{3\phi + 4}{\phi(\phi + 1)} + 1 \right) + O \left( n \left( \frac{x\sigma}{n} \right)^3 \right)$$

$$= - \frac{1}{2} \left( \frac{x\sigma}{n} \right)^2 (\phi + 2) \left( \frac{3\phi + 4 + 2\phi + 1}{\phi(\phi + 1)} \right) + O \left( n \left( \frac{x\sigma}{n} \right)^3 \right)$$

$$= - \frac{1}{2} \frac{x^2 \sigma^2}{\phi n} \left( \frac{5(\phi + 2)}{\phi n} \right) + O \left( n \left( x\sigma / n \right)^3 \right).$$
(Sketch of the) Proof of Gaussianity (cont)

But recall that

\[ \sigma^2 = \frac{\phi n}{5(\phi + 2)}. \]

Also, since \( \sigma \sim n^{-1/2}, n \left( \frac{x\sigma}{n} \right)^3 \sim n^{-1/2} \). So for large \( n \), the \( O \left( n \left( \frac{x\sigma}{n} \right)^3 \right) \) term vanishes. Thus we are left with

\[
\begin{align*}
\log S_n &= -\frac{1}{2} x^2 \\
S_n &= e^{-\frac{1}{2} x^2}.
\end{align*}
\]

Hence, as \( n \) gets large, the density converges to the normal distribution:

\[
\begin{align*}
f_n(k)dk &= N_n S_n dk \\
&= \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2} x^2} \sigma dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx.
\end{align*}
\]
Generalizations
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Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

\[ H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \quad n \geq L \]

with \( H_1 = 1 \), \( H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1 \), \( n < L \), coefficients \( c_i \geq 0 \); \( c_1, c_L > 0 \) if \( L \geq 2 \); \( c_1 > 1 \) if \( L = 1 \).

- **Zeckendorf**: Every positive integer can be written uniquely as \( \sum a_i H_i \) with natural constraints on the \( a_i \)'s (e.g. cannot use the recurrence relation to remove any summand).

- **Lekkerkerkerker**

- **Central Limit Type Theorem**
Generalizing Lekkerkerker

Generalized Lekkerkerker's Theorem

The average number of summands in the generalized Zeckendorf decomposition for integers in \([H_n, H_{n+1})\) tends to \(Cn + d\) as \(n \to \infty\), where \(C > 0\) and \(d\) are computable constants determined by the \(c_i's\).

\[
C = \frac{-y'(1)}{y(1)} = \frac{\sum_{m=0}^{L-1} (s_m + s_{m+1} - 1)(s_{m+1} - s_m)y^m(1)}{2 \sum_{m=0}^{L-1} (m + 1)(s_{m+1} - s_m)y^m(1)}.
\]

\(s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m\).

\(y(x)\) is the root of \(1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}\).

\(y(1)\) is the root of \(1 - c_1 y - c_2 y^2 - \cdots - c_L y^L\).
Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.
Example: the Special Case of $L = 1, c_1 = 10$

$H_{n+1} = 10H_n, H_1 = 1, H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_iH_i$:
  
  $a_i \in \{0, 1, \ldots, 9\}$ (1 ≤ $i$ < $m$), $a_m \in \{1, \ldots, 9\}$.

- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., first term is $a_nH_n = a_n10^{n-1}$.

- $A_i$: the corresponding random variable of $a_i$.
  The $A_i$'s are independent.

- For large $n$, the contribution of $A_n$ is immaterial.
  $A_i$ (1 ≤ $i$ < $n$) are identically distributed random variables with mean 4.5 and variance 8.25.

- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \rightarrow$ Gaussian with mean $4.5n + O(1)$ and variance $8.25n + O(1)$.
Far-difference Representation

**Theorem (Alpert, 2009) (Analogue to Zeckendorf)**

Every integer can be written uniquely as a sum of the \( \pm F_n \)'s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3).

**Example:** \( 1900 = F_{17} - F_{14} - F_{10} + F_6 + F_2 \).

\( K \): # of positive terms, \( L \): # of negative terms.

**Generalized Lekkerkerkerker’s Theorem**

As \( n \to \infty \), \( E[K] \) and \( E[L] \) \( \to n/10 \). \( E[K] - E[L] = \varphi/2 \approx .809 \).

**Central Limit Type Theorem**

As \( n \to \infty \), \( K \) and \( L \) converges to a bivariate Gaussian.

- \( \text{corr}(K, L) = -(21 - 2\varphi)/(29 + 2\varphi) \approx -.551 \), \( \varphi = \frac{\sqrt{5}+1}{2} \).
- \( K + L \) and \( K - L \) are independent.
Method of General Proof
Binet’s Formula

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{-1+\sqrt{5}}{2} \right)^n \right]. \]
Binet’s Formula

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{-1+\sqrt{5}}{2} \right)^n \right]. \]

- Recurrence relation: \( F_{n+1} = F_n + F_{n-1} \) (1)
Generating Function (Example: Binet’s Formula)

Binet’s Formula

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{-1 + \sqrt{5}}{2} \right)^n \right]. \]

- Recurrence relation: \[ F_{n+1} = F_n + F_{n-1} \] (1)
- Generating function: \[ g(x) = \sum_{n>0} F_n x^n. \]
Generating Function (Example: Binet’s Formula)

**Binet’s Formula**

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{-1 + \sqrt{5}}{2} \right)^n \right]. \]

- **Recurrence relation**: \( F_{n+1} = F_n + F_{n-1} \) \hspace{1cm} (1)
- **Generating function**: \( g(x) = \sum_{n>0} F_n x^n \).

(1) \[ \Rightarrow \sum_{n \geq 2} F_{n+1} x^{n+1} = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1} \]
Generating Function (Example: Binet’s Formula)

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\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{-1 + \sqrt{5}}{2} \right)^n \right]. \]

- Recurrence relation: \( F_{n+1} = F_n + F_{n-1} \)  
- Generating function: \( g(x) = \sum_{n>0} F_n x^n. \)

\[ (1) \quad \Rightarrow \quad \sum_{n \geq 2} F_{n+1} x^{n+1} = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1} \]

\[ \Rightarrow \quad \sum_{n \geq 3} F_n x^n = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 1} F_n x^{n+2} \]
Generating Function (Example: Binet’s Formula)

**Binet’s Formula**

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{-1 + \sqrt{5}}{2} \right)^n \right]. \]

- **Recurrence relation**: \( F_{n+1} = F_n + F_{n-1} \)  \( (1) \)
- **Generating function**: \( g(x) = \sum_{n>0} F_n x^n \).

\[
\begin{align*}
\sum_{n \geq 2} F_{n+1} x^{n+1} & = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1} \\
\Rightarrow \sum_{n \geq 3} F_n x^n & = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 1} F_n x^{n+2} \\
& \Rightarrow \sum_{n \geq 3} F_n x^n = x \sum_{n \geq 2} F_n x^n + x^2 \sum_{n \geq 1} F_n x^n
\end{align*}
\]
Generating Function (Example: Binet’s Formula)

**Binet’s Formula**

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{-1+\sqrt{5}}{2} \right)^n \right]. \]

- **Recurrence relation:** \( F_{n+1} = F_n + F_{n-1} \) \hfill (1)
- **Generating function:** \( g(x) = \sum_{n>0} F_n x^n. \)

\[
\begin{align*}
\sum_{n \geq 2} F_{n+1} x^{n+1} &= \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1} \\
\Rightarrow \quad \sum_{n \geq 2} F_n x^n &= \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 1} F_n x^{n+2} \\
\Rightarrow \quad \sum_{n \geq 3} F_n x^n &= x \sum_{n \geq 2} F_n x^n + x^2 \sum_{n \geq 1} F_n x^n \\
\Rightarrow \quad g(x) - F_1 x - F_2 x^2 &= x(g(x) - F_1 x) + x^2 g(x)
\end{align*}
\]
Generating Function (Example: Binet’s Formula)

Binet’s Formula

\[ F_1 = F_2 = 1; \quad F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{-1+\sqrt{5}}{2} \right)^n \right]. \]

- Recurrence relation: \( F_{n+1} = F_n + F_{n-1} \) (1)
- Generating function: \( g(x) = \sum_{n>0} F_n x^n. \)

\[(1) \Rightarrow \sum_{n\geq 2} F_{n+1} x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1} x^{n+1} \]

\[ \Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \]

\[ \Rightarrow \sum_{n\geq 3} F_n x^n = x \sum_{n\geq 2} F_n x^n + x^2 \sum_{n\geq 1} F_n x^n \]

\[ \Rightarrow g(x) - F_1 x - F_2 x^2 = x (g(x) - F_1 x) + x^2 g(x) \]

\[ \Rightarrow g(x) = \frac{x}{1 - x - x^2}. \]
Partial Fraction Expansion (Example: Binet’s Formula)

- Generating function: \( g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}. \)
Partial Fraction Expansion (Example: Binet’s Formula)

- Generating function: \( g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2} \).

- Partial fraction expansion:
Partial Fraction Expansion (Example: Binet’s Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.

- Partial fraction expansion:

$$
\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} x - \frac{-1+\sqrt{5}}{2} x \right).
$$
Partial Fraction Expansion (Example: Binet’s Formula)

- **Generating function**: \( g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2} \).

- **Partial fraction expansion**:

  \[
  \Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left( \frac{\frac{1+\sqrt{5}}{2} x}{1 - \frac{1+\sqrt{5}}{2} x} - \frac{-\frac{1+\sqrt{5}}{2} x}{1 - \frac{-1+\sqrt{5}}{2} x} \right).
  \]

**Coefficient of \( x^n \) (power series expansion)**:

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( -\frac{1+\sqrt{5}}{2} \right)^n \right] \quad \text{- Binet’s Formula!}
\]

(using geometric series: \( \frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots \)).
Differentiating Identities and Method of Moments

- **Differentiating identities**
  Example: Given a random variable $X$ such that
  \[
  \text{Prob}(X = 1) = \frac{1}{2}, \quad \text{Prob}(X = 2) = \frac{1}{4}, \quad \text{Prob}(X = 3) = \frac{1}{8}, \ldots ,
  \]
  then what’s the mean of $X$ (i.e., $E[X]$)?
  
  **Solution:** Let $f(x) = \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots = \frac{1}{1-x/2} - 1$.
  
  \[
  f'(x) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}x + 3 \cdot \frac{1}{8}x^2 + \cdots.
  \]
  
  \[
  f'(1) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots = E[X].
  \]

- **Method of moments:** Random variables $X_1$, $X_2$, 
  
  If the $\ell^{\text{th}}$ moment $E[X_1^\ell]$ converges to that of the standard normal distribution ($\forall \ell$), then $X_n$ converges to a Gaussian.

  **Standard normal distribution:**
  
  $2m^{\text{th}}$ moment: $(2m - 1)!! = (2m - 1)(2m - 3)\ldots 1$,
  
  $(2m - 1)^{\text{th}}$ moment: 0.
New Approach: Case of Fibonacci Numbers

\[ \rho_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} . \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}) : N = F_{n+1} + F_t + \cdots , t \leq n - 1 . \]
  \[ \rho_{n+1,k+1} = \rho_{n-1,k} + \rho_{n-2,k} + \cdots \]
New Approach: Case of Fibonacci Numbers

\( \rho_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} \).

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  \]
  \[
  \rho_{n+1,k+1} = \rho_{n-1,k} + \rho_{n-2,k} + \cdots
  \]
  \[
  \rho_{n,k+1} = \rho_{n-2,k} + \rho_{n-3,k} + \cdots
  \]
New Approach: Case of Fibonacci Numbers

\[ \rho_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \}. \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}) : N = F_{n+1} + F_t + \cdots, t \leq n - 1. \]
  \[ \rho_{n+1,k+1} = \rho_{n-1,k} + \rho_{n-2,k} + \cdots \]
  \[ \rho_{n,k+1} = \rho_{n-2,k} + \rho_{n-3,k} + \cdots \]
  \[ \Rightarrow \rho_{n+1,k+1} = \rho_{n,k+1} + \rho_{n-1,k}. \]
New Approach: Case of Fibonacci Numbers

\[ p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} . \]

- **Recurrence relation:**
  \[ N \in [F_{n+1}, F_{n+2}) : N = F_{n+1} + F_t + \cdots, \ t \leq n - 1. \]
  \[ p_{n+1,k+1} = p_{n-1,k} + p_{n-2,k} + \cdots \]
  \[ p_{n,k+1} = p_{n-2,k} + p_{n-3,k} + \cdots \]
  \[ \Rightarrow p_{n+1,k+1} = p_{n,k+1} + p_{n-1,k}. \]

- **Generating function:**
  \[ \sum_{n,k>0} p_{n,k} x^k y^n = \frac{y}{1 - y - xy^2}. \]

- **Partial fraction expansion:**
  \[ \frac{y}{1 - y - xy^2} = -\frac{y}{y_1(x) - y_2(x)} \left( \frac{1}{y - y_1(x)} - \frac{1}{y - y_2(x)} \right) \]
  where \( y_1(x) \) and \( y_2(x) \) are the roots of \( 1 - y - xy^2 = 0. \)

**Coefficient of \( y^n \):**
\[ g(x) = \sum_{k>0} p_{n,k} x^k. \]
New Approach: Case of Fibonacci Numbers (Continued)

\(K_n\): the corresponding random variable associated with \(k\).

\[g(x) = \sum_{k>0} p_{n,k} x^k.\]

- **Differentiating identities:**
  
  \[g(1) = \sum_{k>0} p_{n,k} = F_{n+1} - F_n,\]
  
  \[g'(x) = \sum_{k>0} kp_{n,k} x^{k-1}, \quad g'(1) = g(1)E[K_n],\]
  
  \[(xg'(x))' = \sum_{k>0} k^2 p_{n,k} x^{k-1},\]
  
  \[(xg'(x))' \big|_{x=1} = g(1)E[K_n^2], \quad (x (xg'(x))')' \big|_{x=1} = g(1)E[K_n^3], \ldots\]

  Similar results hold for the centralized \(K_n\): \(K_n' = K_n - E[K_n]\).

- **Method of moments** (for normalized \(K_n'\)):
  
  \[E[(K_n')^{2m}] / (SD(K_n'))^{2m} \to (2m-1)!!,\]
  
  \[E[(K_n')^{2m-1}] / (SD(K_n'))^{2m-1} \to 0. \quad \Rightarrow K_n \to \text{Gaussian}.\]
New Approach: General Case

Let $\rho_{n,k} = \# \{N \in [H_n, H_{n+1}) : \text{the generalized Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\}$. 

- **Recurrence relation:**
  - Fibonacci: $\rho_{n+1,k+1} = \rho_{n,k+1} + \rho_{n,k}$.  
  - General: $\rho_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \rho_{n-m,k-j}$. 
  - where $s_0 = 0$, $s_m = c_1 + c_2 + \cdots + c_m$.

- **Generating function:**
  - Fibonacci: $\frac{y}{1-y-xy^2}$. 
  - General: 
    \[
    \sum_{n \leq L} \rho_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n \leq L-m} \rho_{n,k} x^k y^n \\
    \frac{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}.
    \]
New Approach: General Case (Continued)

- **Partial fraction expansion:**

  **Fibonacci:** \[- \frac{y}{y_1(x) - y_2(x)} \left( \frac{1}{y - y_1(x)} - \frac{1}{y - y_2(x)} \right).\]

  **General:** \[- \frac{1}{\sum_{j=s_{L-1}}^{s_{L-1}} x^j} \sum_{i=1}^{L} \frac{B(x, y)}{(y - y_i(x)) \prod_{j \neq i} (y_j(x) - y_i(x))}.\]

  \[B(x, y) = \sum_{n \leq L} \sum_{k \geq 0} \rho_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} \rho_{n,k} x^k y^n,\]

  \[y_i(x): \text{root of } 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} = 0.\]

  **Coefficient of** \(y^n\): \[g(x) = \sum_{n,k>0} \rho_{n,k} x^k.\]

- **Differentiating identities**

- **Method of moments** \(\Rightarrow K_n \rightarrow \text{Gaussian}\).
Gaps Between Summands
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$. 
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$. 
Distribution of Gaps

For \( F_{i_1} + F_{i_2} + \cdots + F_{i_n} \), the gaps are the differences
\[ i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1. \]

Example: For \( F_1 + F_8 + F_{18} \), the gaps are 7 and 10.

Let \( P_n(k) \) be the probability that a gap for a decomposition in
\([F_n, F_{n+1})\) is of length \( k \).

What is \( P(k) = \lim_{n \to \infty} P_n(k) \)?
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$.

What is $P(k) = \lim_{n \to \infty} P_n(k)$?

Can ask similar questions about binary or other expansions: $2011 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0$. 
Main Results (Beckwith-Miller 2011)

**Theorem (Base B Gap Distribution)**

For base $B$ decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$,

$$P(k) = c_B B^{-k}, \text{ with } c_B = \frac{(B-1)(3B-2)}{B^2}.$$

**Theorem (Zeckendorf Gap Distribution)**

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ for $k \geq 2$, with

$$\phi = \frac{1+\sqrt{5}}{2} \text{ the golden mean.}$$
Proof of Fibonacci Result

Lekkerkerker ⇒ total number of gaps \( \sim F_{n-1} \frac{n}{\phi^2 + 1} \).
Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1} \frac{n}{\phi^2 + 1}$.

Let $x_{i,j} = \#\{m \in [F_n, F_{n+1}]):$ decomposition of $m$ includes $F_i, F_j$, but not $F_q$ for $i < q < j\}$. 
Proof of Fibonacci Result

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1} \frac{n}{\phi^2 + 1}$.

Let $x_{i,j} = \# \{ m \in [F_n, F_{n+1}) : \text{decomposition of } m \text{ includes } F_i, F_j, \text{ but not } F_q \text{ for } i < q < j \}$. 

$$P(k) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n-k} x_{i,i+k}}{F_{n-1} \frac{n}{\phi^2 + 1}}.$$
Calculating $x_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?
Calculating $x_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

$1 \leq i \leq n - k - 2$: 
Calculating $x_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

$1 \leq i \leq n - k - 2$:

For the indices less than $i$: $F_{i-1}$ choices.

For the indices greater than $i + k$: $F_{n-k-2-i}$ choices.
Calculating $x_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

$1 \leq i \leq n - k - 2$:

For the indices less than $i$: $F_{i-1}$ choices.

For the indices greater than $i + k$: $F_{n-k-2-i}$ choices.

So total choices number of choices is $F_{n-k-2-i}F_{i-1}$. 
Calculating $x_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

$1 \leq i \leq n - k - 2$:

For the indices less than $i$: $F_{i-1}$ choices.

For the indices greater than $i + k$: $F_{n-k-2-i}$ choices.

So total choices number of choices is $F_{n-k-2-i}F_{i-1}$.

$i = n - k - 1$: 0.

$i = n - k$: $F_{n-k-1}$. 
\[ \sum_{i=1}^{n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2} \]

\[ \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3} \text{ is the } x^{n-k-3} \text{ coefficient of } (g(x))^2! \]
\[ \sum_{i=1}^{n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1} F_{n-k-i-2} \]

\[ \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3} \] is the \( x^{n-k-3} \) coefficient of \((g(x))^2!\)

\[ g(x)^2 = \left( \frac{1}{\sqrt{5}} \left( \frac{1}{(1-\phi x)} - \frac{1}{(1-(1-\phi)x)} \right) \right)^2 \]
\[ \sum_{i=1}^{n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1} F_{n-k-i-2} \]

\[ \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3} \text{ is the } x^{n-k-3} \text{ coefficient of } (g(x))^2! \]

\[ g(x)^2 = \left( \frac{1}{\sqrt{5}} \left( \frac{1}{1-\phi x} - \frac{1}{(1-(1-\phi)x)} \right) \right)^2 \]

\[ (g(x))^2 = \frac{A}{(1-\phi x)^2} + \frac{B}{(1-x(1-\phi))^2} + \frac{C}{1-\phi x} + \frac{D}{1-(1-\phi)x} \]
\[ \sum_{i=1}^{n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2} \]

\[ \sum_{i=0}^{n-k-3} F_iF_{n-k-i-3} \text{ is the } x^{n-k-3} \text{ coefficient of } (g(x))^2! \]

\[ g(x)^2 = \left( \frac{1}{\sqrt{5}} \left( \frac{1}{(1-\phi x)} - \frac{1}{(1-(1-\phi)x)} \right) \right)^2 \]

\[ (g(x))^2 = \frac{A}{(1-\phi x)^2} + \frac{B}{(1-x(1-\phi))^2} + \frac{C}{1-\phi x} + \frac{D}{1-(1-\phi)x} \]

Using geometric series / differentiating identities, the \( x^m \) coefficient is

\[ A(m+1)\phi^m + B(m+1)(1-\phi)^m + C\phi^m + D(1-\phi)^m. \]
Consider the ratio:

\[
\frac{P(k+1)}{P(k)} = \lim_{n \to \infty} \frac{P_n(k+1)}{P_n(k)}
\]

\[
= \lim_{n \to \infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}
\]

\[
= \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}}
\]

\[
= \frac{A(n-k-3)\phi^{n-k-4}}{A(n-k-2)\phi^{n-k-3}} = \frac{1}{\phi}.
\]
Consider the ratio:

\[
\frac{P(k + 1)}{P(k)} = \lim_{n \to \infty} \frac{P_n(k + 1)}{P_n(k)}
\]

\[
= \lim_{n \to \infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}
\]

\[
= \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-2)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}}
\]

\[
= \frac{A(n-k-3)\phi^{n-k-4}}{A(n-k-2)\phi^{n-k-3}} = \frac{1}{\phi}.
\]

\[
P(k) = \frac{C}{\phi^k}, \text{ for some constant } C.
\]
Consider the ratio:

\[
\frac{P(k+1)}{P(k)} = \lim_{n \to \infty} \frac{P_n(k+1)}{P_n(k)} = \lim_{n \to \infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}
\]

\[
= \frac{F_{n-k-2} + A(n - k - 3)\phi^{n-k-4} + B(n - k - 3)(1 - \phi)^{n-k-4} + C\phi^{n-k-4} + D(1 - \phi)^{n-k-4}}{F_{n-k-1} + A(n - k - 2)\phi^{n-k-3} + B(n - k - 2)(1 - \phi)^{n-k-3} + C\phi^{n-k-3} + D(1 - \phi)^{n-k-3}}
\]

\[
= \frac{A(n - k - 3)\phi^{n-k-4}}{A(n - k - 2)\phi^{n-k-3}} = \frac{1}{\phi}.
\]

\[
P(k) = \frac{C}{\phi^k}, \text{ for some constant } C.
\]

\[
1 = C \sum_{k=2}^{\infty} \frac{1}{\phi^k} = \frac{C}{\phi^2} \sum_{k=0}^{\infty} \frac{1}{\phi^k} = \frac{C}{\phi^2(1 - \frac{1}{\phi})} \quad \rightarrow \quad C = \phi(\phi - 1).
\]
Consider the ratio:

\[
\frac{P(k+1)}{P(k)} = \lim_{n \to \infty} \frac{P_n(k+1)}{P_n(k)}
\]

\[
= \lim_{n \to \infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}
\]

\[
= \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}}
\]

\[
= \frac{A(n-k-3)\phi^{n-k-4}}{A(n-k-2)\phi^{n-k-3}} = \frac{1}{\phi}.
\]

\[P(k) = \frac{C}{\phi^k}, \text{ for some constant } C.\]

\[1 = C \sum_{k=2}^{\infty} \frac{1}{\phi^k} = \frac{C}{\phi^2} \sum_{k=0}^{\infty} \frac{1}{\phi^k} = \frac{C}{\phi^2(1-\frac{1}{\phi})} \quad \rightarrow \quad C = \phi(\phi - 1).\]

\[P(k) = \frac{\phi(\phi-1)}{\phi^k}.\]
Future Research
Further Research

1. Are there similar results for linearly recursive sequences with arbitrary integer coefficients (i.e. negative coefficients are allowed in the defining relation)?

2. Lekkerkerker’s theorem, and the Gaussian extension, are for the behavior in intervals \([F_n, F_{n+1})\). Do the limits exist if we consider other intervals, say \([F_n + g_1(F_n), F_n + g_2(F_n))\) for some functions \(g_1\) and \(g_2\)?

3. For the generalized recurrence relations, what happens if instead of looking at \(\sum_{i=1}^{n} a_i\) we study \(\sum_{i=1}^{n} \min(1, a_i)\)? In other words, we only care about how many distinct \(H_i\)’s occur in the decomposition.

4. What else can we say about the distribution of the gaps / largest gap between summands in the Zeckendorf decomposition? Appropriately normalized, how do they behave?
Appendix:
Combinatorial Identities and Lekkerkerker’s Theorem
Needed Binomial Identity

Binomial identity involving Fibonacci Numbers

Let $F_m$ denote the $m^{th}$ Fibonacci number, with $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$ and so on. Then

$$\left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{n-1-k} \binom{n-1-k}{k} = F_{n-1}.$$

**Proof by induction:** The base case is trivially verified. Assume our claim holds for $n$ and show that it holds for $n + 1$. We may extend the sum to $n - 1$, as $\binom{n-1-k}{k} = 0$ whenever $k > \left\lfloor \frac{n-1}{2} \right\rfloor$. Using the standard identity that

$$\binom{m}{\ell} + \binom{m}{\ell+1} = \binom{m+1}{\ell+1},$$

and the convention that $\binom{m}{\ell} = 0$ if $\ell$ is a negative integer, we find

$$\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=0}^{n} \left[ \binom{n-1-k}{k-1} + \binom{n-1-k}{k} \right]$$

$$= \sum_{k=1}^{n} \binom{n-1-k}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k}$$

$$= \sum_{k=1}^{n} \binom{n-2-(k-1)}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-2} + F_{n-1}$$

by the inductive assumption; noting $F_{n-2} + F_{n-1} = F_n$ completes the proof.
Preliminaries for Lekkerkerker’s Theorem

\[ E(n) := \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \binom{n-1-k}{k}. \]

Average number of summands in \([F_n, F_{n+1})\) is

\[ \frac{E(n)}{F_{n-1}} + 1. \]

Recurrence Relation for \(E(n)\)

\[ E(n) + E(n-2) = (n-2)F_{n-3}. \]
Recurrence Relation

Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n - 2) = (n - 2)F_{n-3}.$$ 

Proof by algebra (details later):

$$\mathcal{E}(n) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} k \binom{n-1-k}{k}$$

$$= (n - 2) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell \binom{n-3-\ell}{\ell}$$

$$= (n - 2)F_{n-3} - \mathcal{E}(n - 2).$$
## Formula for $E(n)$ (i.e., Lekkerkerker’s Theorem)

$$E(n) = \frac{nF_{n-1}}{\varphi^2 + 1} + O(F_{n-2}).$$

$$\left\lfloor \frac{n-3}{2} \right\rfloor \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (E(n - 2\ell) + E(n - 2(\ell + 1)))$$

$$= \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (n - 2 - 2\ell) F_{n-3-2\ell}.$$

Result follows from Binet’s formula, the geometric series formula, and differentiating identities: $\sum_{j=0}^{m} jx^j = x^{m+1}(x-1) - \frac{x^m(x^m - 1)}{(x-1)^2}$. Details later in the appendix.
Derivation of Recurrence Relation for $\mathcal{E}(n)$

$$
\mathcal{E}(n) = \frac{1}{2} \sum_{k=0}^{n-1} k \binom{n-1-k}{k}
$$

$$
= \frac{1}{2} \sum_{k=1}^{n-1} k \frac{(n-1-k)!}{k!(n-1-2k)!}
$$

$$
= \frac{1}{2} \sum_{k=1}^{n-1} (n-1-k) \frac{(n-2-k)!}{(k-1)!(n-1-2k)!}
$$

$$
= \frac{1}{2} \sum_{k=1}^{n-1} (n-2-(k-1)) \frac{(n-3-(k-1))!}{(k-1)!(n-3-2(k-1))!}
$$

$$
= \frac{1}{2} \sum_{\ell=0}^{n-3} (n-2-\ell) \binom{n-3-\ell}{\ell}
$$

$$
= (n-2) \sum_{\ell=0}^{n-3} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{n-3} \ell \binom{n-3-\ell}{\ell}
$$

$$
= (n-2)F_{n-3} - \mathcal{E}(n-2),
$$

which proves the claim (note we used the binomial identity to replace the sum of binomial coefficients with a Fibonacci number).
**Formula for $\mathcal{E}(n)$**

\[ \mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^2 + 1} + O(F_{n-2}). \]

**Proof:** The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by inputting Binet’s formula and differentiating identities. Explicitly, consider

\[
\left\lfloor \frac{n-3}{2} \right\rfloor \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (\mathcal{E}(n - 2\ell) + \mathcal{E}(n - 2(\ell + 1))) = \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (n - 2 - 2\ell)F_{n-3-2\ell}
\]

\[
= \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (n - 3 - 2\ell)F_{n-3-2\ell} + \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (2\ell)F_{n-3-2\ell}
\]

\[
= \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} (-1)^\ell (n - 3 - 2\ell)F_{n-3-2\ell} + O(F_{n-2});
\]

while we could evaluate the last sum exactly, trivially estimating it suffices to obtain the main term (as we have a sum of every other Fibonacci number, the sum is at most the next Fibonacci number after the largest one in our sum).
We now use Binet’s formula to convert the sum into a geometric series. Letting \( \varphi = \frac{1+\sqrt{5}}{2} \) be the golden mean, we have

\[
F_n = \frac{\varphi}{\sqrt{5}} \cdot \varphi^n - \frac{1-\varphi}{\sqrt{5}} \cdot (1-\varphi)^n
\]

(our constants are because our counting has \( F_1 = 1, F_2 = 2 \) and so on). As \( |1 - \varphi| < 1 \), the error from dropping the \( (1 - \varphi)^n \) term is \( O(\sum_{\ell \leq n} n^2) = o(F_{n-2}) \), and may thus safely be absorbed in our error term. We thus find

\[
\mathcal{E}(n) = \frac{\varphi^{n-2}}{\sqrt{5}} \left[ (n-3) \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \left( -\varphi^{-2} \right)^{\ell} \left( -2 \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell(-\varphi^{-2})^{\ell} \right) \right] + O(F_{n-2}).
\]
We use the geometric series formula to evaluate the first term. We drop the upper boundary term of 
\(-\varphi^{-1}\left\lfloor \frac{n-3}{2} \right\rfloor\), as this term is negligible since \(\varphi > 1\). We may also move the 3 from the \(n - 3\) into the error term, and are left with

\[
\mathcal{E}(n) = \frac{\varphi^{n-2}}{\sqrt{5}} \left[ \frac{n}{1 + \varphi^{-2}} - 2 \sum_{\ell=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} \ell(-\varphi^{-2})^{\ell} \right] + O(F_{n-2})
\]

\[
= \frac{\varphi^{n-2}}{\sqrt{5}} \left[ \frac{n}{1 + \varphi^{-2}} - 2S\left(\left\lfloor \frac{n-3}{2} \right\rfloor, -\varphi^{-2}\right) \right] + O(F_{n-2}),
\]

where

\[
S(m, x) = \sum_{j=0}^{m} jx^{j}.
\]

There is a simple formula for \(S(m, x)\). As

\[
\sum_{j=0}^{m} x^{j} = \frac{x^{m+1} - 1}{x - 1},
\]

applying the operator \(x \frac{d}{dx}\) gives

\[
S(m, x) = \sum_{j=0}^{m} jx^{j} = x \frac{(m+1)x^{m}(x - 1) - (x^{m+1} - 1)}{(x - 1)^{2}} = \frac{mx^{m+2} - (m + 1)x^{m+1} + x}{(x - 1)^{2}}.
\]
Formula for $\mathcal{E}(n)$ (continued)

Taking $x = -\varphi^{-2}$, we see that the contribution from this piece may safely be absorbed into the error term $O(F_{n-2})$, leaving us with

$$
\mathcal{E}(n) = \frac{n\varphi^{n-2}}{\sqrt{5}(1 + \varphi^{-2})} + O(F_{n-2}) = \frac{n\varphi^n}{\sqrt{5}(\varphi^2 + 1)} + O(F_{n-2}).
$$

Noting that for large $n$ we have $F_{n-1} = \frac{\varphi^n}{\sqrt{5}} + O(1)$, we finally obtain

$$
\mathcal{E}(n) = \frac{nF_{n-1}}{\varphi^2 + 1} + O(F_{n-2}). \quad \Box
$$