Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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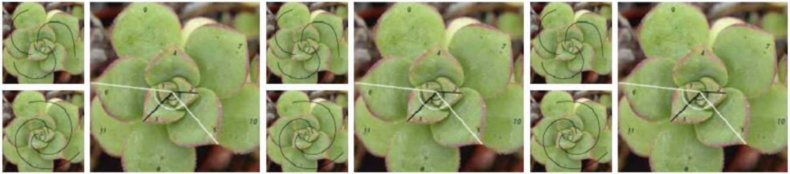
Introduction

Goals of the Talk

- Some linear recursions and decompositions.
- Uncover some of the secrets of gaps.
- Methods: Combinatorial vantage, Binet-like formulas.
- Specific open problems.



Thanks to my advisor and his colleagues from the Williams College 2010 and 2011 SMALL REU programs.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

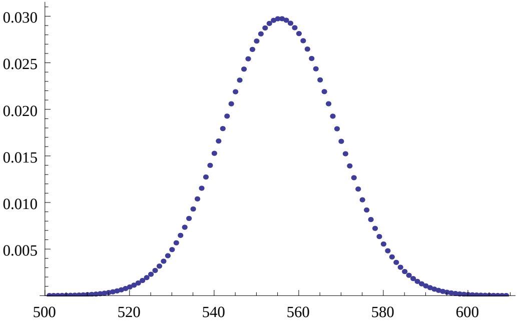

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results

Theorem (Zeckendorf Gap Distribution (BM))

For Zeckendorf decompositions, $P(k) = \frac{\varphi(\varphi-1)}{\varphi^k}$ for $k \geq 2$, with $\varphi = \frac{1+\sqrt{5}}{2}$ the golden mean.

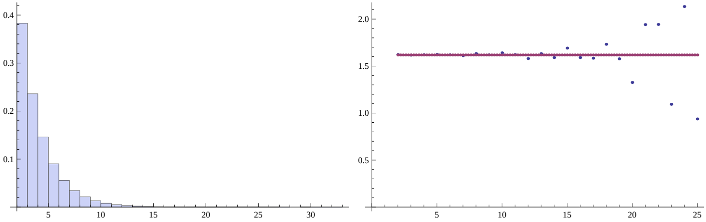

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

Preliminaries: The Cookie Problem

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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

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Let $p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

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For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \; i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \; d_j := i_j - i_{j-1} - 2 \; (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \; d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.

Generalizations

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Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1},\ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1,\ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).
- Lekkerkerker
- Central Limit Type Theorem

Generalizing Lekkerkerker

Generalized Lekkerkerker's Theorem

The average number of summands in the generalized Zeckendorf decomposition for integers in $[H_n, H_{n+1})$ tends to $Cn + d$ as $n \to \infty$, where $C > 0$ and d are computable constants determined by the c_i's.

$$C = -\frac{y'(1)}{y(1)} = \frac{\sum_{m=0}^{L-1}(s_m + s_{m+1} - 1)(s_{m+1} - s_m)y^m(1)}{2\sum_{m=0}^{L-1}(m+1)(s_{m+1} - s_m)y^m(1)}.$$

$$s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m.$$

$y(x)$ is the root of $1 - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}$.

$y(1)$ is the root of $1 - c_1 y - c_2 y^2 - \cdots - c_L y^L$.

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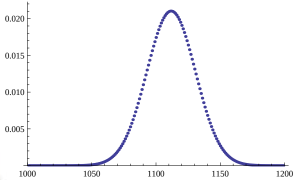
As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.



Gaps Between Summands

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

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Can ask similar questions about binary or other expansions: $2011 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0$.

Main Results (Beckwith-Miller 2011)

Theorem (Base B Gap Distribution)

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

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Lekkerkerker $\Rightarrow$ $Y(n) \sim F_{n-1}\frac{n}{\varphi^2+1}$.

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$$Y(n) = F_{n-1}\frac{n}{\varphi^2+1} + \text{smaller} = \frac{n}{\varphi^2+1} c_1 r_1^{n-1} + \text{smaller}$$

Finding $P(k)$

$$
\begin{aligned}
P(k) &= \lim_{n\to\infty} \frac{1}{Y(n)} \sum_{i=1}^{n-k} X_{i,i+k}(n) \\
&= \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} c_1^2 r_1^{n-k-3}}{\frac{n}{\varphi^2+1} c_1 r_1^{n-1}} \\
&= \frac{\varphi(\varphi-1)}{\varphi^k} \text{ for } k \geq 2
\end{aligned}
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Gaps for other Linear Recurrences

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Counting:

$$X_{i,i+k}(n) = \begin{cases} T_{i-1}(T_{n-i-3} + T_{n-i-4}) & \text{if } k = 1 \\ (T_{i-1} + T_{i-2})(T_{n-k-i-1} + T_{n-k-i-3}) & \text{if } k \geq 2 \end{cases}$$

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$$\textstyle\sum_{k=1}^{\infty} P(k) = 1 \Rightarrow C = c_1 \left(\frac{3\lambda_1^2 - 1}{(\lambda_1^2 - 1)\lambda_1^3} \right)$$

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Generalize to all positive linear recurrences?