# From Zombies to Fibonaccis: An Introduction to the Theory of Games

**Steven J. Miller, Williams College**

http://www.williams.edu/Mathematics/sjmiller/public_html

Manhattan Education: Summer 2019

$\sqrt{2}$

## $\sqrt{2}$ Is Irrational

Standard Proof: Assume $\sqrt{2} = a/b$.

WLOG, assume $b$ is the smallest denominator among all fractions that equal $\sqrt{2}$.

$2b^2 = a^2$ thus $a = 2m$ is even.

Then $2b^2 = 4m^2$ so $b^2 = 2m^2$ so $b = 2n$ is even.

Thus $\sqrt{2} = a/b = 2m/2n = m/n$, contradicts minimality of $n$.

(Could also do by contradiction from $a, b$ relatively prime.)

## Tennenbaum's Proof

Assume $\sqrt{2} = a/b$ with $b$ minimal.

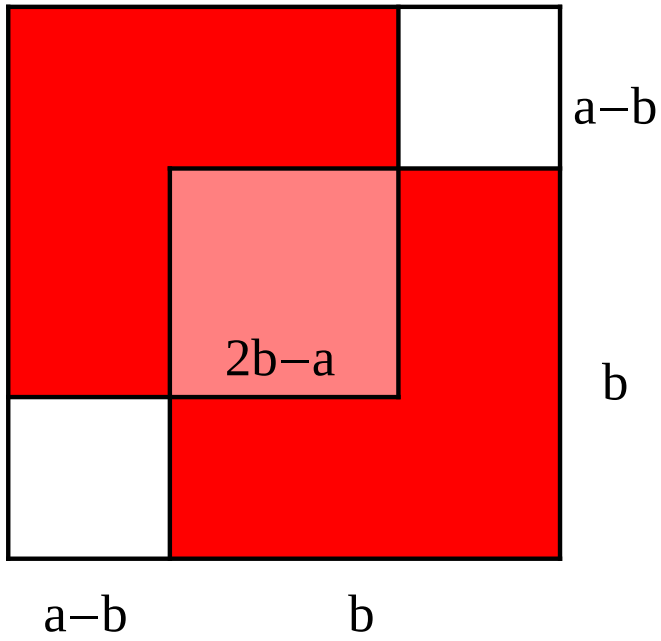



**Figure:** $2b^2 = a^2$ so $(2b-a)^2 = 2(a-b)^2$ and $\sqrt{2} = \frac{2b-a}{a-b}$.

As $0 < a - b < b$ (if not, $a - b \geq b$ so $a \geq 2b$ and $\sqrt{2} = a/b \geq 2$), contradicts minimality of $b$.

## Challenge

WHAT OTHER NUMBERS HAVE GEOMETRIC IRRATIONALITY PROOFS?

# Tic-Tac-Toe

# Tic-Tac-Toe

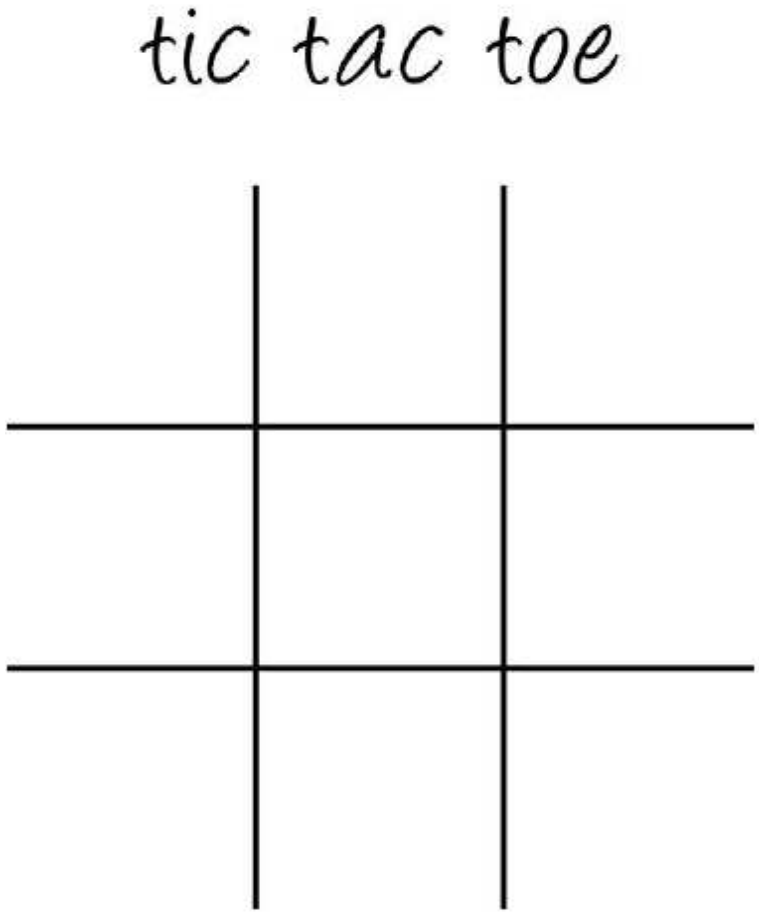



**Figure:** How many opening moves? How many first two moves?

# Tic-Tac-Toe: First Move

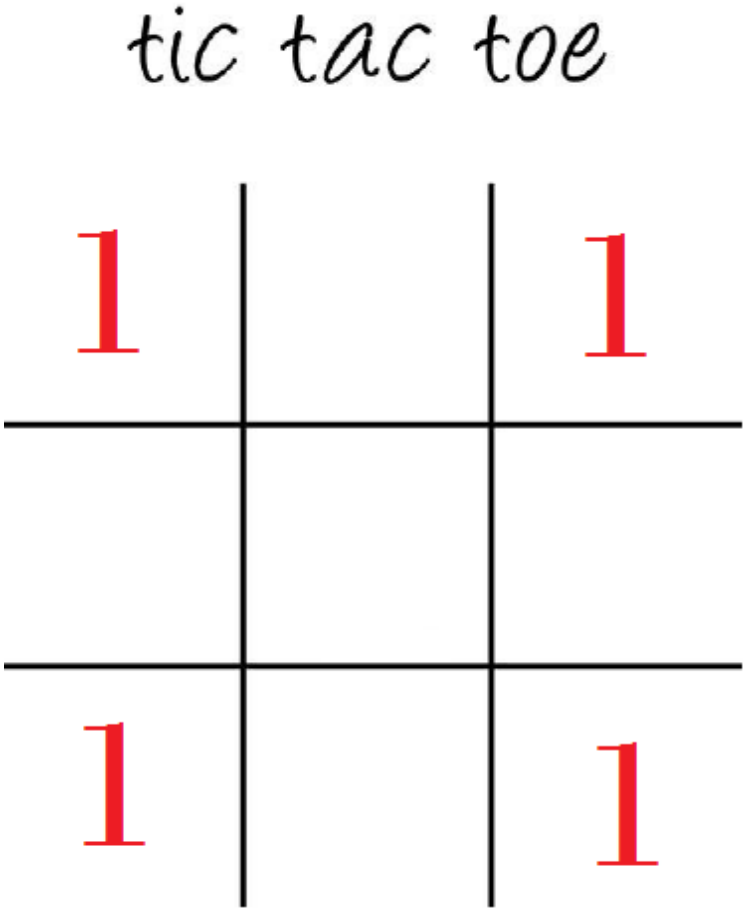



**Figure:** Analyzing Opening Moves: Corners all equivalent.

# Tic-Tac-Toe: First Move

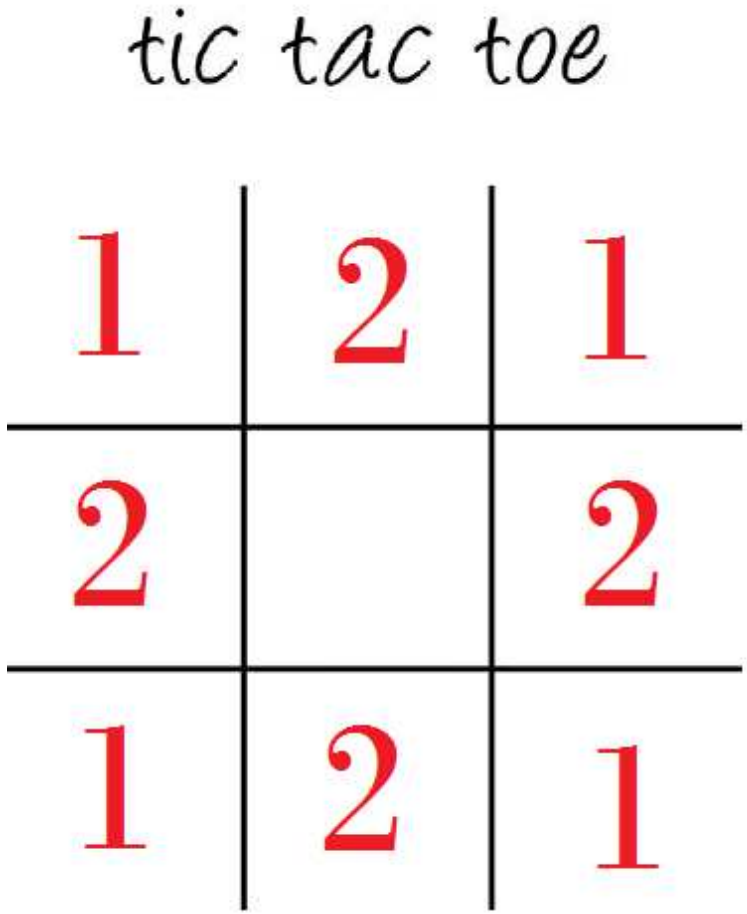



**Figure:** Analyzing Opening Moves: Middles all equivalent.

# Tic-Tac-Toe: First Move

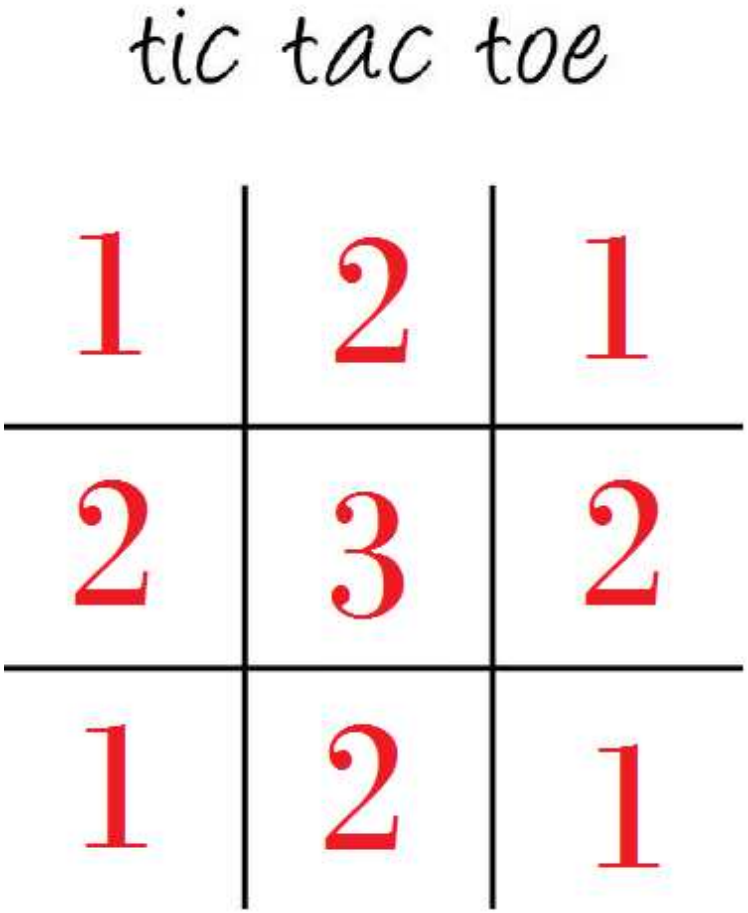



**Figure:** Analyzing Opening Moves: Only one center: 3 classes of moves.

# Tic-Tac-Toe: Second Move

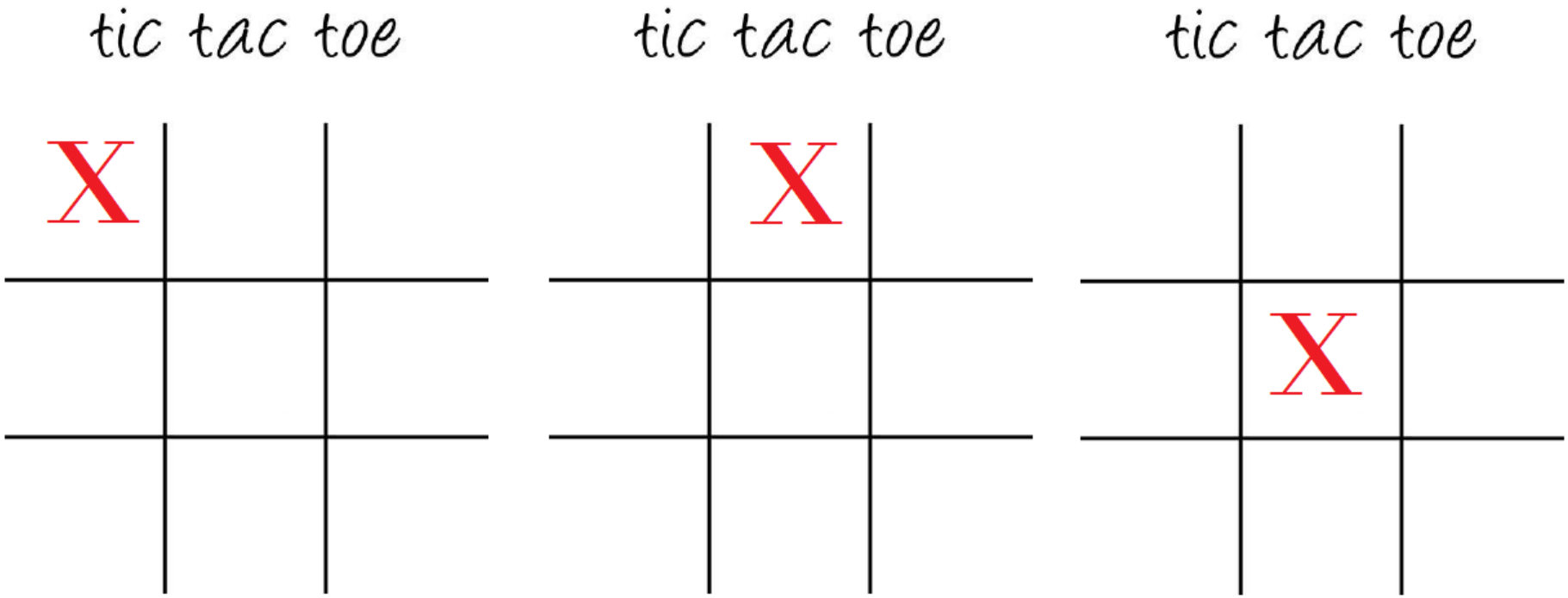



**Figure:** Analyzing Second Player Response.

# Tic-Tac-Toe: Second Move

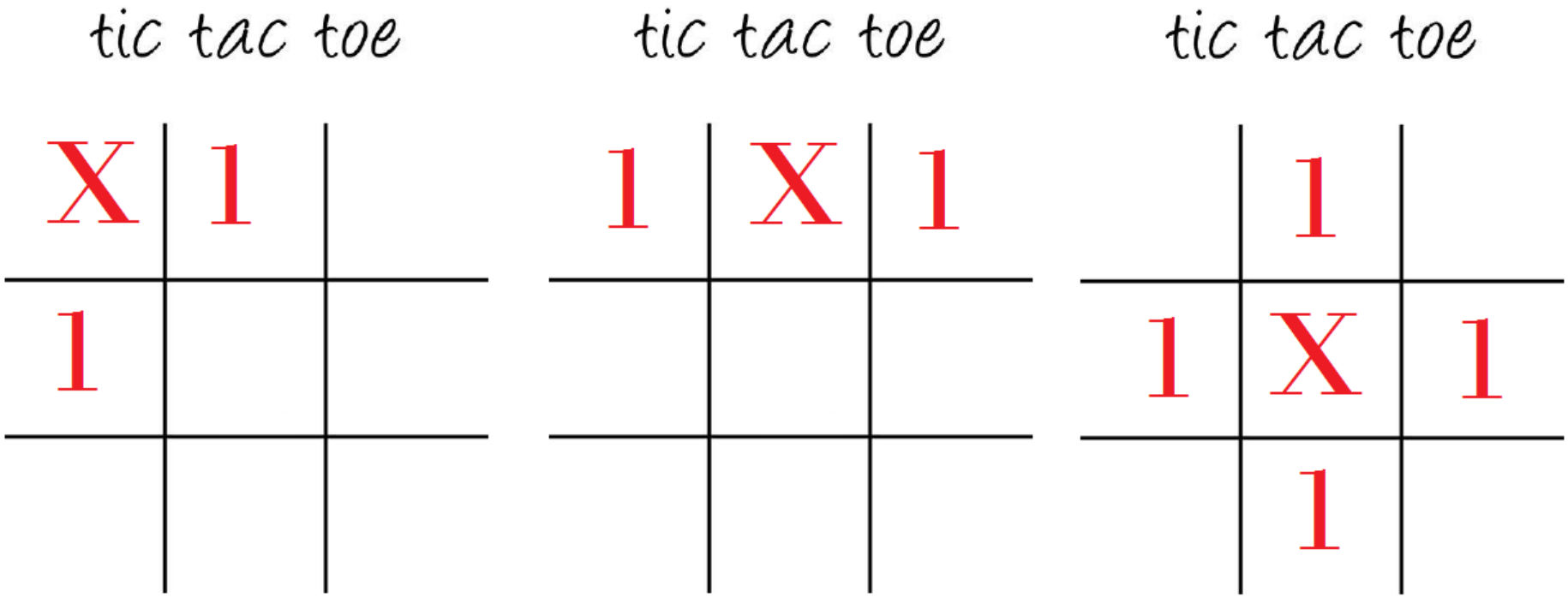



**Figure:** Analyzing Second Player Response.

# Tic-Tac-Toe: Second Move

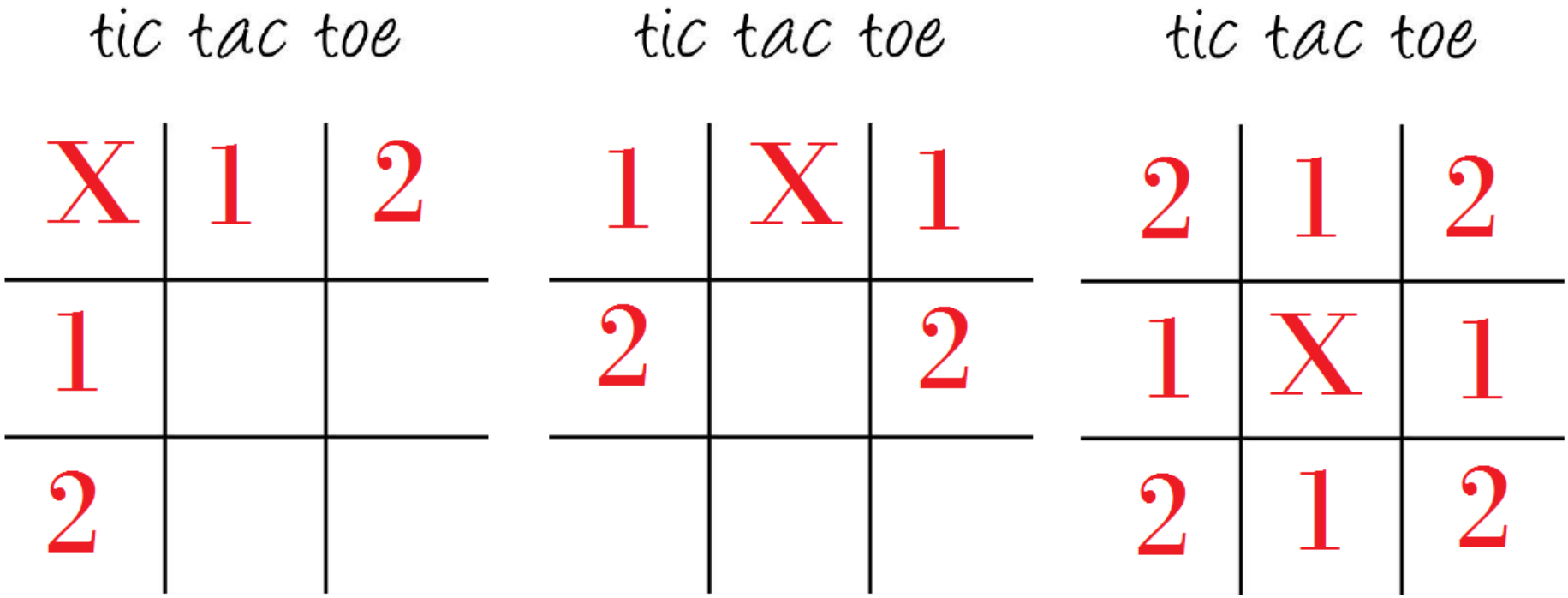



**Figure:** Analyzing Second Player Response.

# Tic-Tac-Toe: Second Move

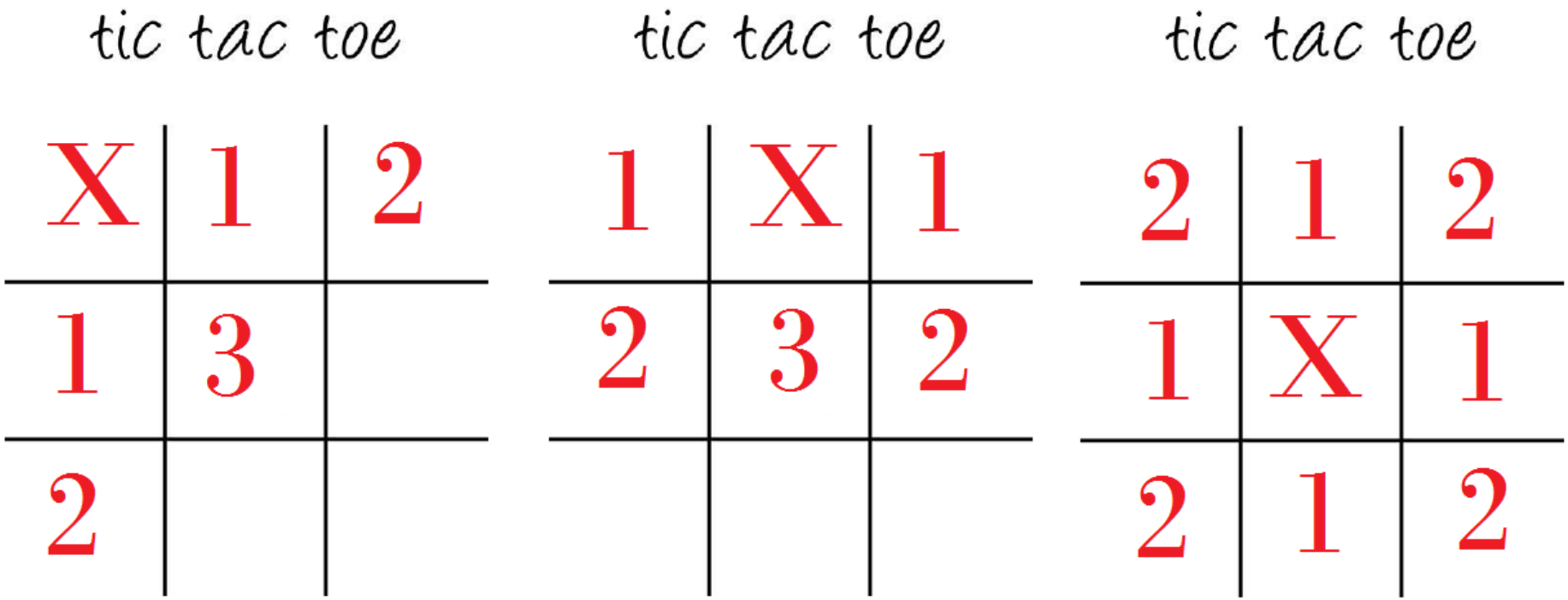



**Figure:** Analyzing Second Player Response.

# Tic-Tac-Toe: Second Move

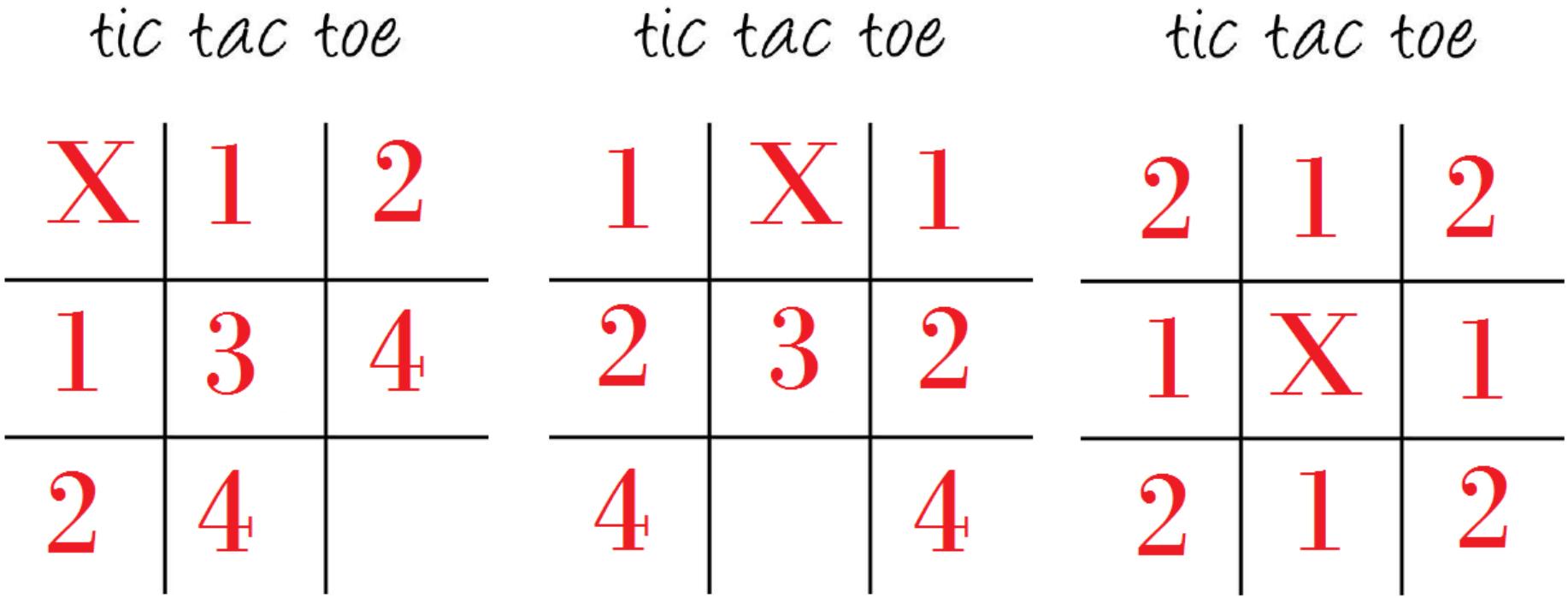



**Figure:** Analyzing Second Player Response.

# Tic-Tac-Toe: Second Move

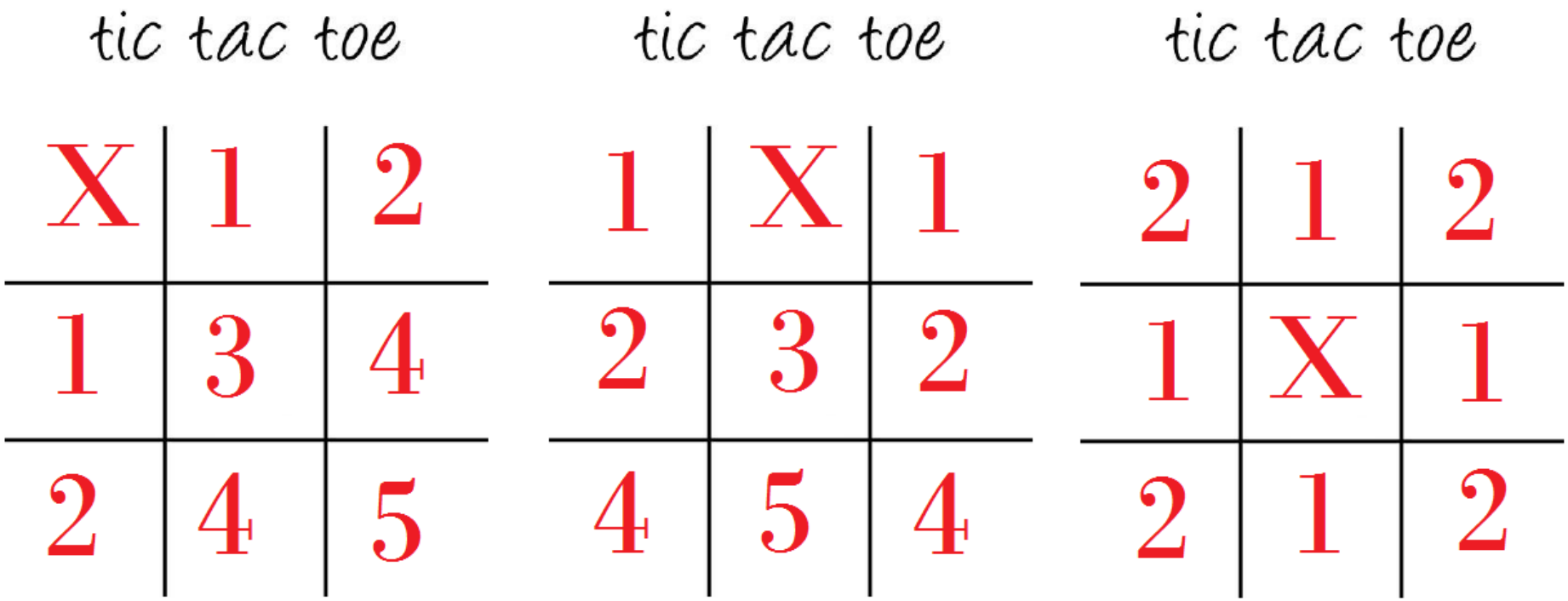



**Figure:** Analyzing Second Player Response: Thus there are 12 possible pairs of first two moves.

# Tic-Tac-Toe: Questions

- Does either player have a winning strategy?
- What happens if we play randomly? Chance of Player 1 winning?
- How can we make it interesting?

# Tic-Tac-Toe: Interesting Variants

- Gobble Tic-Tac-Toe
- Larger board (and handicaps)
- Tic-Tac-Toe in Tic-Tac-Toe
- Bidding Tic-Tac-Toe

# Tic-Tac-Toe in Tic-Tac-Toe

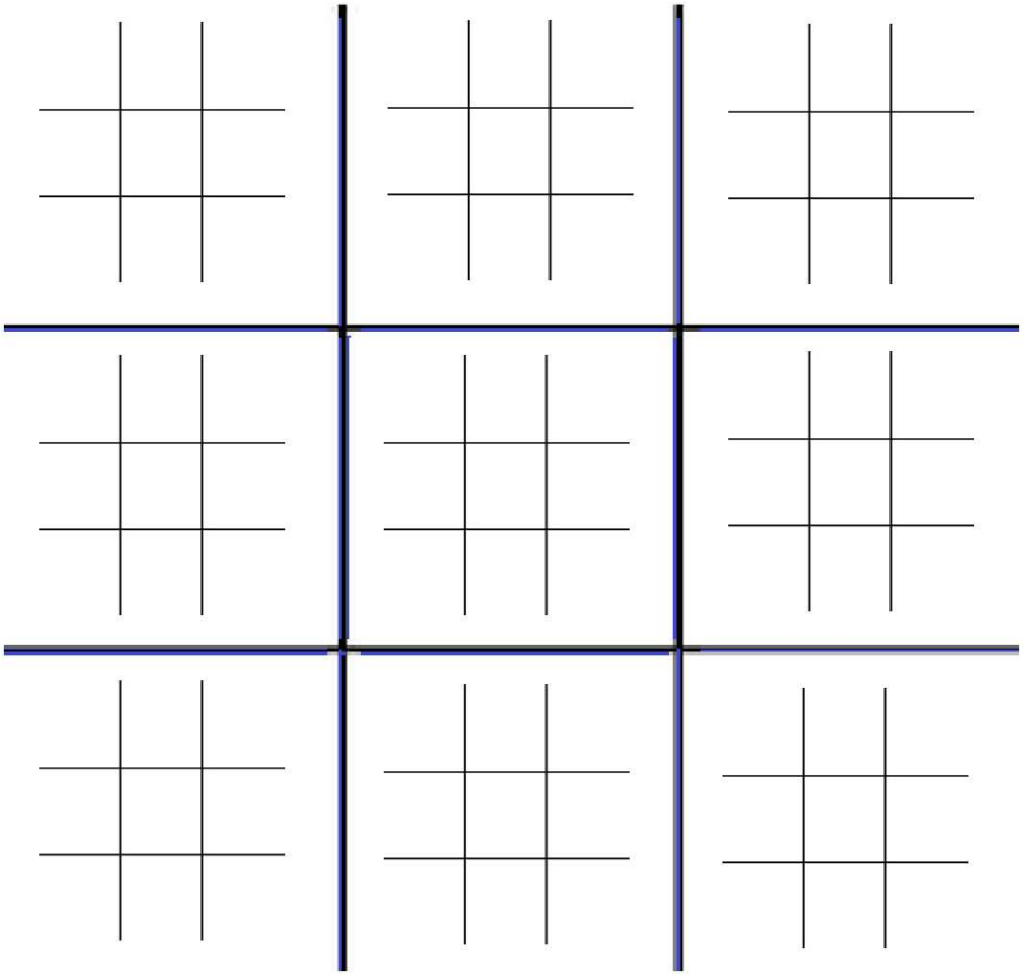


**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

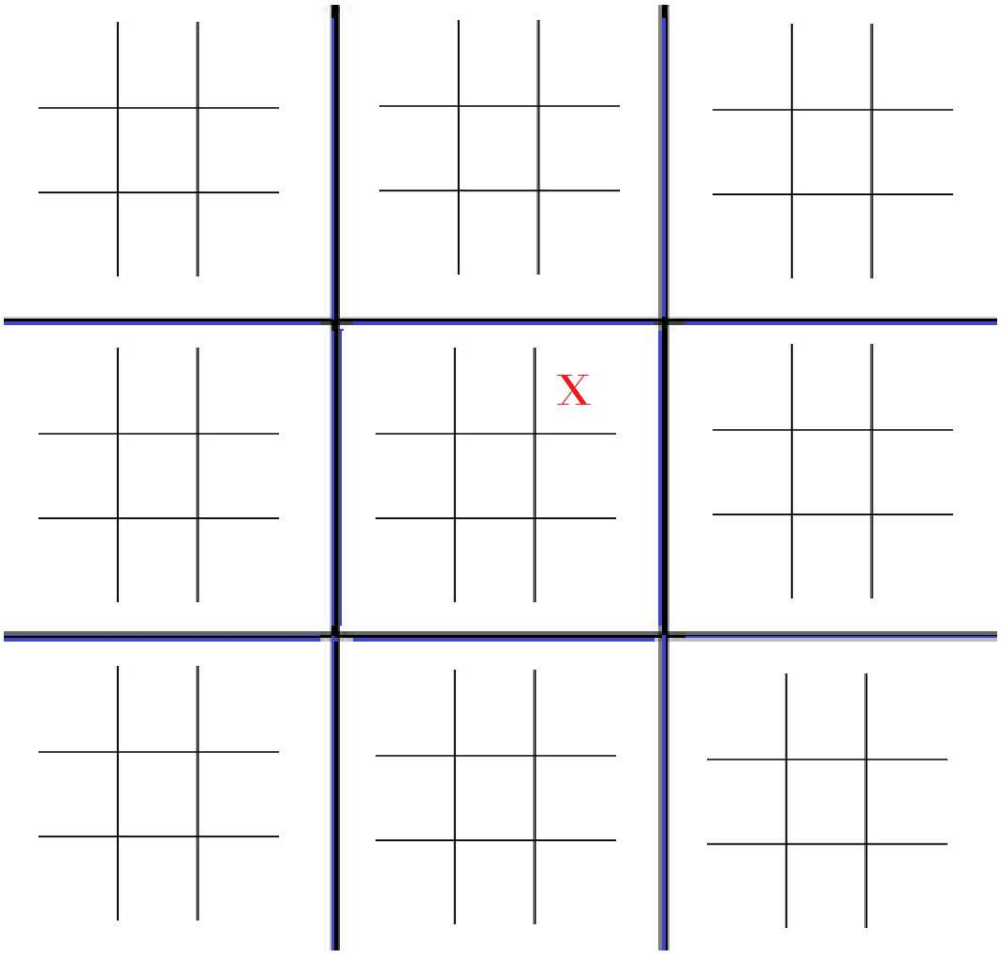



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

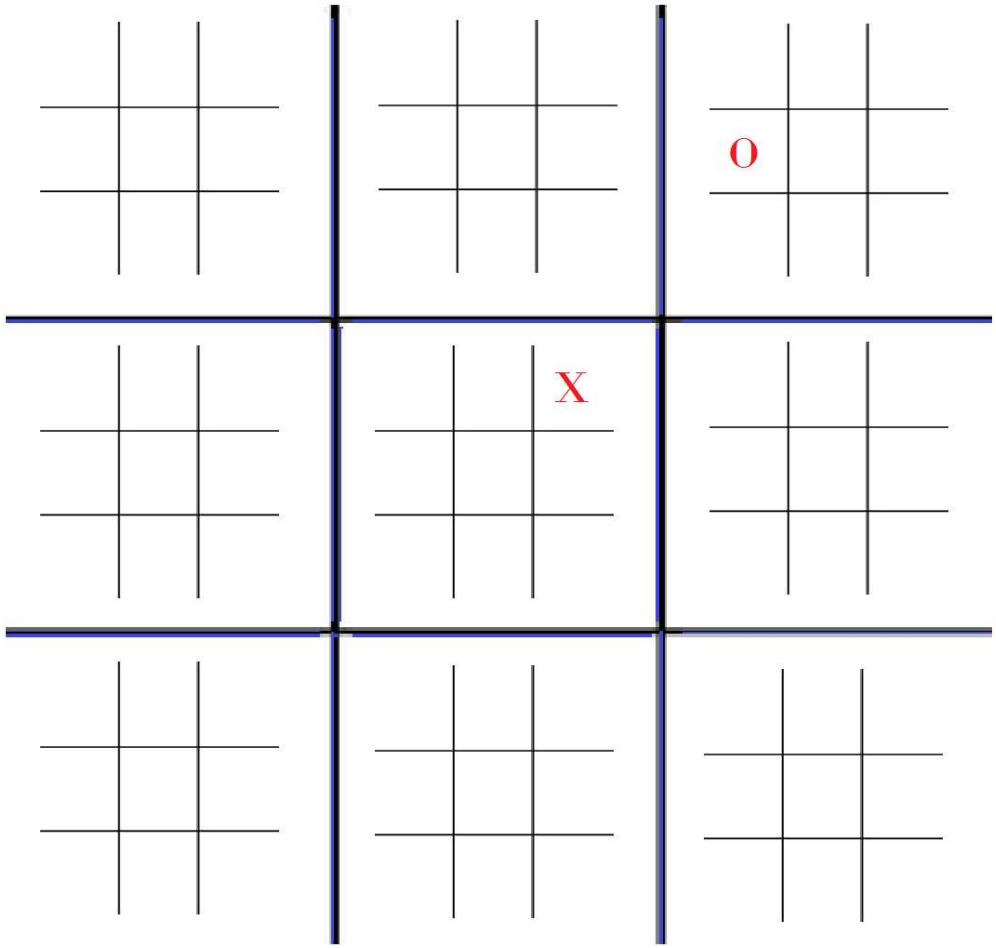



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

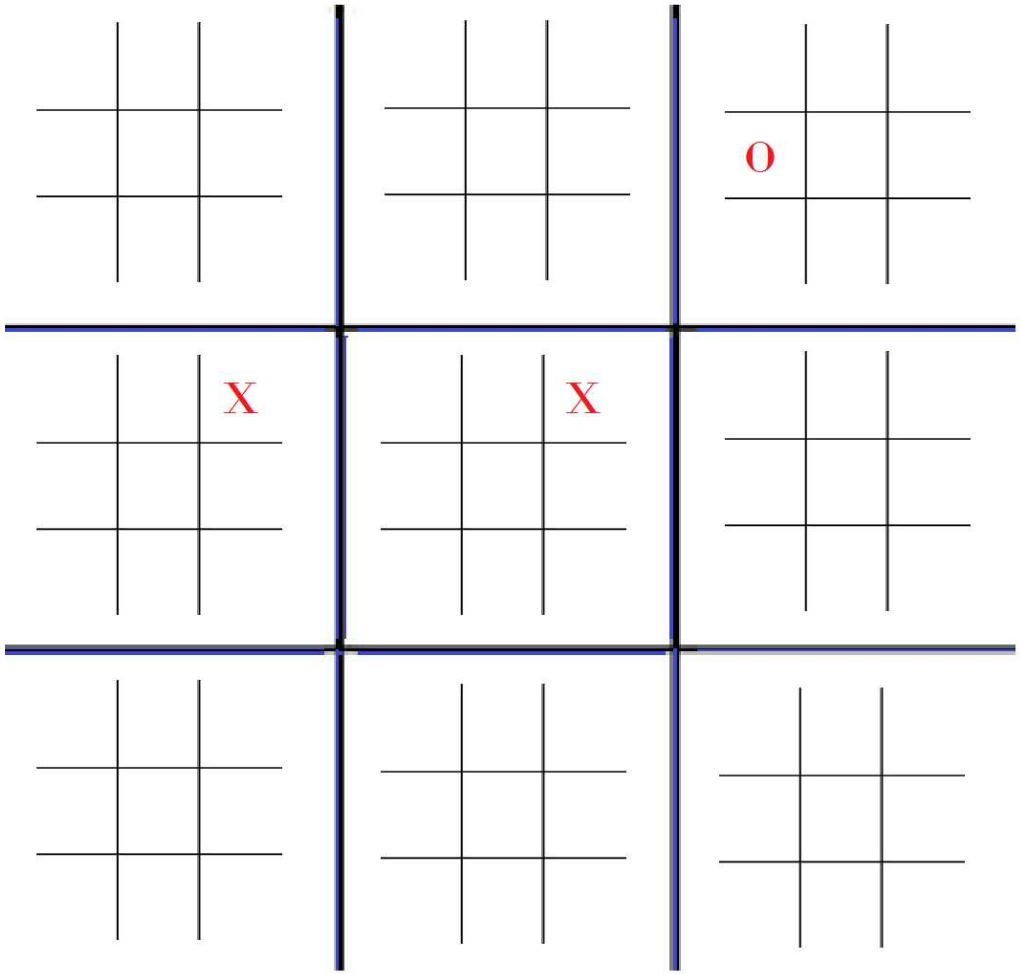



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

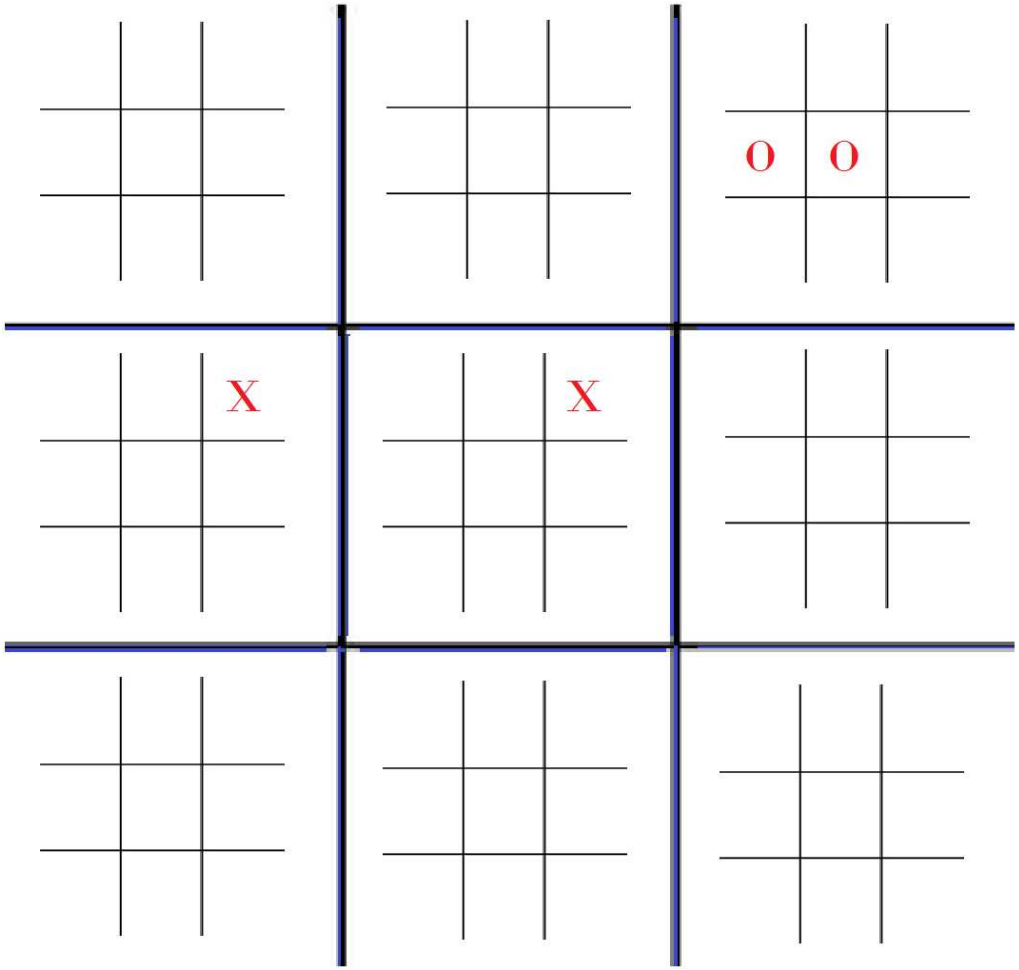



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

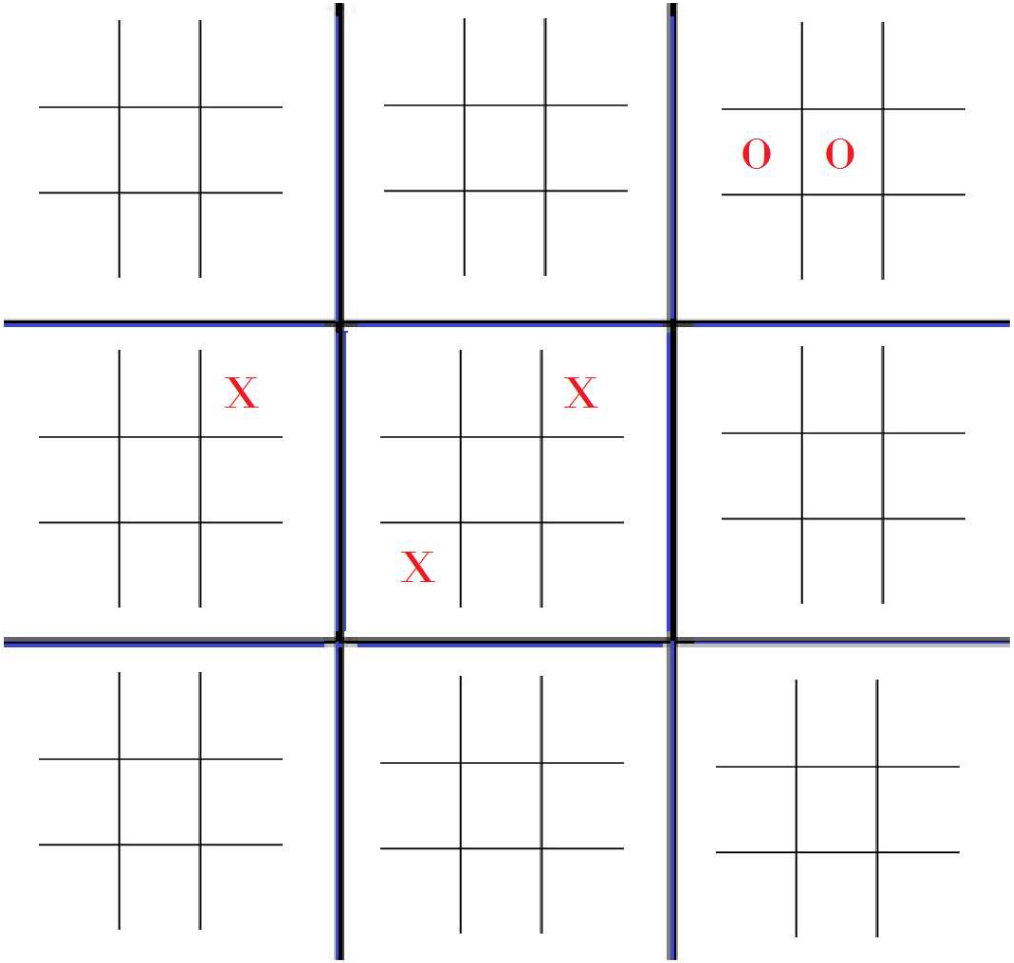



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

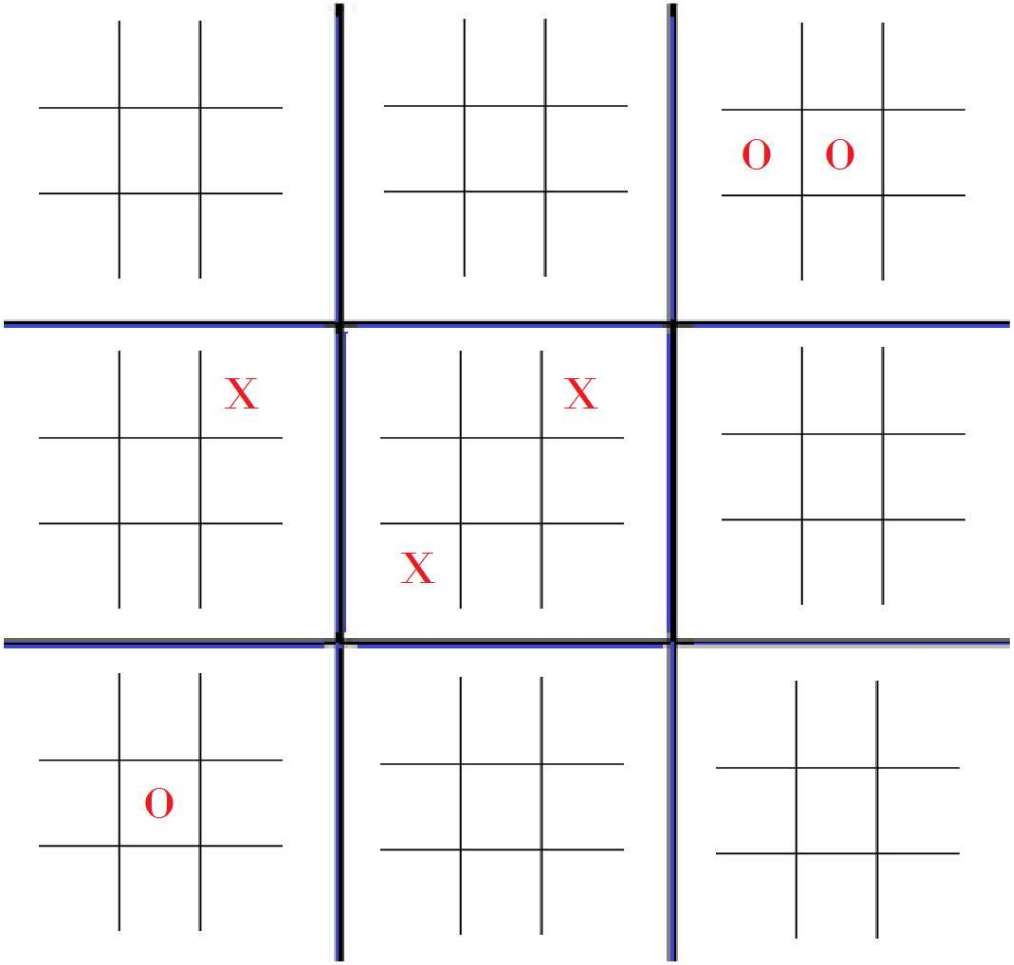



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

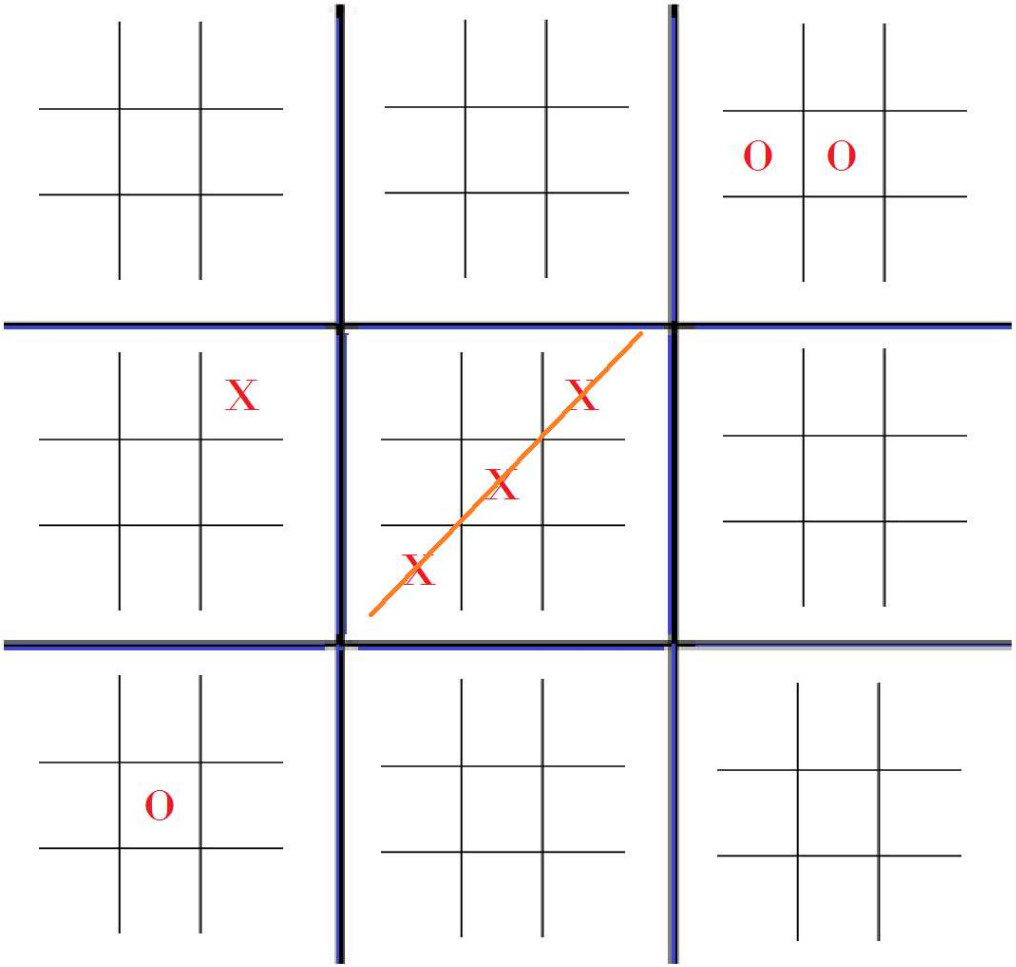



**Figure:** Rule: next move from position of previous.

# Tic-Tac-Toe in Tic-Tac-Toe

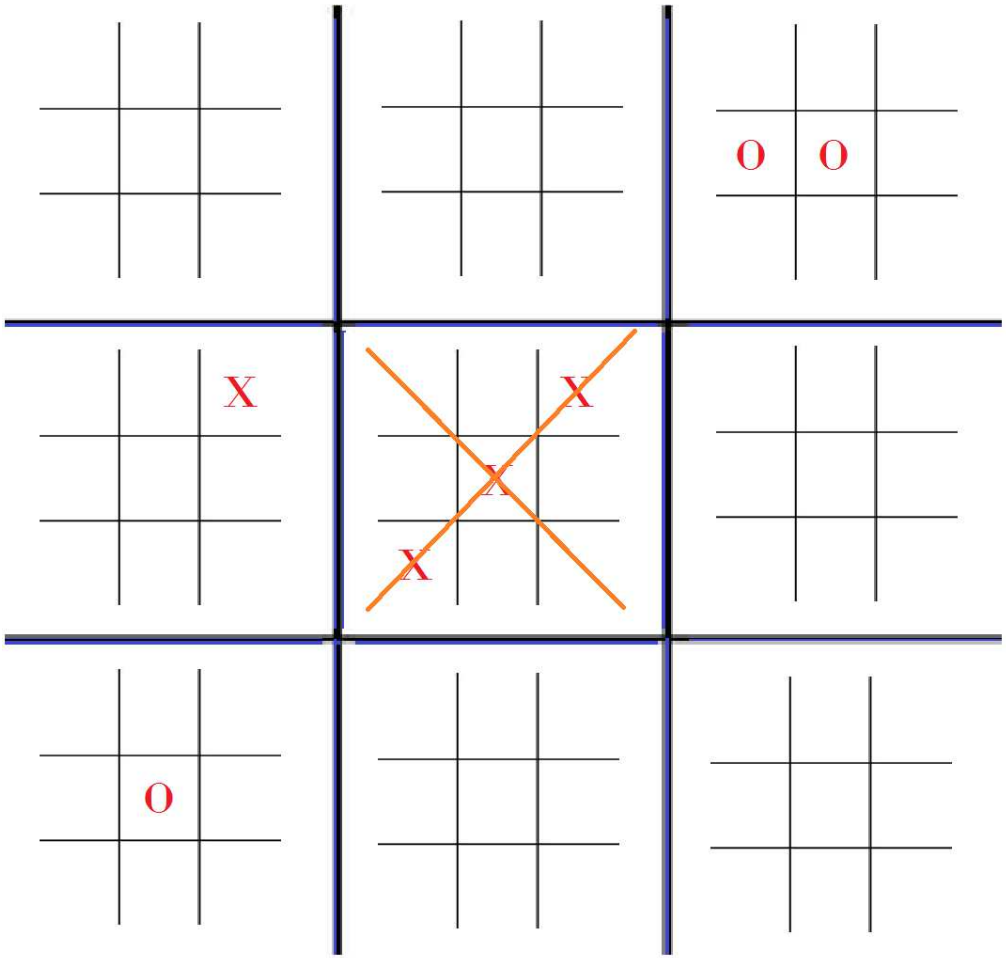
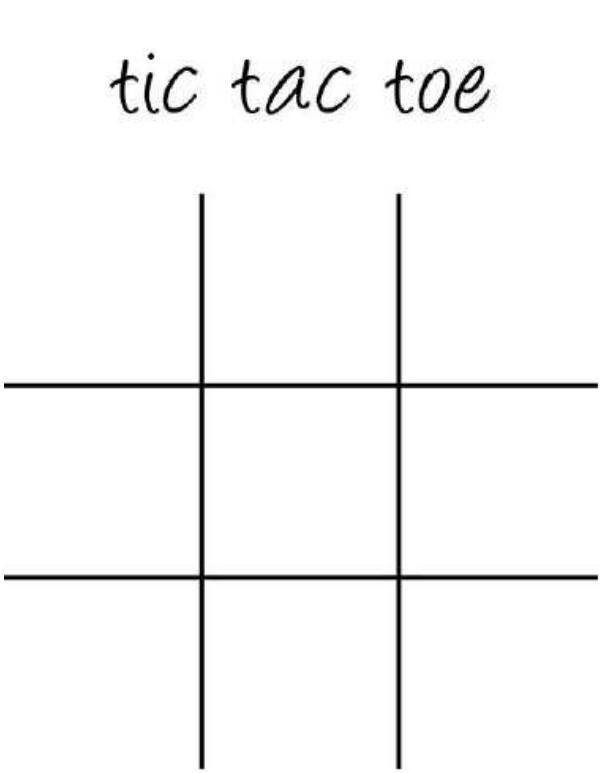



**Figure:** Rule: next move from position of previous.

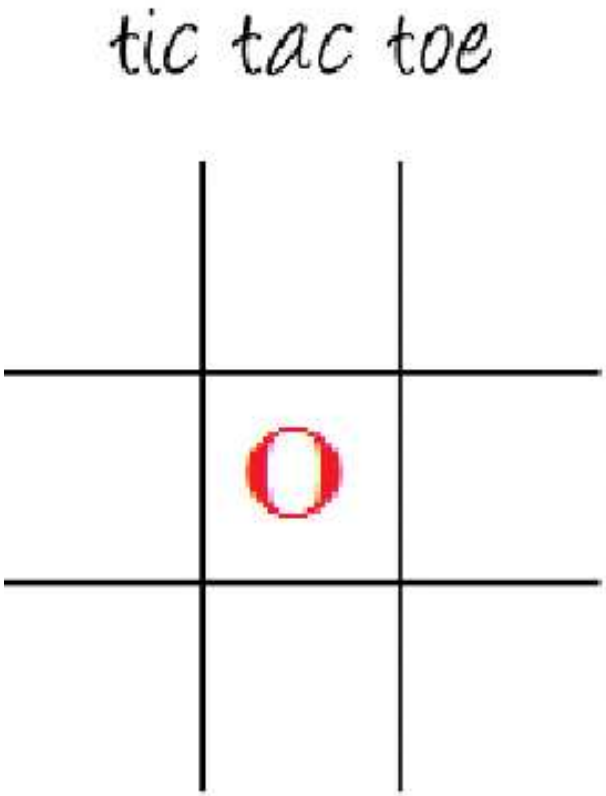
# Bidding Tic-Tac-Toe




| Move | #1's Money | #2's Money | #1's Bid | #2's Bid |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |

**Figure:** Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

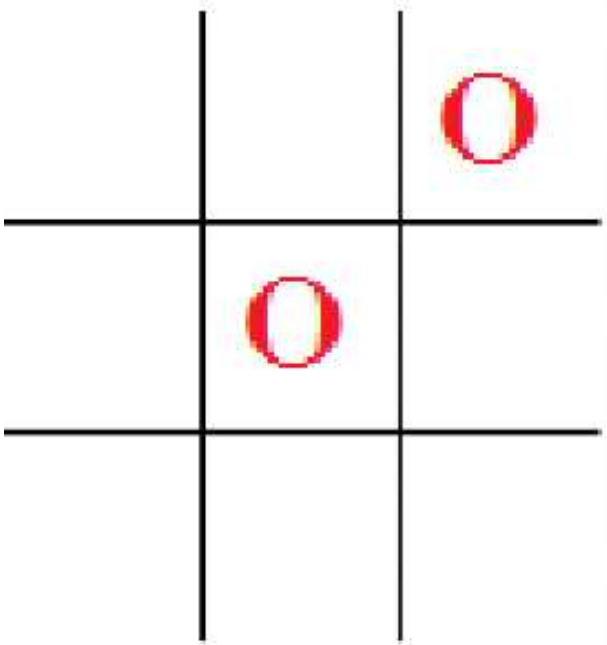
# Bidding Tic-Tac-Toe




| Move | #1's Money | #2's Money | #1's Bid | #2's Bid |
|---|---|---|---|---|
| 1 | 1000 | 1000 | 300 | 700 |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |

**Figure:** Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

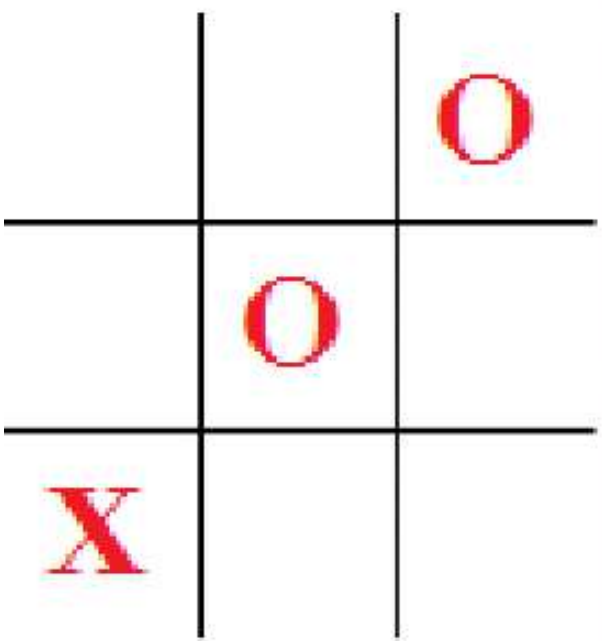
# Bidding Tic-Tac-Toe




| Move | #1's Money | #2's Money | #1's Bid | #2's Bid |
|---|---|---|---|---|
| 1 | 1000 | 1000 | 300 | 700 |
| 2 | 1700 | 300 | 99 | 100 |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |

**Figure:** Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

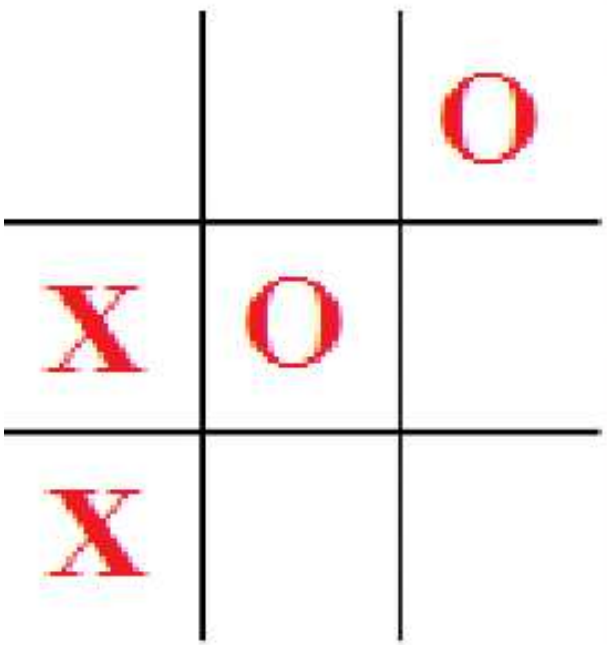
# Bidding Tic-Tac-Toe




| Move | #1's Money | #2's Money | #1's Bid | #2's Bid |
|---|---|---|---|---|
| 1 | 1000 | 1000 | 300 | 700 |
| 2 | 1700 | 300 | 99 | 100 |
| 3 | 1800 | 200 | 201 | 200 |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |

**Figure:** Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

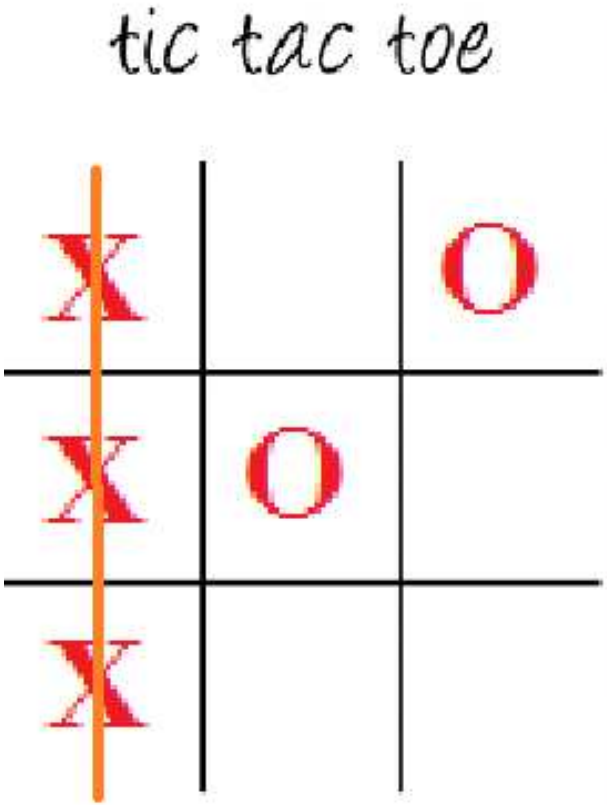
# Bidding Tic-Tac-Toe




| Move | #1's Money | #2's Money | #1's Bid | #2's Bid |
|---|---|---|---|---|
| 1 | 1000 | 1000 | 300 | 700 |
| 2 | 1700 | 300 | 99 | 100 |
| 3 | 1800 | 200 | 201 | 200 |
| 4 | 1599 | 401 | 402 | 400 |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |

**Figure:** Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

# Bidding Tic-Tac-Toe




| Move | #1's Money | #2's Money | #1's Bid | #2's Bid |
|---|---|---|---|---|
| 1 | 1000 | 1000 | 300 | 700 |
| 2 | 1700 | 300 | 99 | 100 |
| 3 | 1800 | 200 | 201 | 200 |
| 4 | 1599 | 401 | 402 | 400 |
| 5 | 1197 | 803 | 804 | 803 |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |

**Figure:** Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

# Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have \$1000, how much is too much to bid for the first move?

Easy answer:

# Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have \$1000, how much is too much to bid for the first move?

Easy answer: \$1000.

# Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have \$1000, how much is too much to bid for the first move?

Easy answer: \$1000.

Can we do better? Assume if tie Player 1 gets the move.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids $x$ and wins:

$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids $x$ and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids $x$ and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids $x$ and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Player 1 bids $4000 - 4x$ and just wins:
$(4x - 2000, 4000 - 4x) \longrightarrow (8x - 6000, 8000 - 8x)$.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids $x$ and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Player 1 bids $4000 - 4x$ and just wins:
$(4x - 2000, 4000 - 4x) \longrightarrow (8x - 6000, 8000 - 8x)$.

Need $4x - 2000 \geq 4000 - 4x$ or $8x \geq 6000$ or $x \geq 750$.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids 750 and wins: $(1000, 1000) \longrightarrow (1750, 250)$.

Player 1 bids 250 and just wins: $(1750, 250) \longrightarrow (1500, 500)$.

Player 1 bids 500 and just wins: $(1500, 500) \longrightarrow (1000, 1000)$.

Player 1 bids 1000 and just wins: $(1000, 1000) \longrightarrow (0, 2000)$.

# Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids 750 and wins: $(1000, 1000) \longrightarrow (1750, 250)$.

Player 1 bids 250 and just wins: $(1750, 250) \longrightarrow (1500, 500)$.

Player 1 bids 500 and just wins: $(1500, 500) \longrightarrow (1000, 1000)$.

Player 1 bids 1000 and just wins: $(1000, 1000) \longrightarrow (0, 2000)$.

Note: If Player 2 spends \$750 to win the first two moves then Player 1 can win!

# Rectangle Game

# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

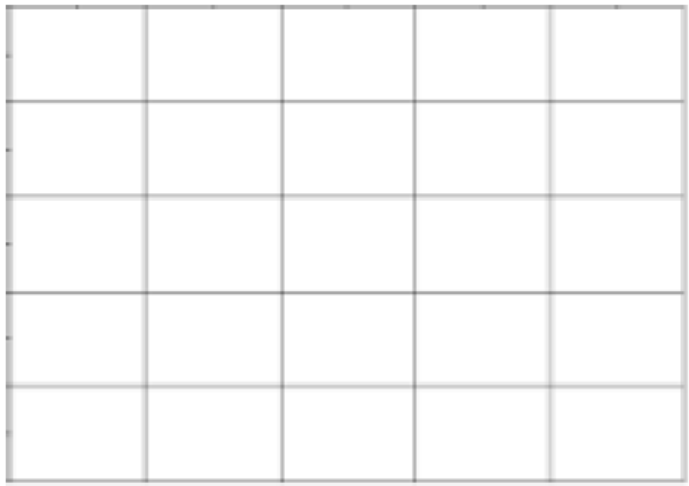


**Figure:** Winning strategy? Function of board dimension?

# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

# Rectangle Game

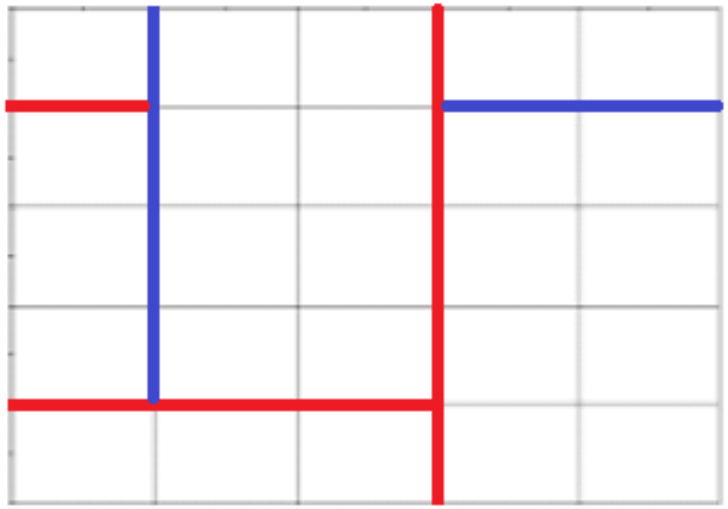
**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

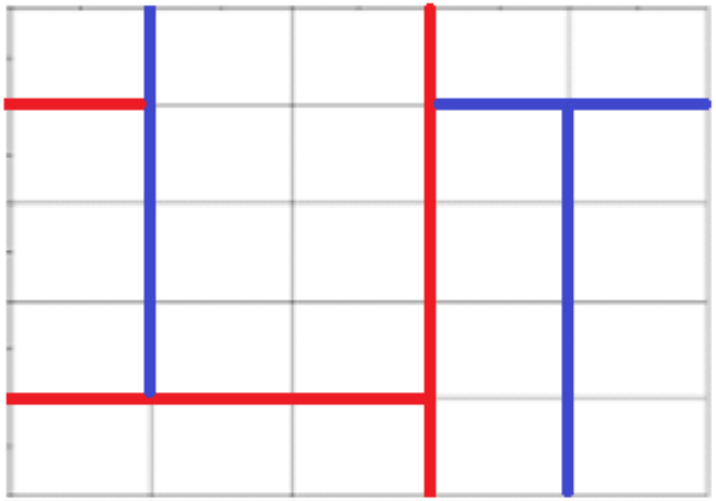
# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



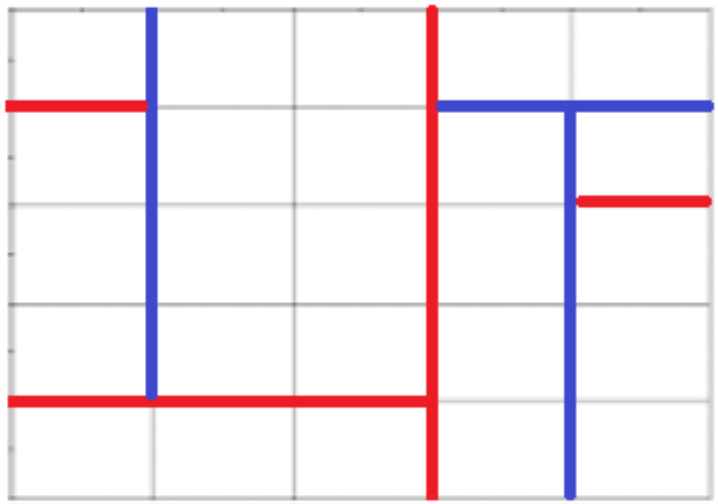
# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



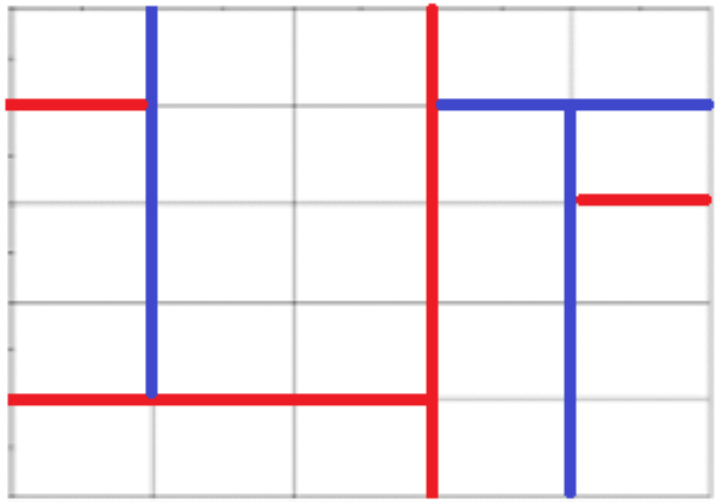
# Rectangle Game

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



# Rectangle Game

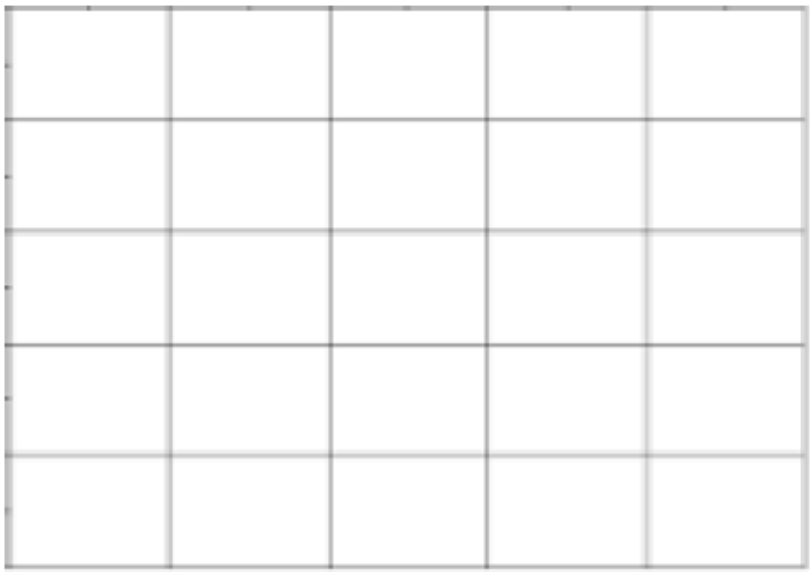
**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Gather data! Try various sized boards, strategies.

## Rectangle Game: Data

**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



| Length | Width | Winner |
|---|---|---|
| 2 | 2 | --1-- |
| 2 | 3 | --1-- |
| 3 | 3 | --2-- |
| 2 | 4 | --1-- |
| 3 | 4 | --1-- |
| 4 | 4 | --1-- |
| 3 | 5 | --2-- |

**Figure:** Do you see a pattern?

## Mono-variant

A mono-variant is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the rectangle game....

# Rectangle Game: Solution

Every time move, increase number of pieces by 1!

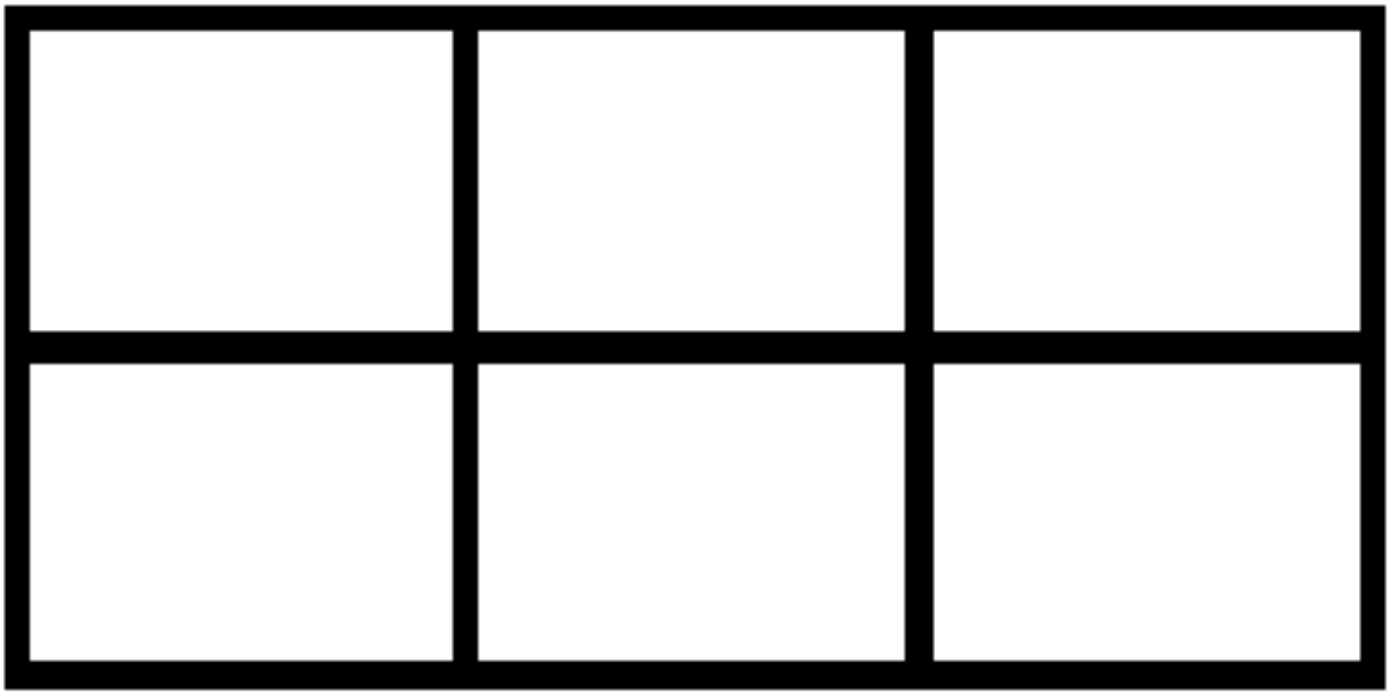


**Figure:** Move: 0; Pieces: 1.

# Rectangle Game: Solution

Every time move, increase number of pieces by 1!

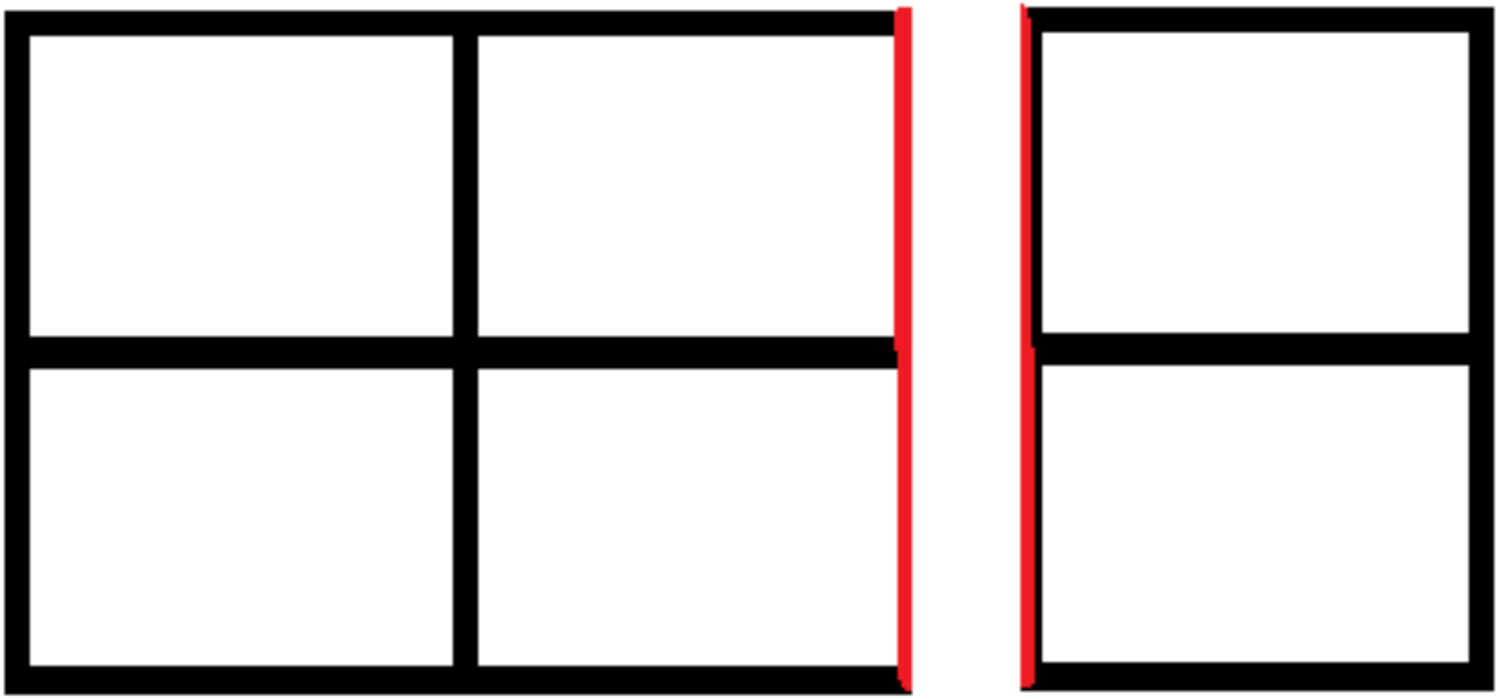


**Figure:** Move: 1; Pieces: 2.

# Rectangle Game: Solution

Every time move, increase number of pieces by 1!

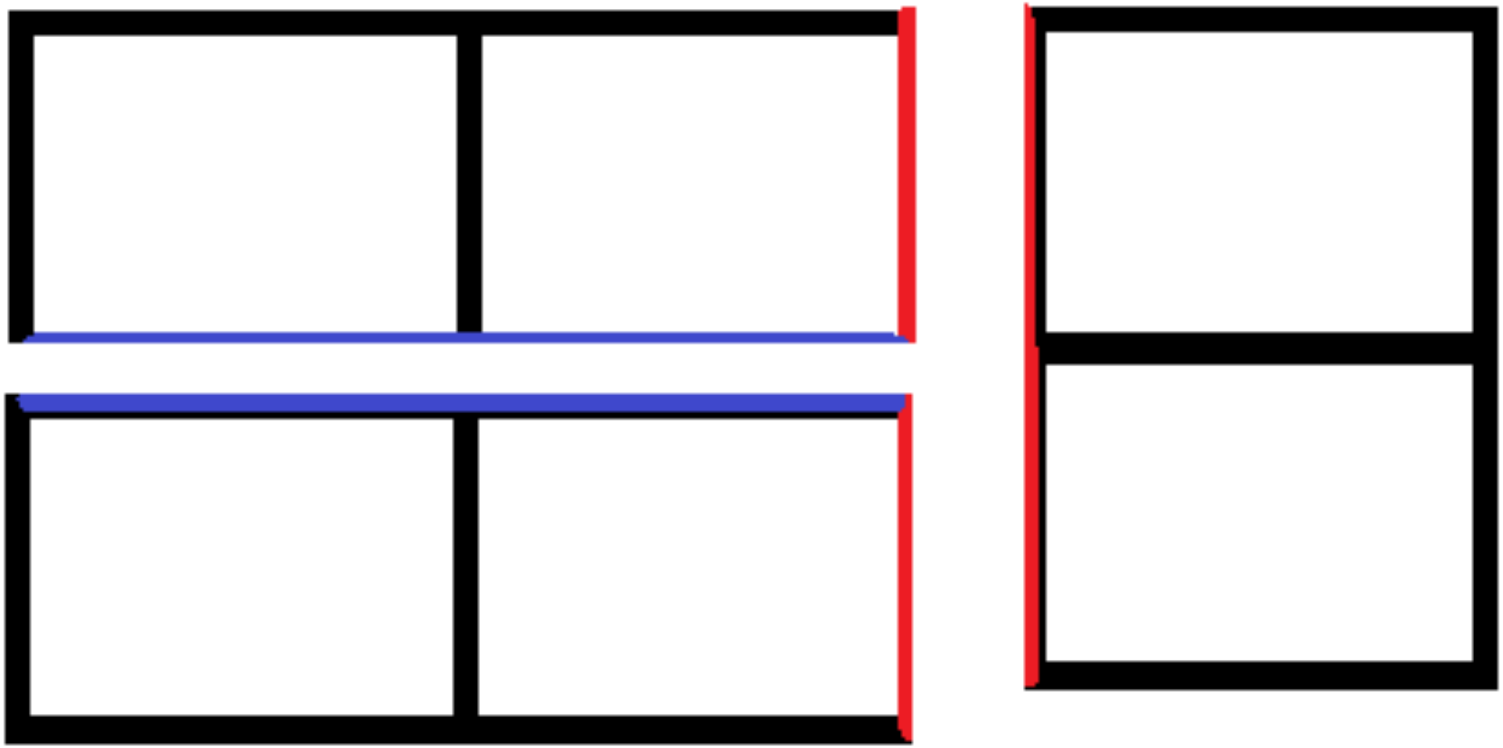


**Figure:** Move: 2; Pieces: 3.

# Rectangle Game: Solution

Every time move, increase number of pieces by 1!

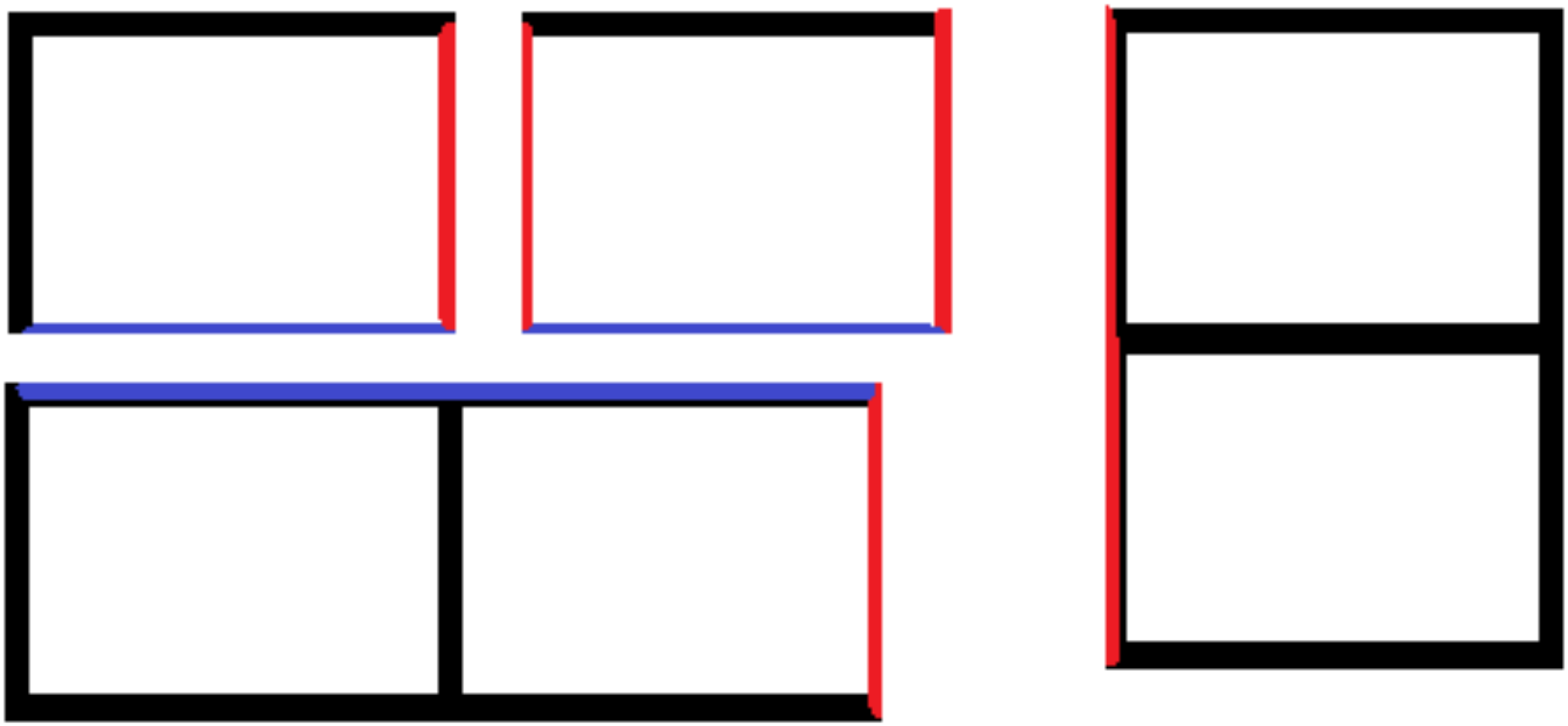


**Figure:** Move: 3; Pieces: 4.

# Rectangle Game: Solution

Every time move, increase number of pieces by 1!

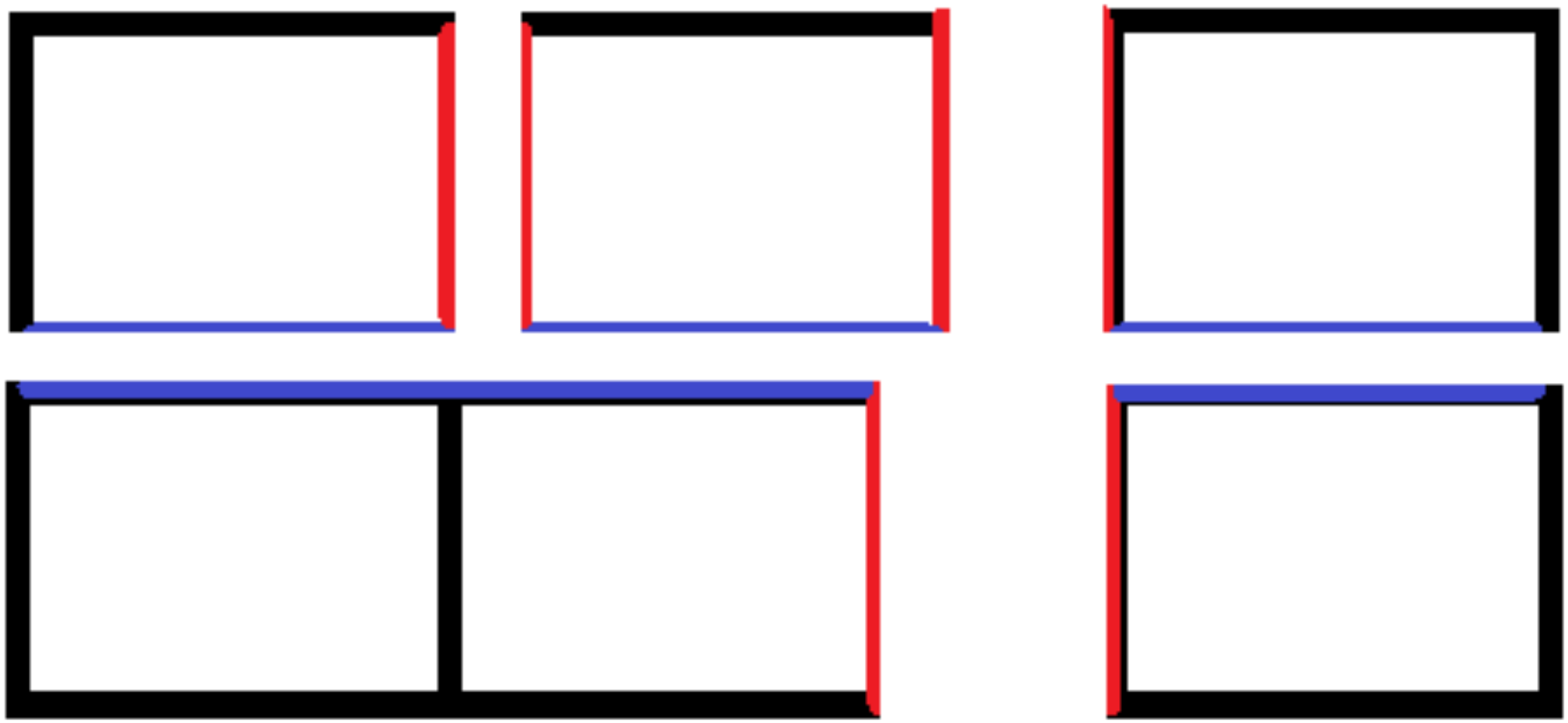


**Figure:** Move: 4; Pieces: 5.

# Rectangle Game: Solution

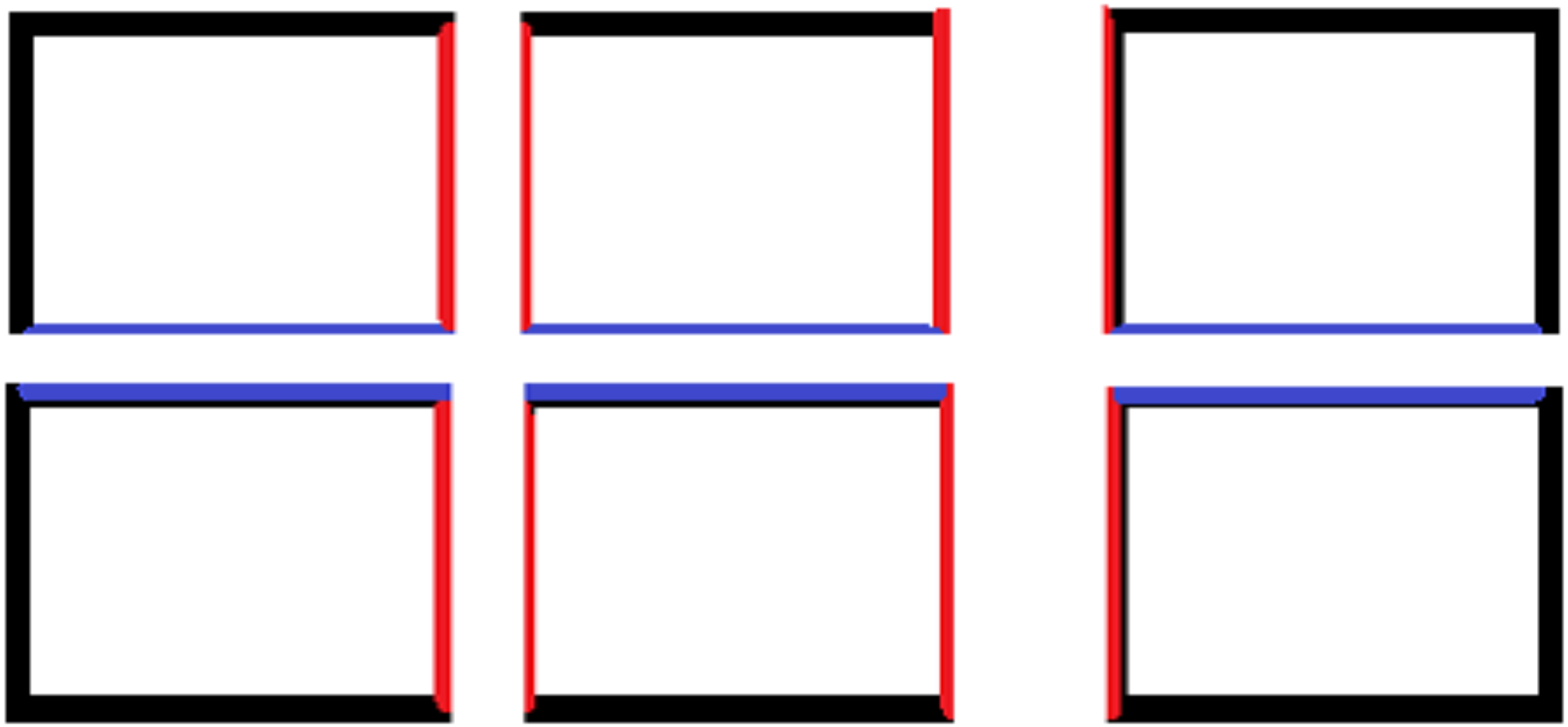


**Figure:** Move: 5; Pieces: 5. Player 1 Wins.

# Rectangle Game: Solution (Continued)

Mono-variant is the number of pieces.

If board is $m \times n$, game ends with $mn$ pieces.

Thus takes $mn - 1$ moves.

If $mn$ even then Player 1 wins else Player 2 wins.

# Zombies

# General Advice: What are your tools and how can they be used?

**Law of the Hammer:**

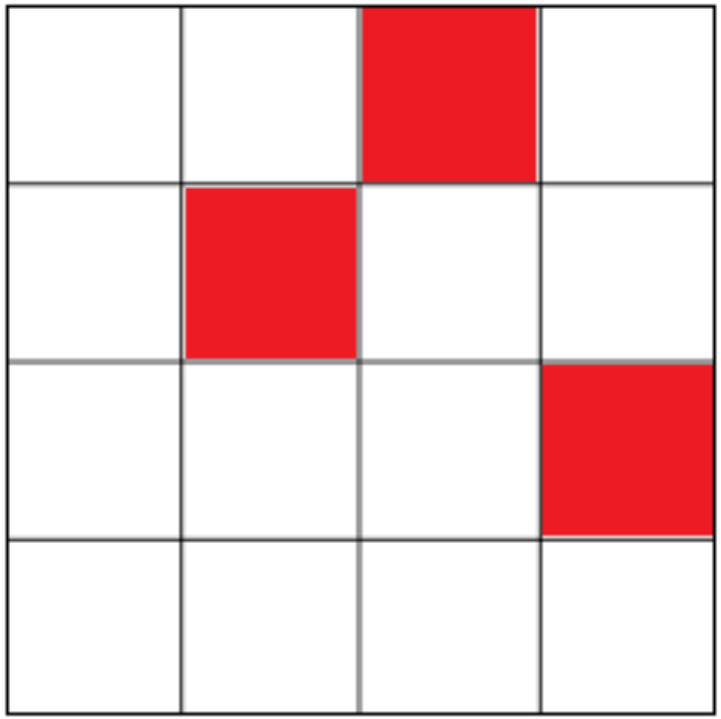
- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

## Zombine Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

# Zombine Infection: Rules

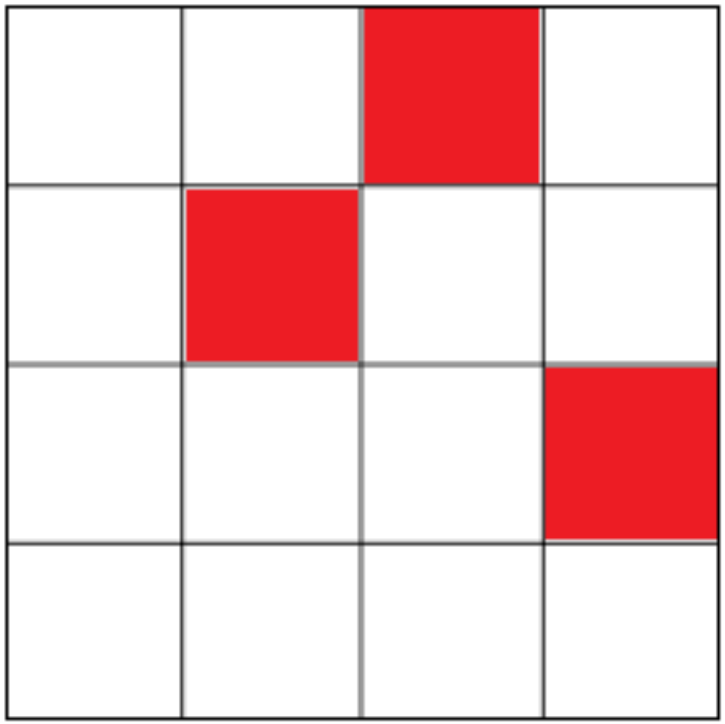
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



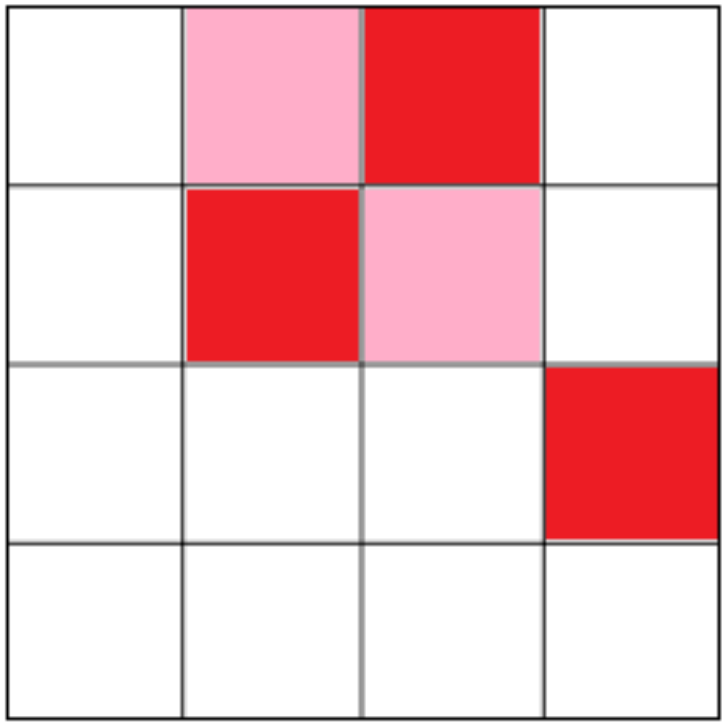
*Initial Configuration*

# Zombine Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



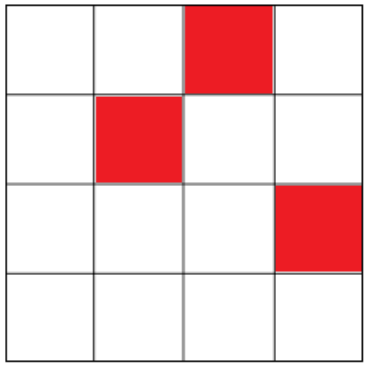
*Initial Configuration*



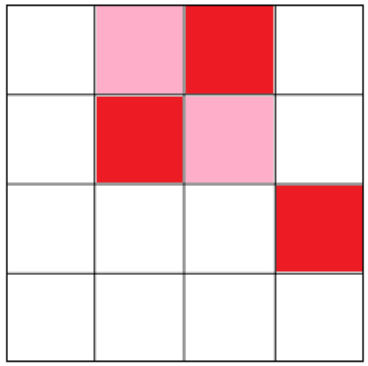
*One moment later*

# Zombine Infection: Rules

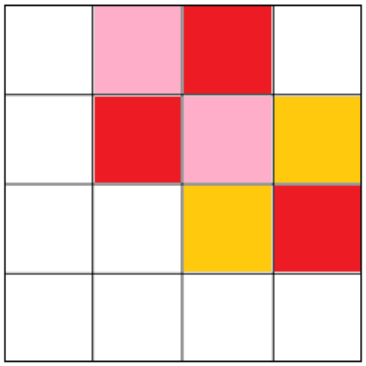
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



*Initial Configuration*



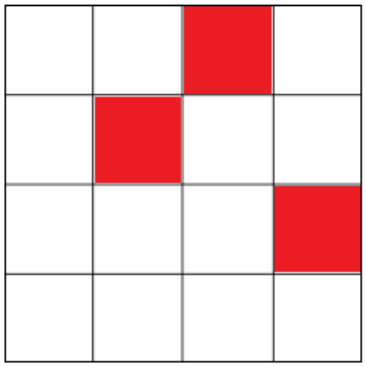
*One moment later*



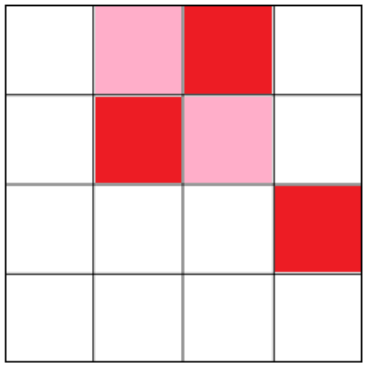
*Two moments later*

# Zombine Infection: Rules

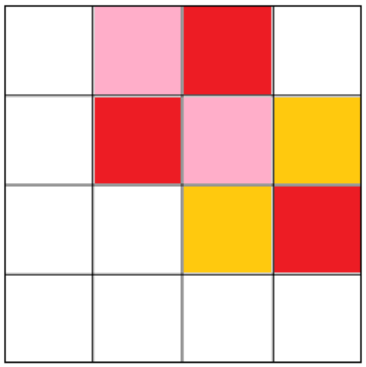
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



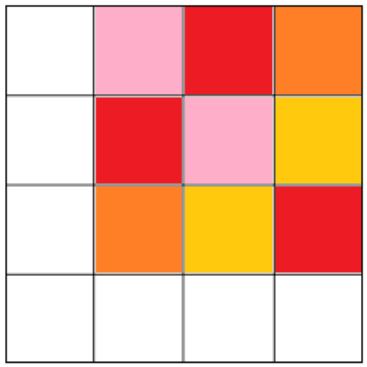
*Initial Configuration*



*One moment later*



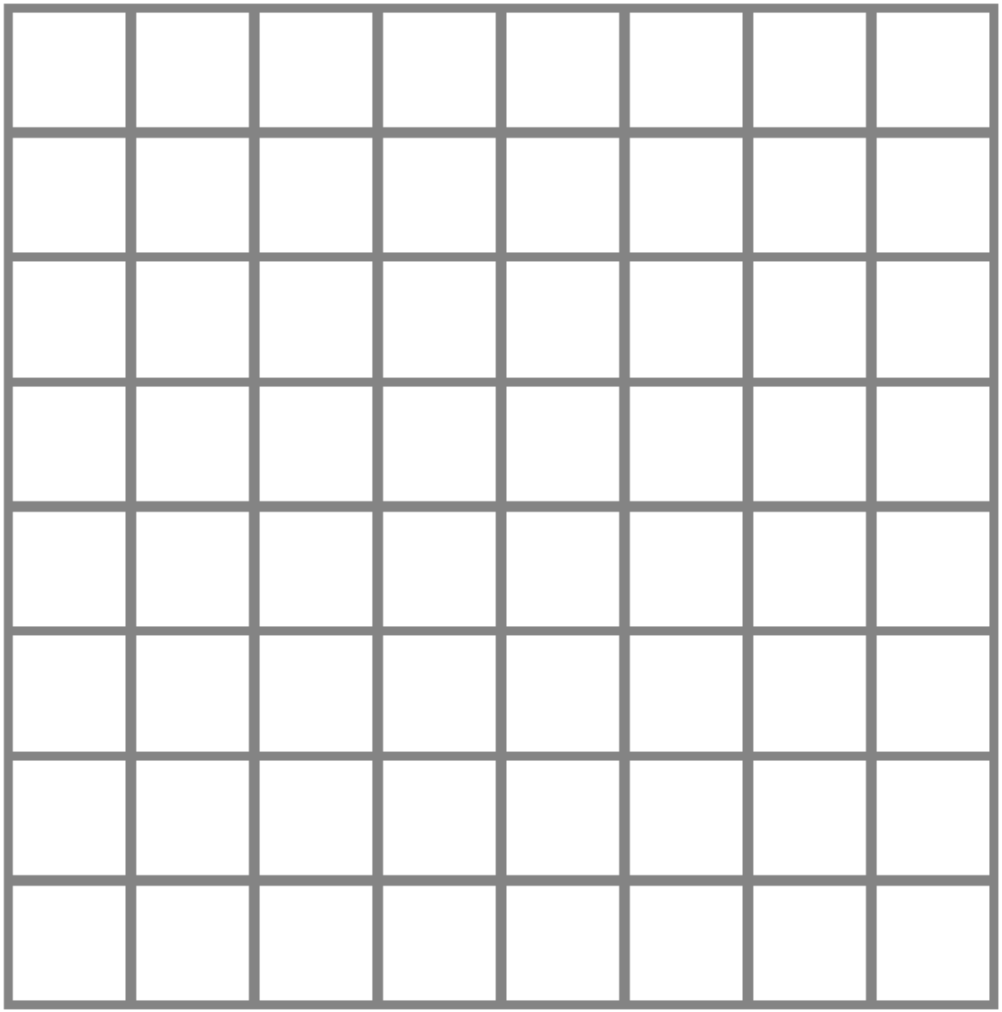
*Two moments later*



*Three moments later*

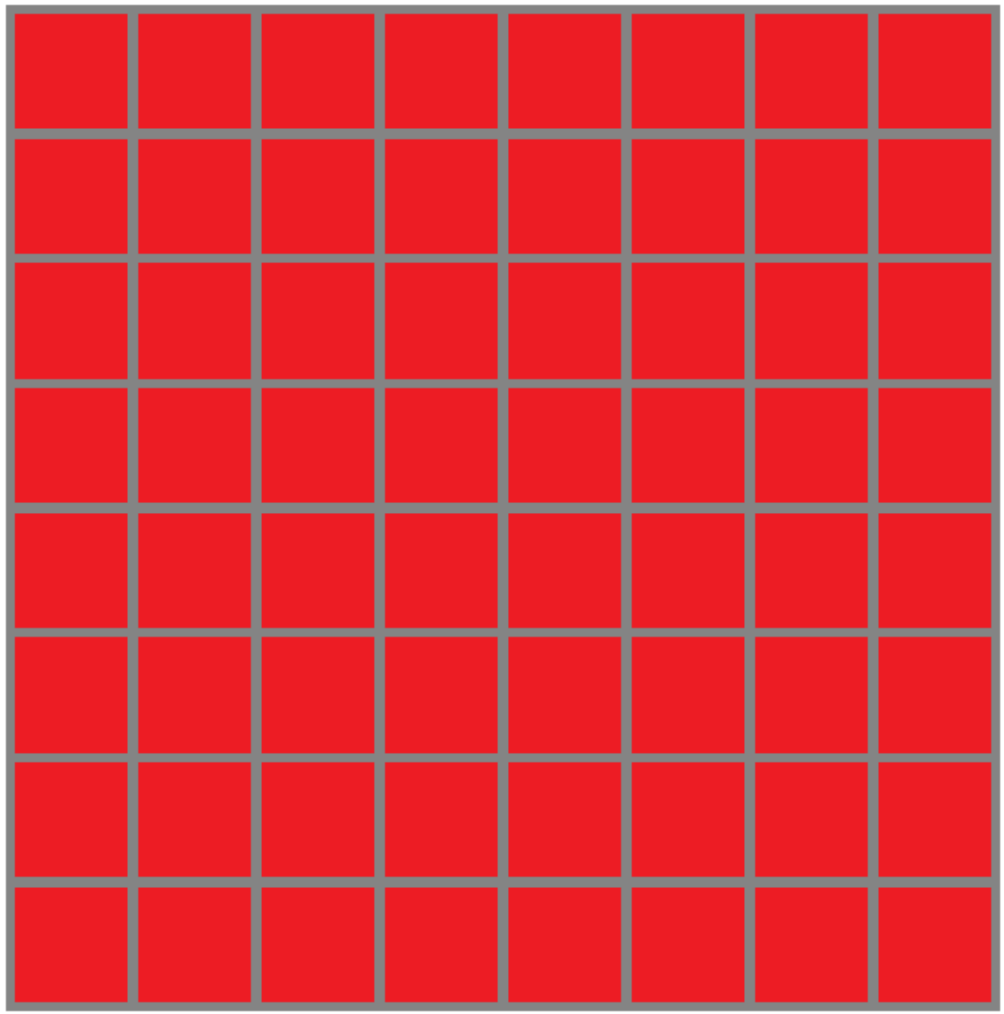
# Zombine Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



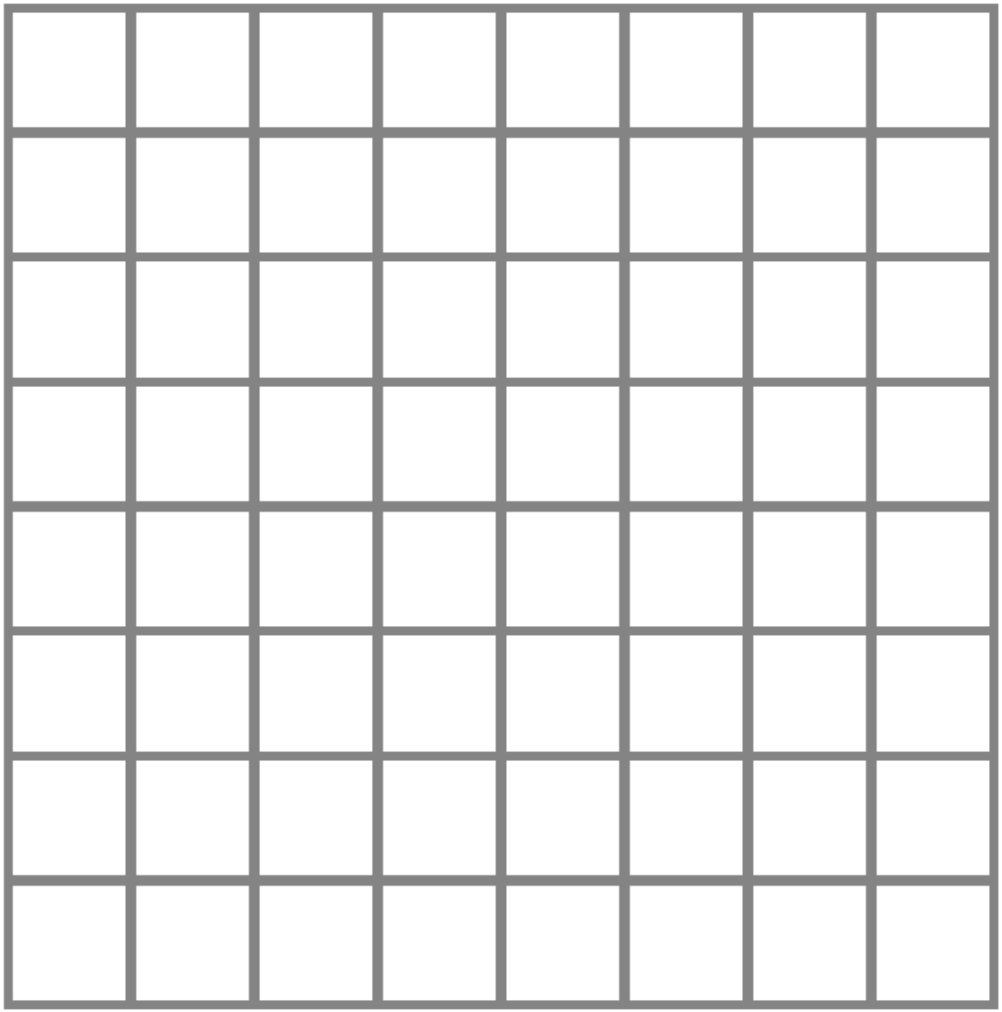
# Zombine Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



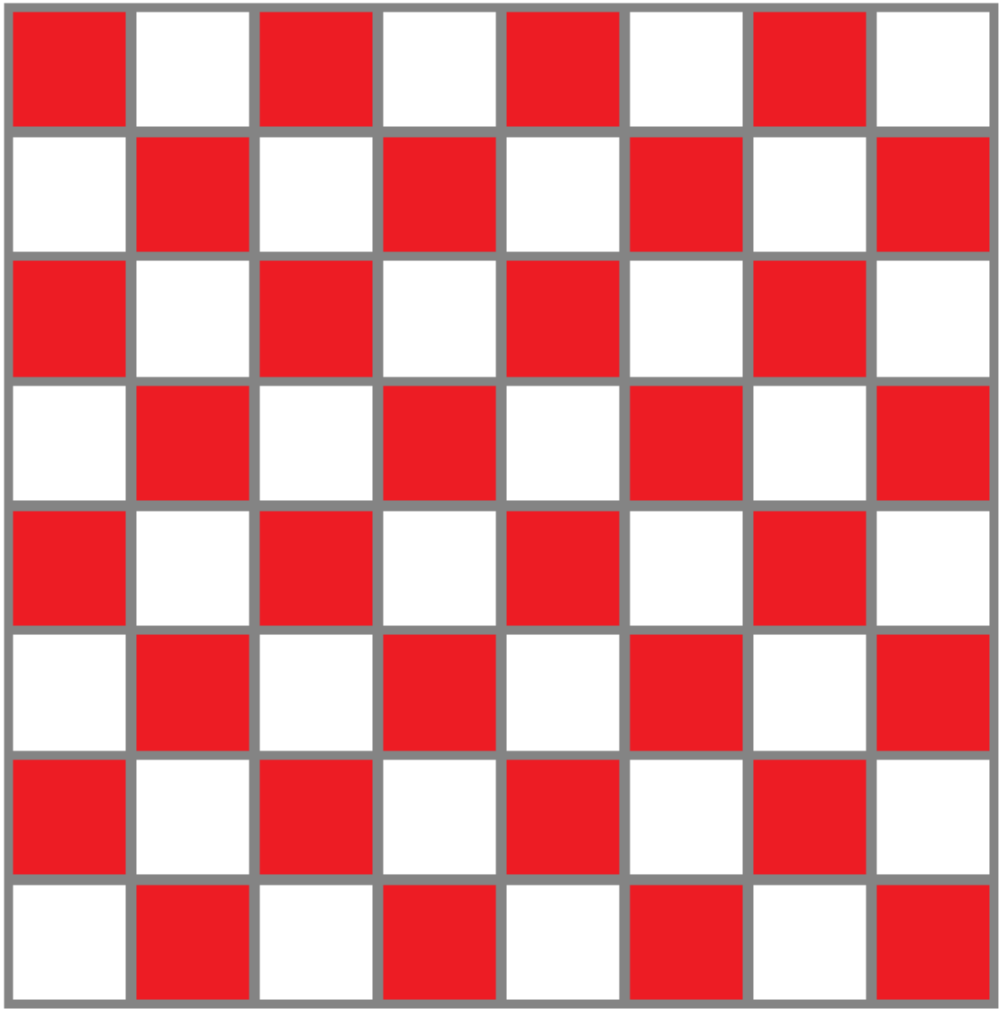
# Zombine Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



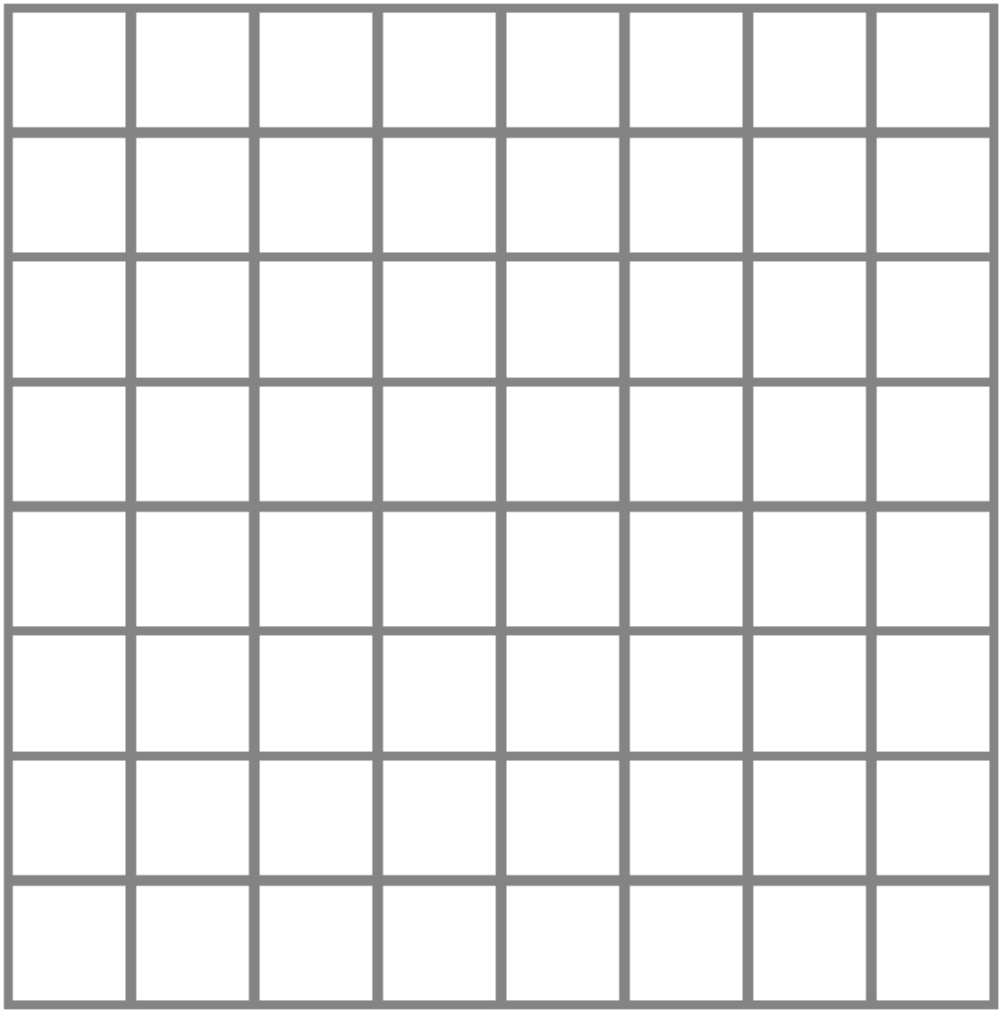
# Zombine Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



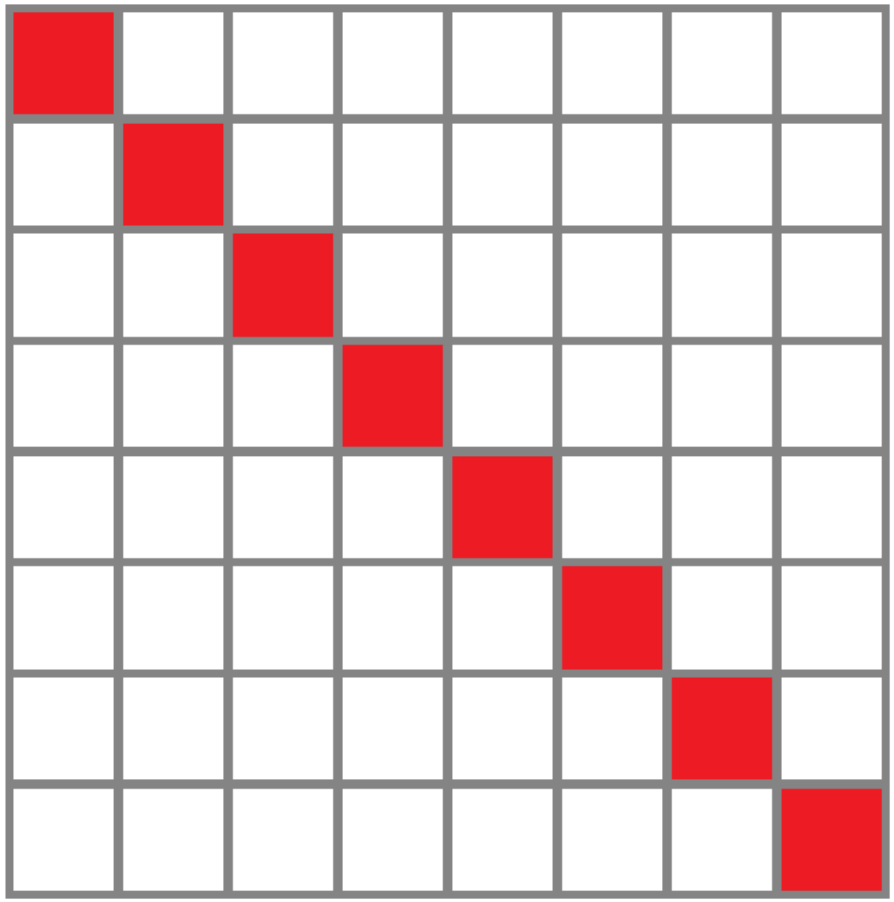
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



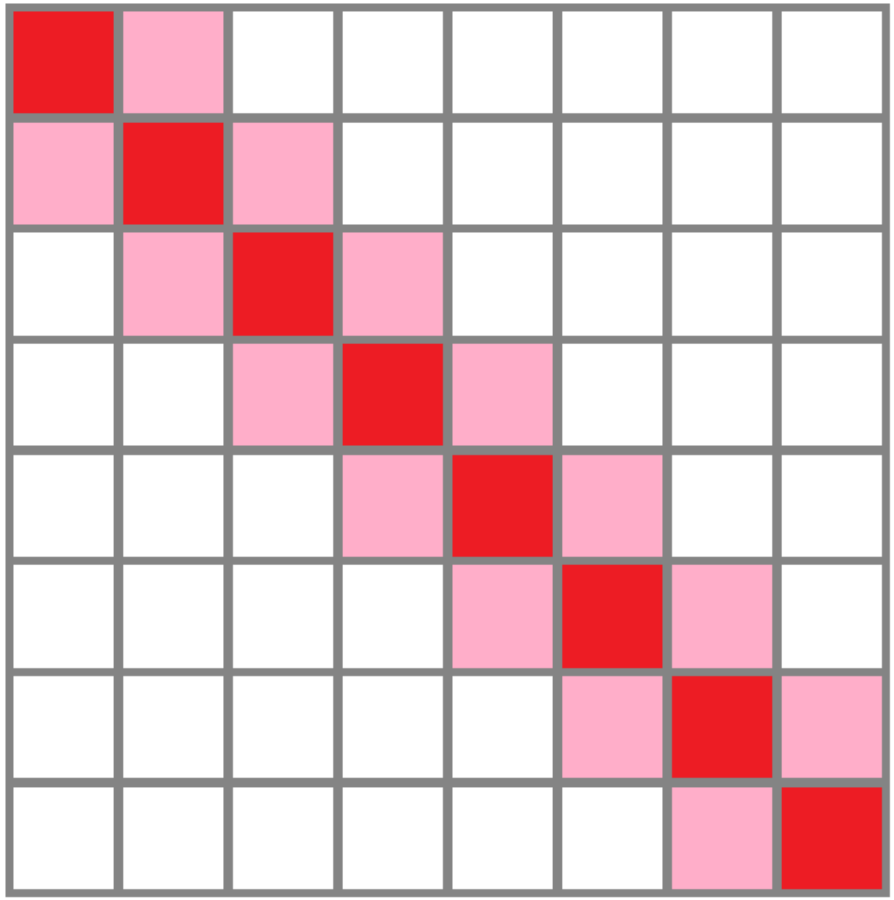
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



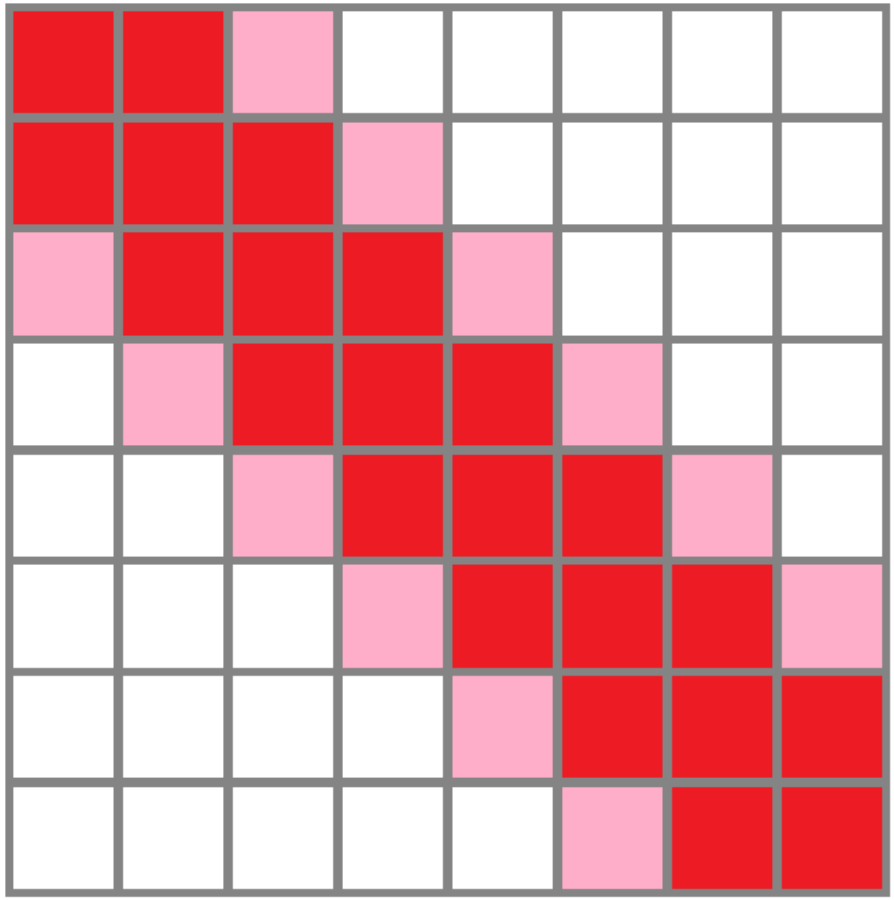
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



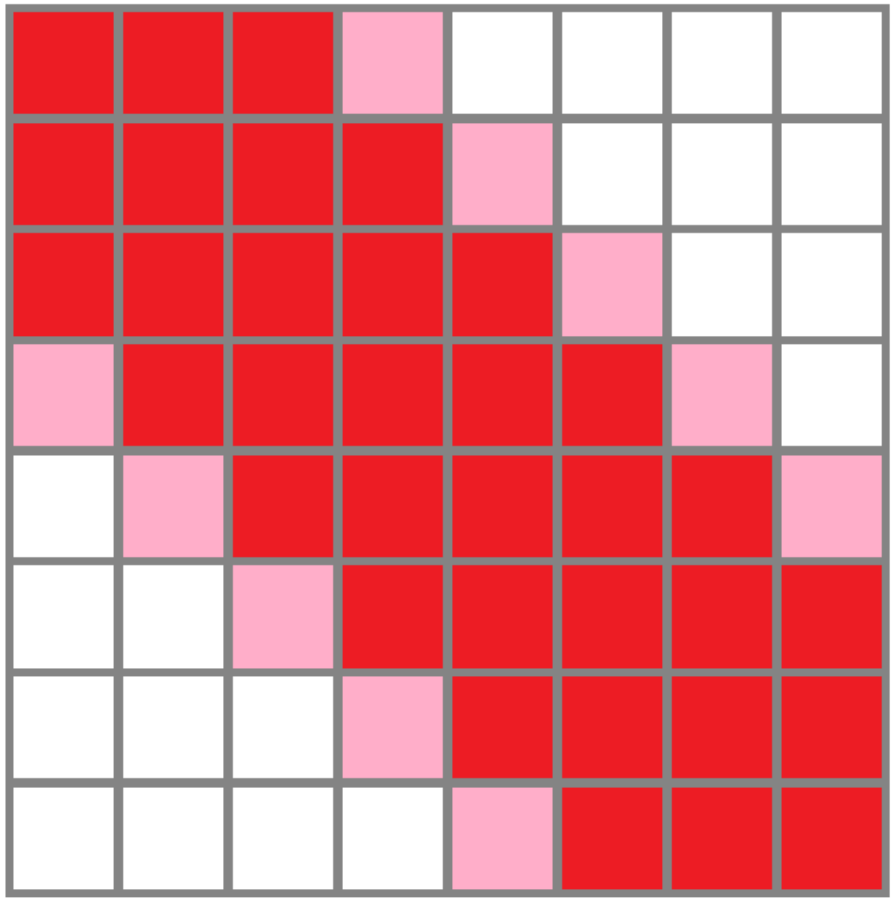
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



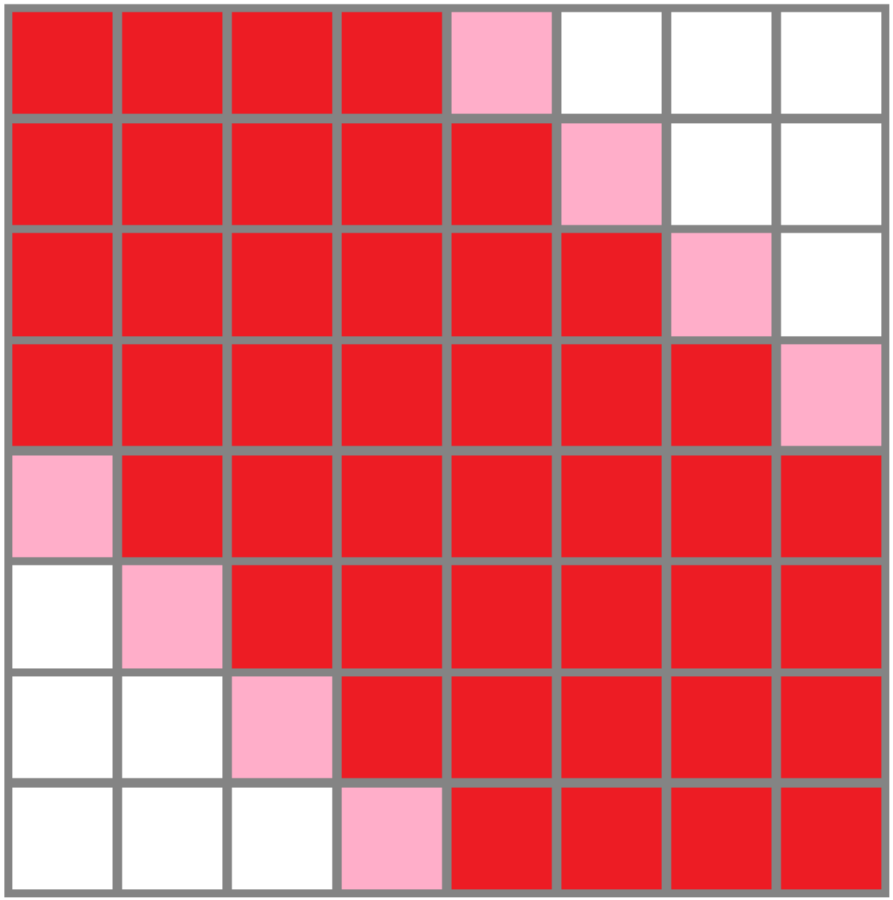
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



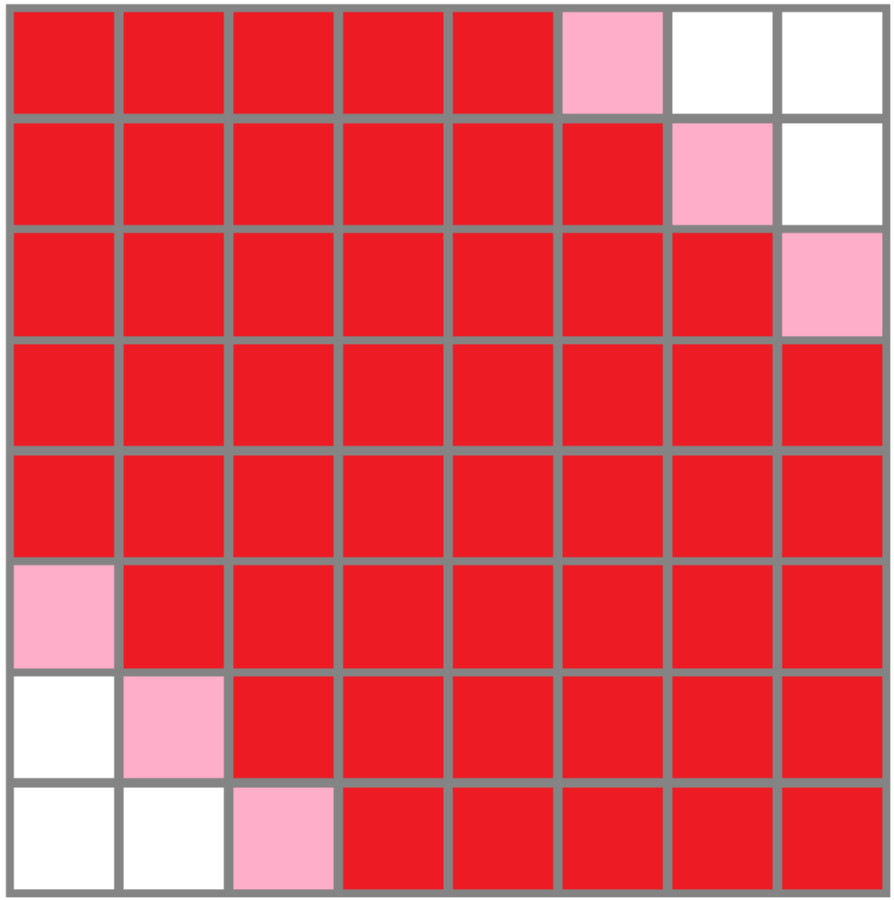
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



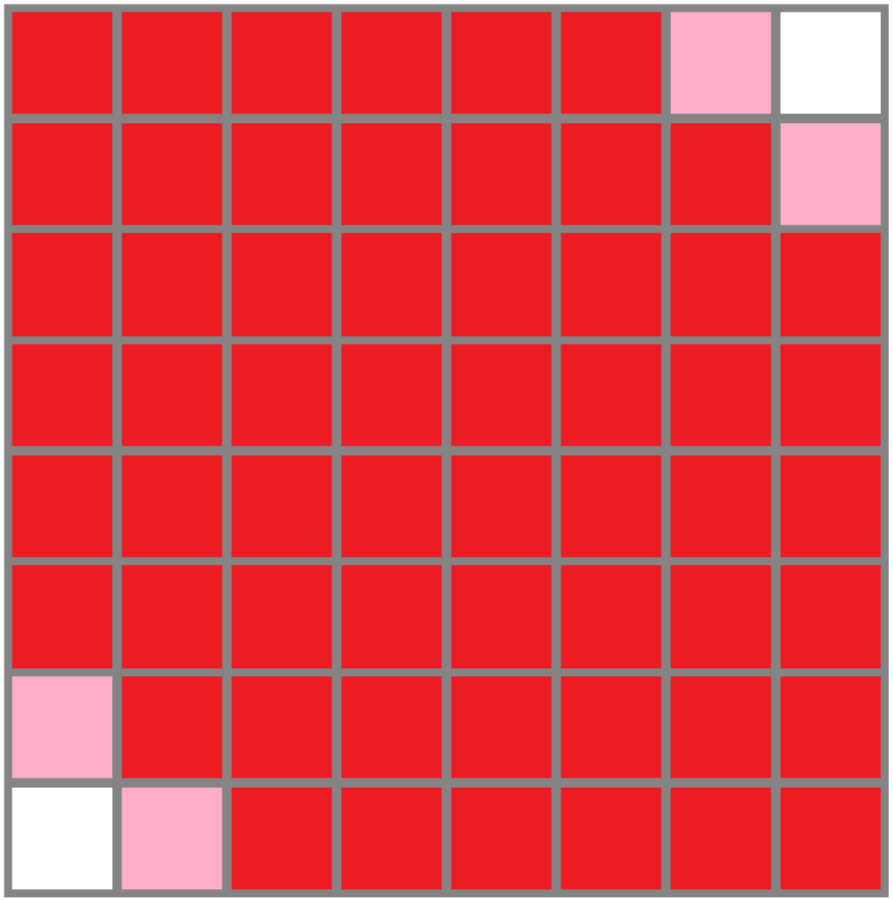
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



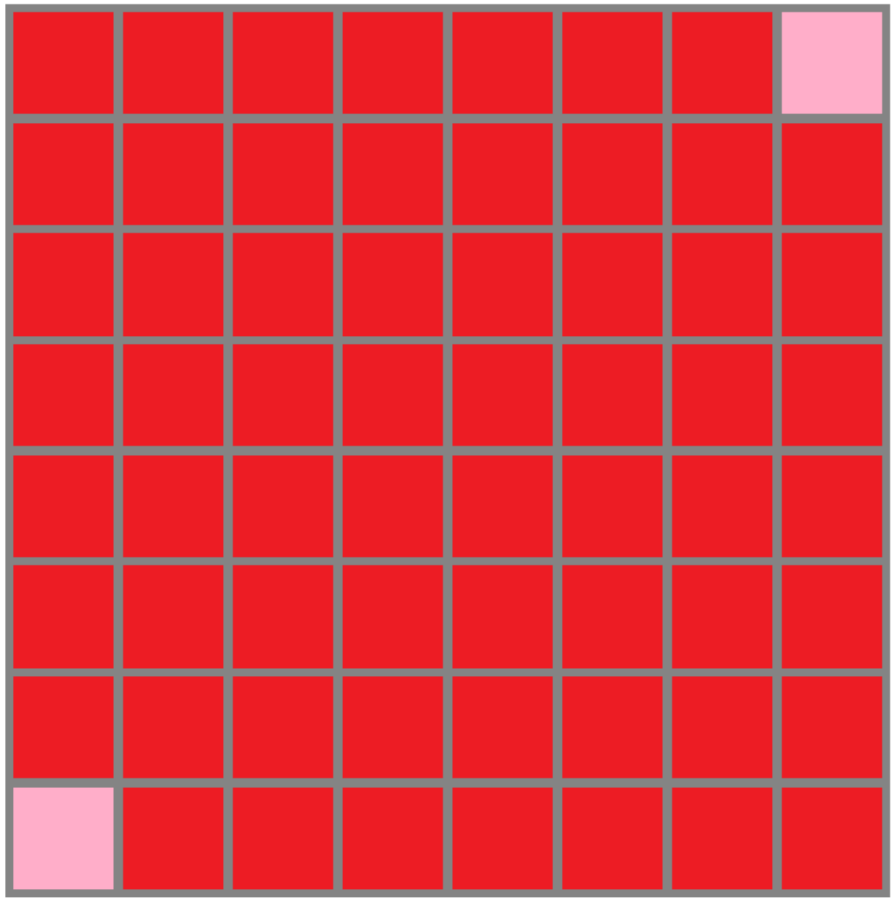
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



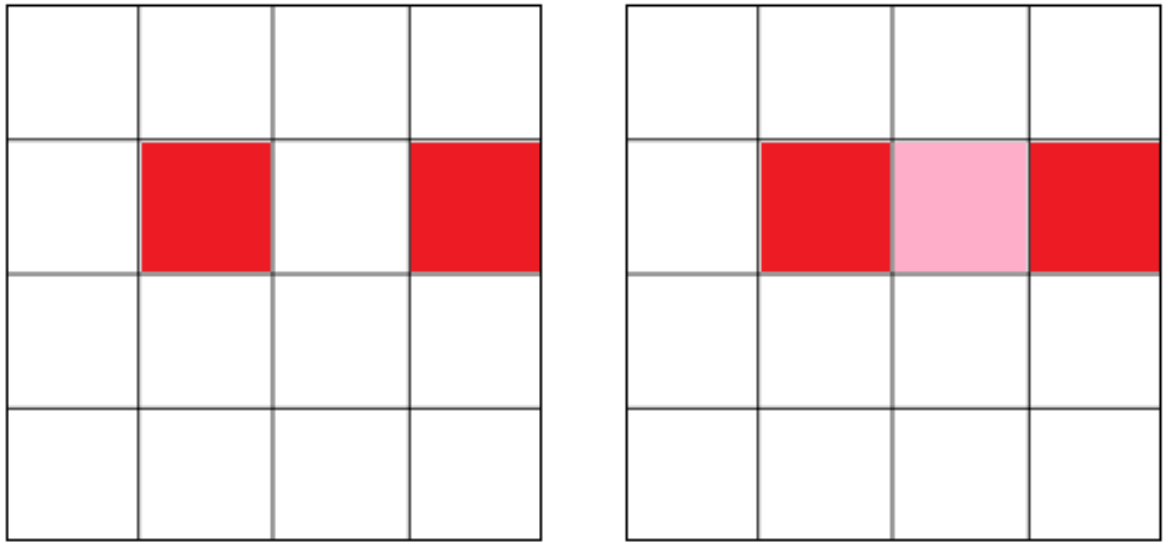
# Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



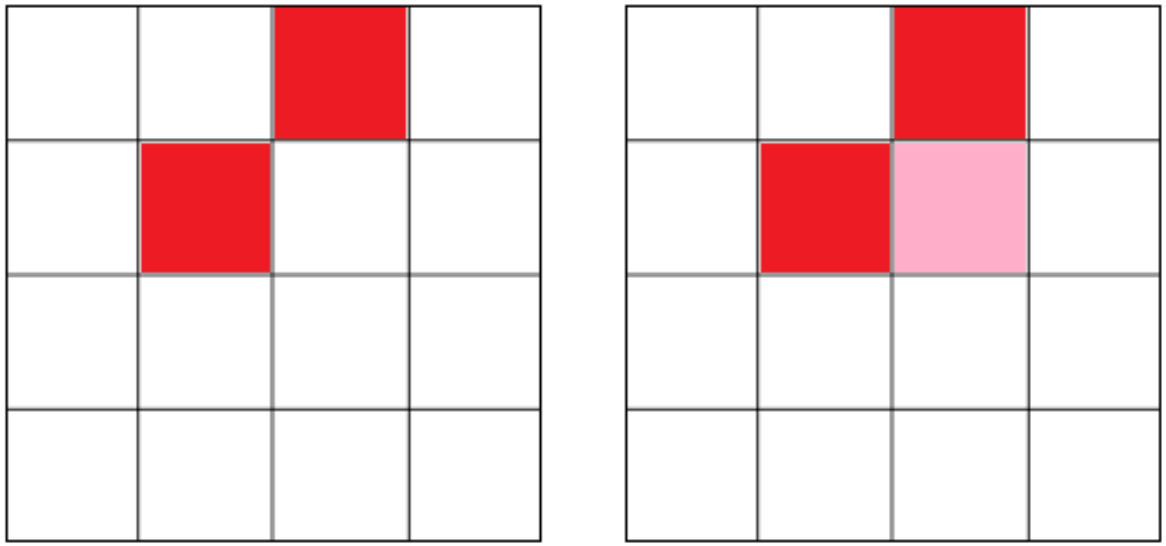
# Zombie Infection: Can $n-1$ infect all?

# Zombie Infection: Can $n-1$ infect all?



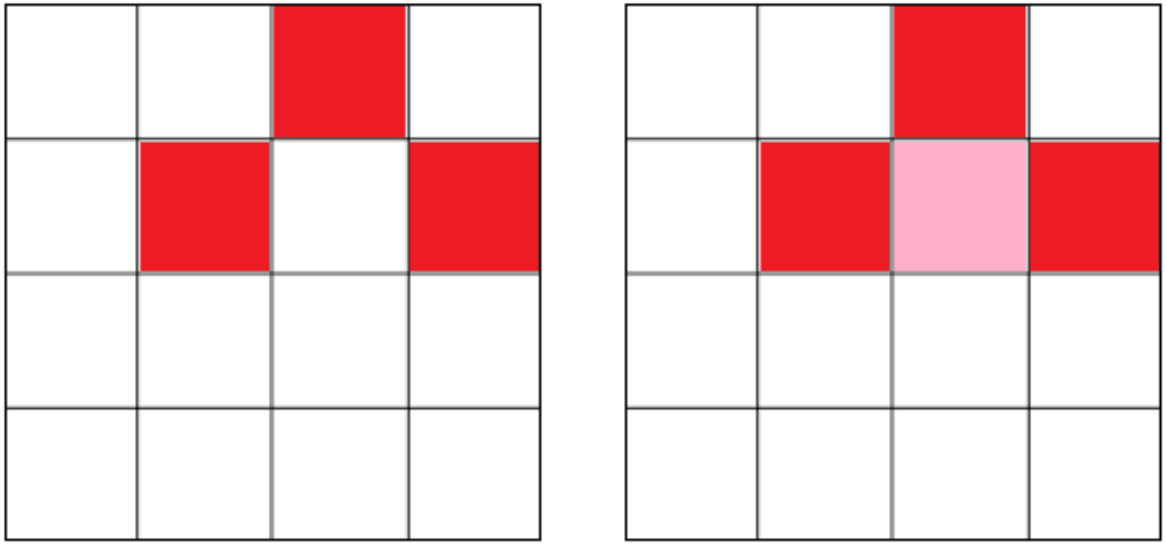
Perimeter of infection unchanged.

# Zombie Infection: Can $n-1$ infect all?



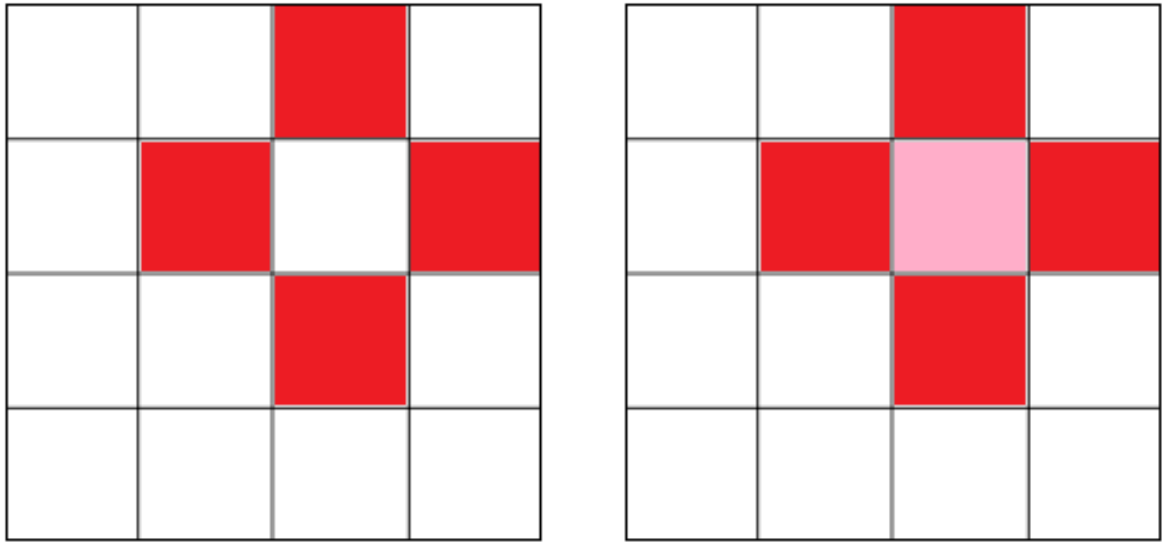
Perimeter of infection unchanged.

# Zombie Infection: Can $n-1$ infect all?



Perimeter of infection decreases by 2.

# Zombie Infection: Can $n-1$ infect all?



Perimeter of infection decreases by 4.

## Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.

## Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.
- Mono-variant: As time passes, perimeter of infection never increases.

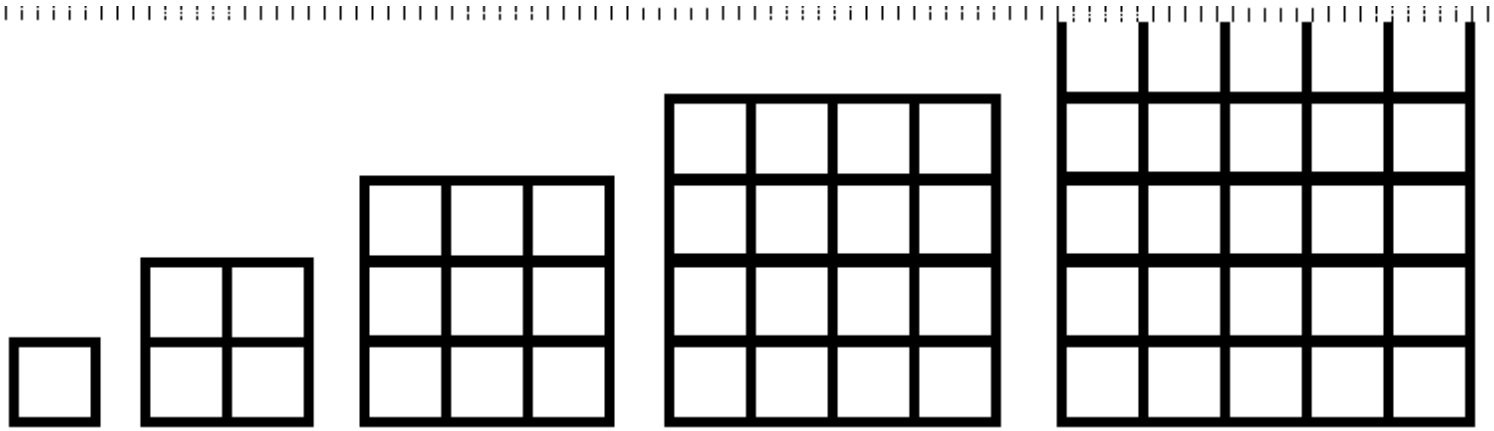
## Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.
- Mono-variant: As time passes, perimeter of infection never increases.
- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!

# I Love Rectangles

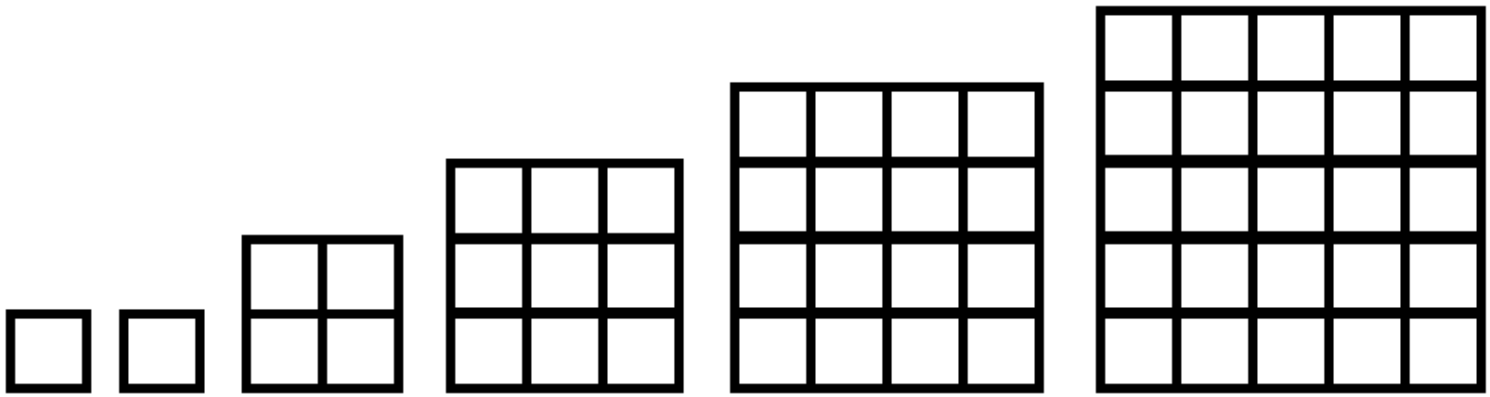
# Tiling the Plane with Squares

Have $n \times n$ square for each $n$, place one at a time so that shape formed is always connected and a rectangle.



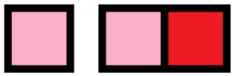
## Tiling the Plane with Squares

Have $n \times n$ square for each $n$, extra $1 \times 1$ square, place one at a time so that shape formed is always connected and a rectangle.

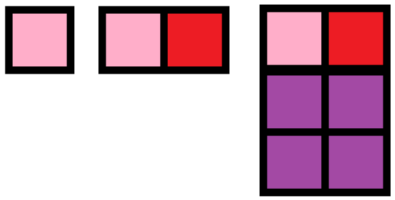


## Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.

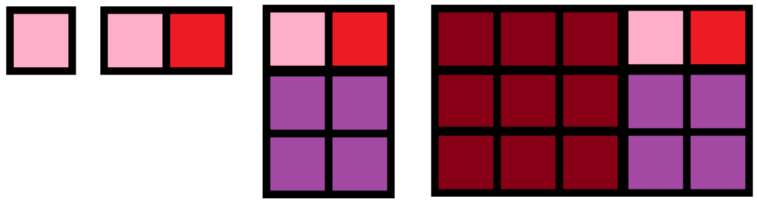
## Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.



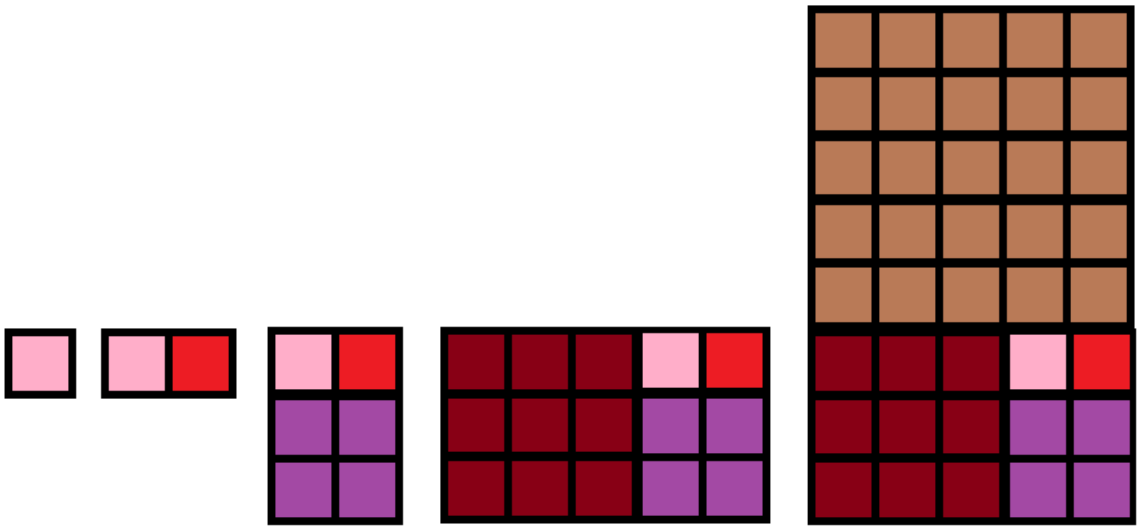
## Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.



# Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.



# Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.



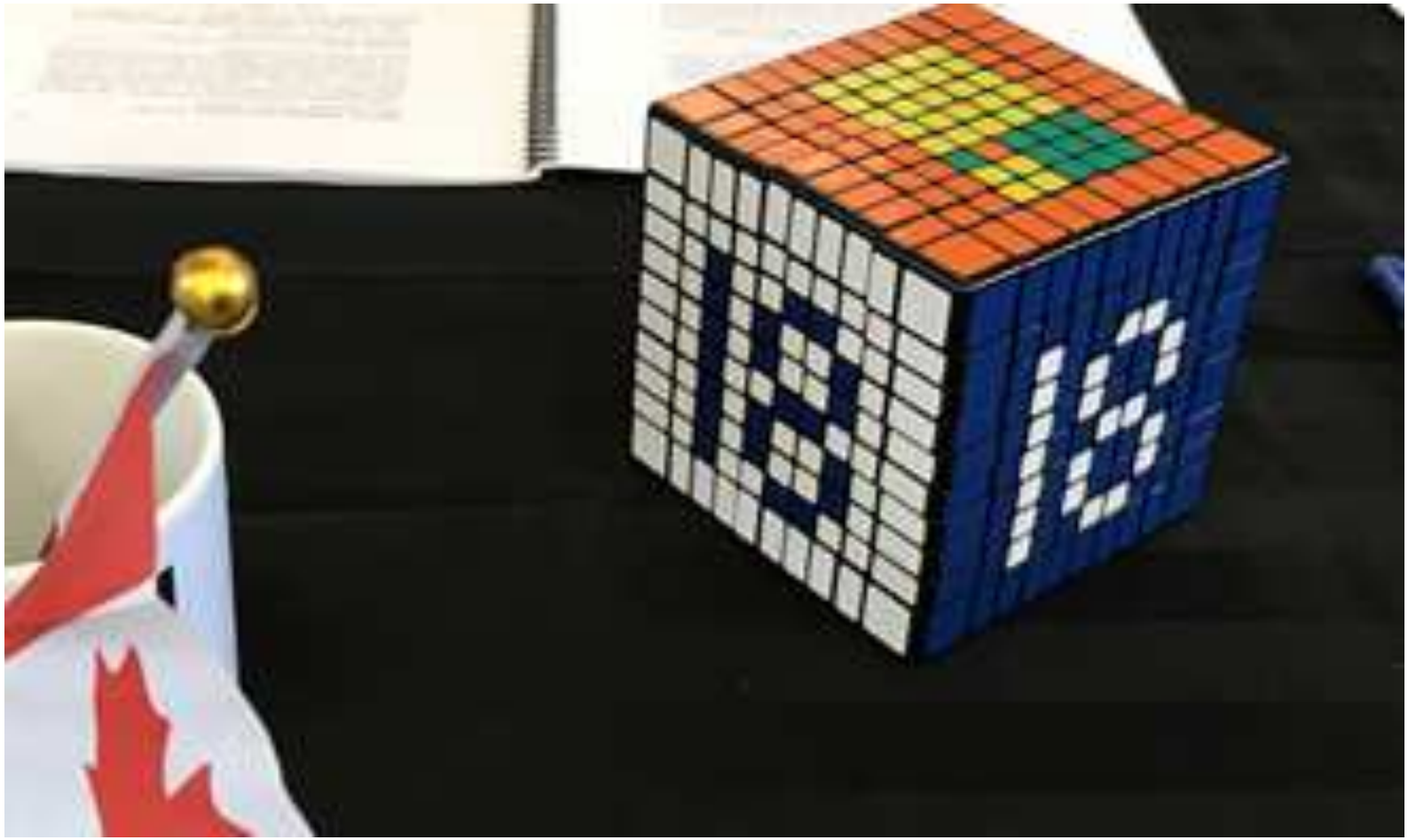
1, 1, 2, 3, 5, . . . .

## Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA

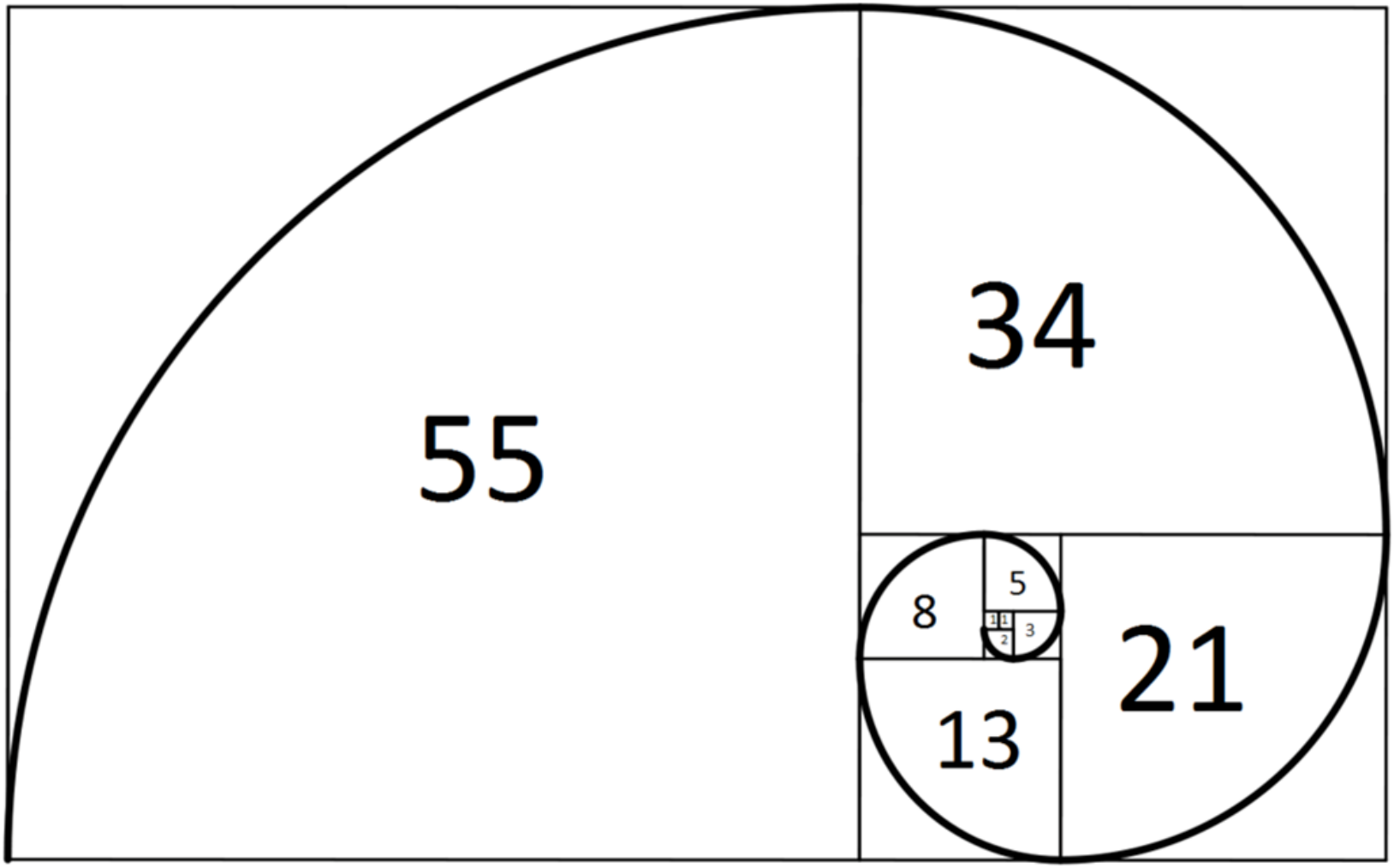
## Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA



## Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA



## Triangle Game

## Rules for Triangle Game

Take an equilateral triangle, label corners 0, 1 and 2.

Subdivide however you wish into triangles.

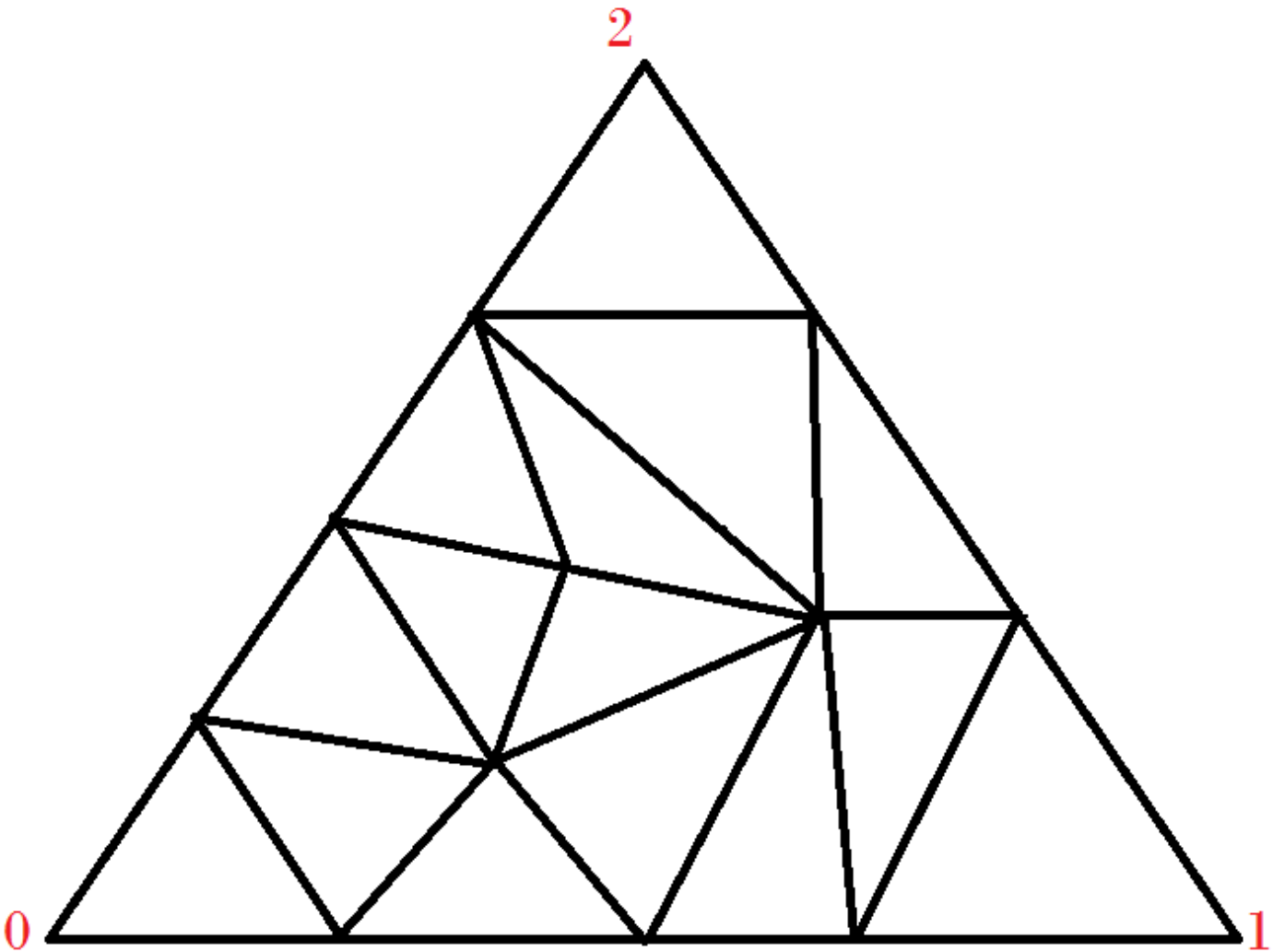
Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

Take turns adding labels, subject to:

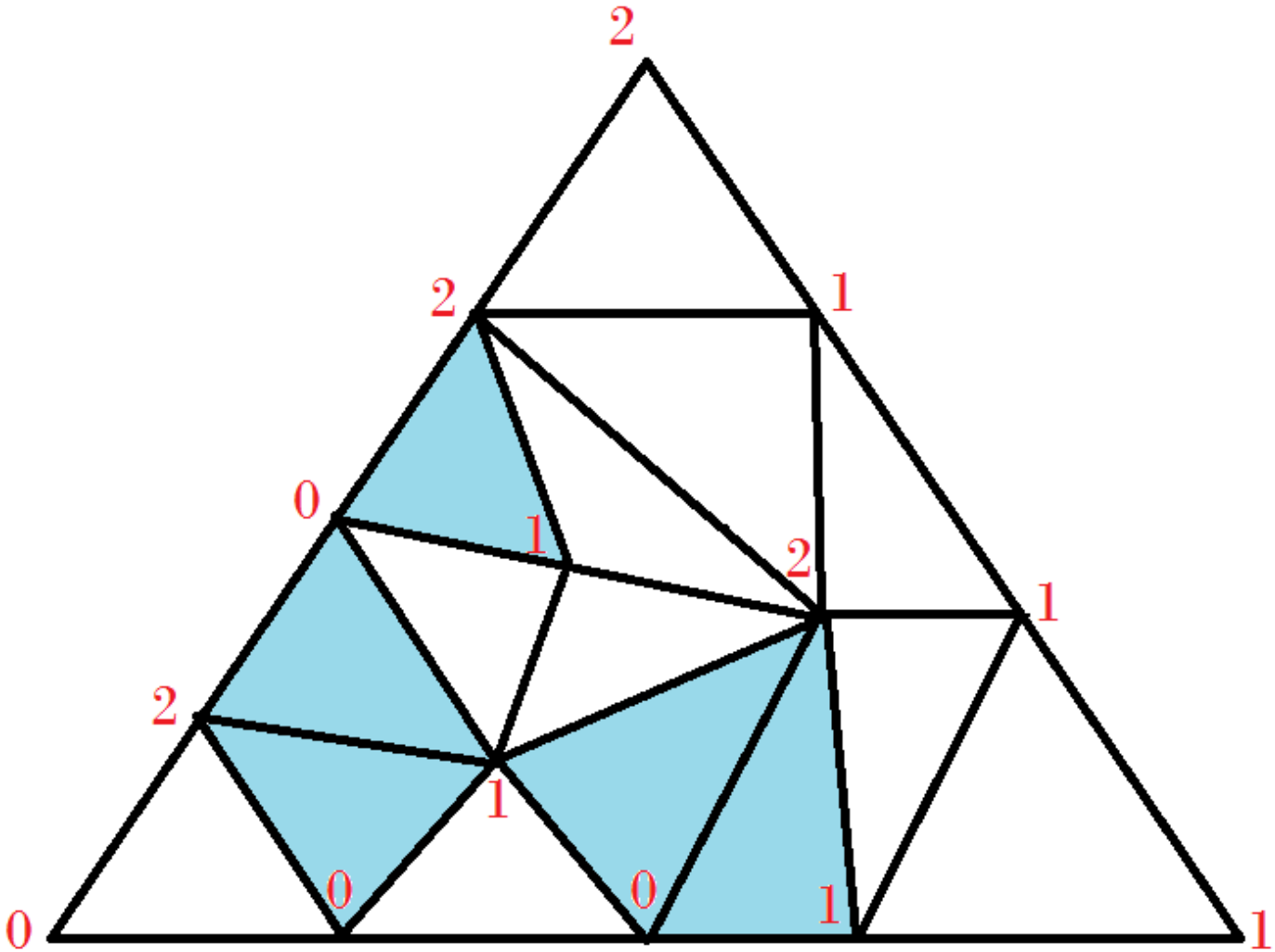
- On 0–1 boundary must use 0 or 1
- On 1–2 boundary must use 1 or 2
- On 0–2 boundary must use 0 or 2

Who has the winning strategy? What is it?

# Rules for Triangle Game

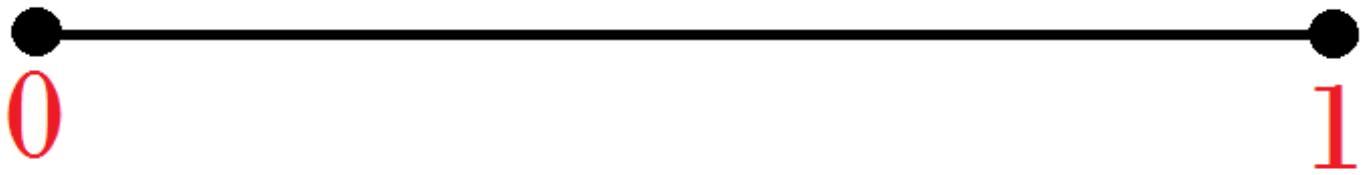


# Rules for Triangle Game



## The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.

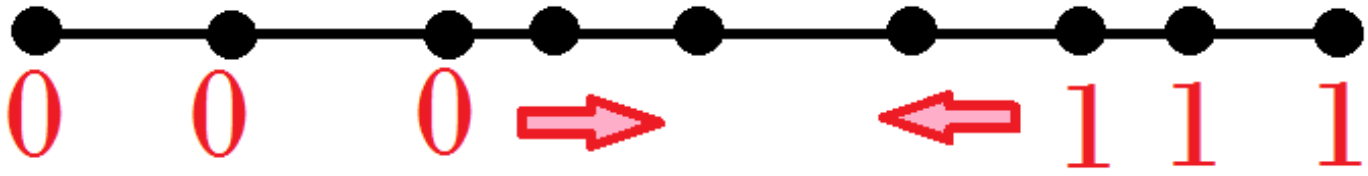


## The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.

## The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.




Cannot prevent at least one 0–1 segment.

# The Line Game (cont)

Mono-variant: as add labels, number of 0–1 segments stays the same or increases by 2.

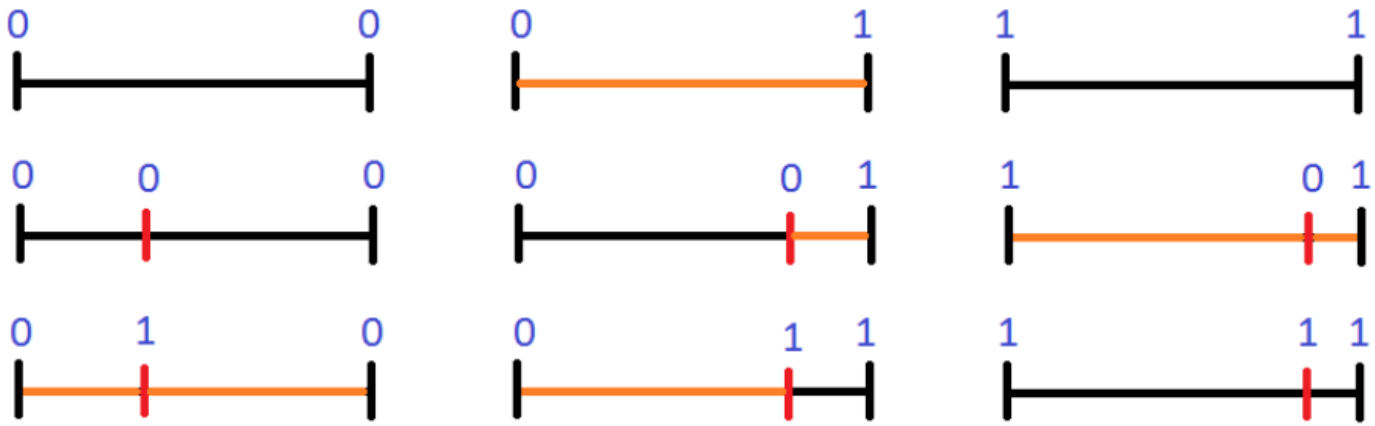



**Figure 3.** The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

Can also view this as a parity argument.

Zeckendorf Games
with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8

# Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

# Summand Minimality

## Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

## Theorem

*The Zeckendorf decomposition is **summand minimal**.*

## Overall Question

What other recurrences are summand minimal?

# Positive Linear Recurrence Sequences

## Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence $\{a_n\}$ with

$$a_n := c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \ldots, c_t)$ the **signature of the sequence**.

## Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

*For a PLRS with signature $(c_1, c_2, \ldots, c_t)$, the Generalized Zeckendorf Decompositions are summand minimal if and only if*

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

## Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of $N$, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

## Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of $N$, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

- Move to $\mathcal{D}'$ by
  - $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\mathrm{Ind}(\mathcal{D}') \leq \mathrm{Ind}(\mathcal{D})$.
  - $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k-1$.
  - $F_k + F_{k+1} = F_{k+2}$: $2k+1$ vs $k+2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\mathrm{Ind}'(\mathcal{D}) = b_1\sqrt{1} + \cdots + b_n\sqrt{n}$.

## Rules

- Two player game, alternate turns, last to move wins.

## Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1$, $F_2$, $F_3$, ..., start with $N$ pieces in $F_1$ and others empty.

## Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1$, $F_2$, $F_3$, ..., start with $N$ pieces in $F_1$ and others empty.
- A turn is one of the following moves:
  - ◇ If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$
    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - ◇ If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$.

## Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1$, $F_2$, $F_3$, ..., start with $N$ pieces in $F_1$ and others empty.
- A turn is one of the following moves:
  - ◇ If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$
    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - ◇ If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$.

Questions:

- Does the game end? How long?
- For each $N$ who has the winning strategy?
- What is the winning strategy?

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 10 | 0 | 0 | 0 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 1: $F_1 + F_1 = F_2$

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 8 | 1 | 0 | 0 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 2: $F_1 + F_1 = F_2$

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 6 | 2 | 0 | 0 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 1: $2F_2 = F_3 + F_1$

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 7 | 0 | 1 | 0 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 2: $F_1 + F_1 = F_2$

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 5 | 1 | 1 | 0 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 1: $F_2 + F_3 = F_4$.

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 5 | 0 | 0 | 1 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 2: $F_1 + F_1 = F_2$.

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 0 | 1 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 1: $F_1 + F_1 = F_2$.

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 0 | 1 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 2: $F_1 + F_2 = F_3$.

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

Next move: Player 1: $F_3 + F_4 = F_5$.

## Sample Game

Start with 10 pieces at $F_1$, rest empty.

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 1 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

No moves left, Player One wins.

## Sample Game

Player One won in 9 moves.

| | | | | |
|---|---|---|---|---|
| 10 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 0 | 0 |
| 6 | 2 | 0 | 0 | 0 |
| 7 | 0 | 1 | 0 | 0 |
| 5 | 1 | 1 | 0 | 0 |
| 5 | 0 | 0 | 1 | 0 |
| 3 | 1 | 0 | 1 | 0 |
| 1 | 2 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

## Sample Game

Player Two won in 10 moves.

| | | | | |
|---|---|---|---|---|
| 10 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 0 | 0 |
| 6 | 2 | 0 | 0 | 0 |
| 7 | 0 | 1 | 0 | 0 |
| 5 | 1 | 1 | 0 | 0 |
| 5 | 0 | 0 | 1 | 0 |
| 3 | 1 | 0 | 1 | 0 |
| 1 | 2 | 0 | 1 | 0 |
| 2 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| $[F_1 = 1]$ | $[F_2 = 2]$ | $[F_3 = 3]$ | $[F_4 = 5]$ | $[F_5 = 8]$ |

## Games end

**Theorem**

*All games end in finitely many moves.*

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

## Games Lengths: I

Upper bound: At most $n \log_\phi \left(n\sqrt{5} + 1/2\right)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in $n$'s Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

# Games Lengths: II

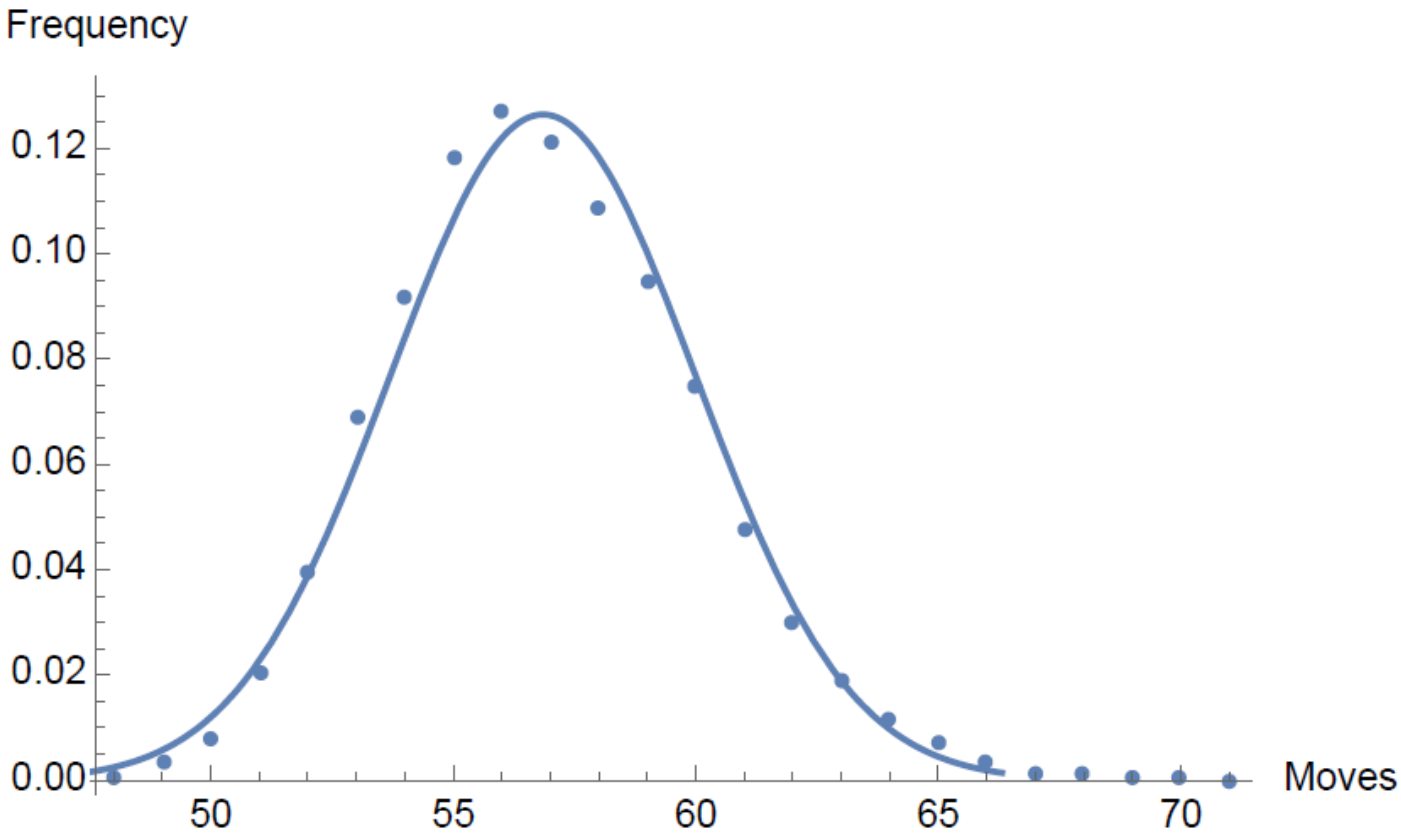



**Figure:** Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

## Winning Strategy

**Theorem**

*Payer Two Has a Winning Strategy*

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

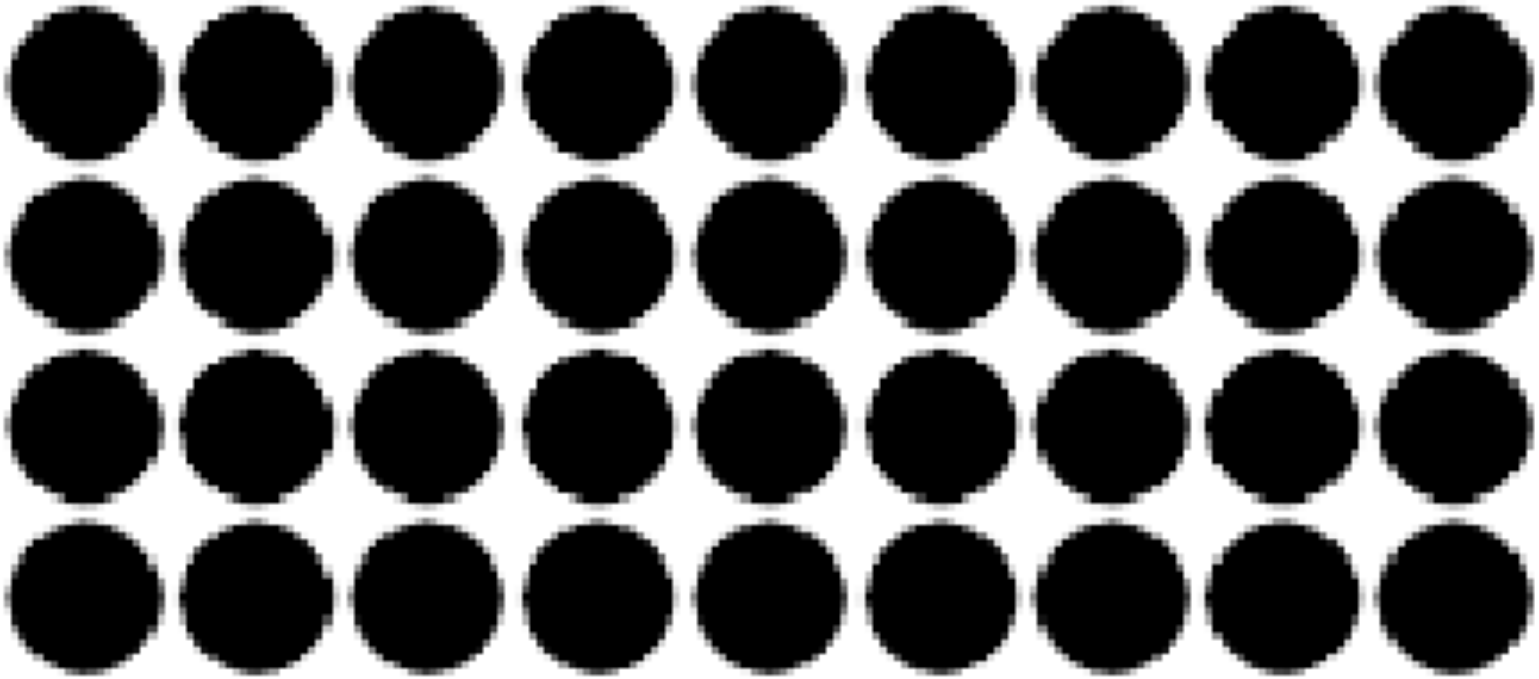
Will highlight idea with a simpler game.

## Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at $(i, j)$ and coloring every dot $(m, n)$ with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!

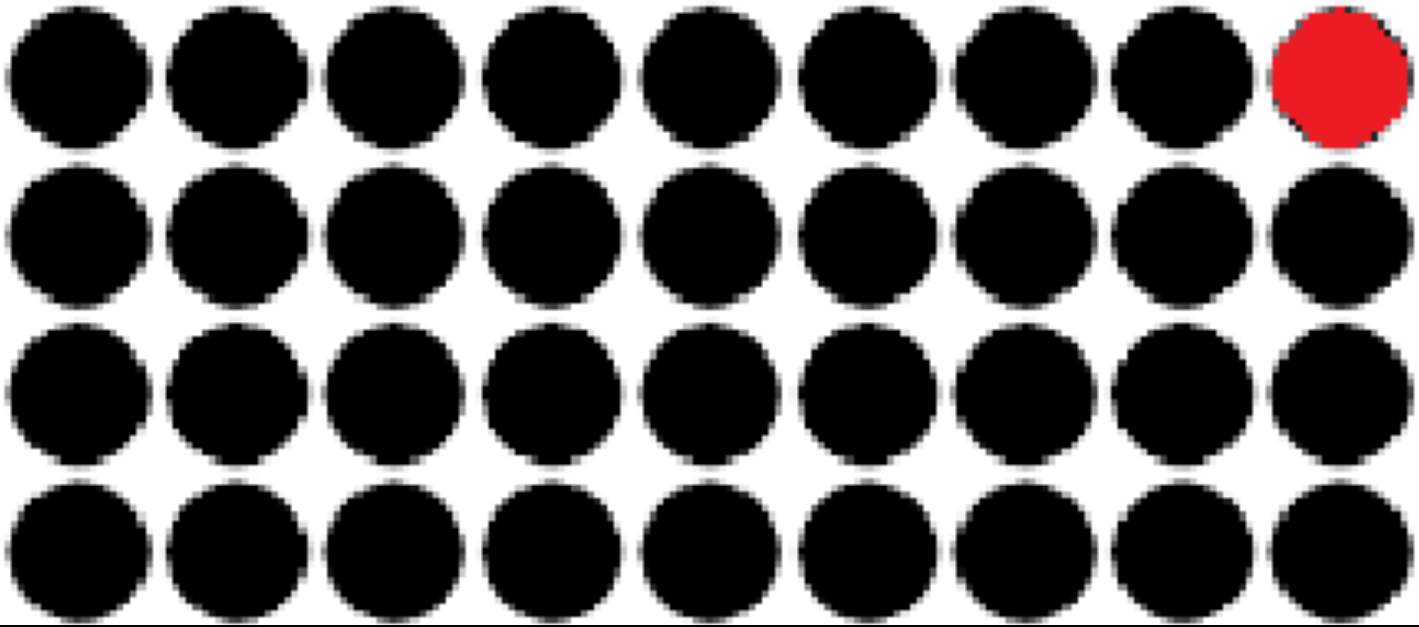


## Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at $(i, j)$ and coloring every dot $(m, n)$ with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.

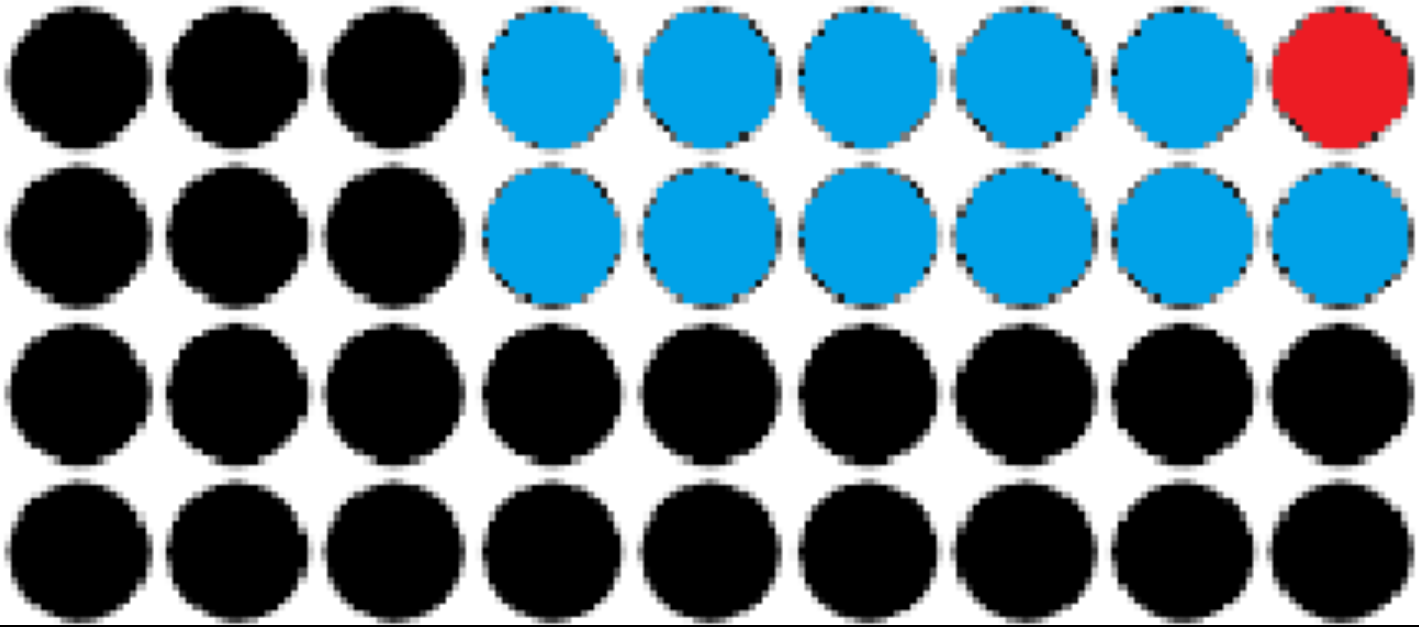


## Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at $(i, j)$ and coloring every dot $(m, n)$ with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.

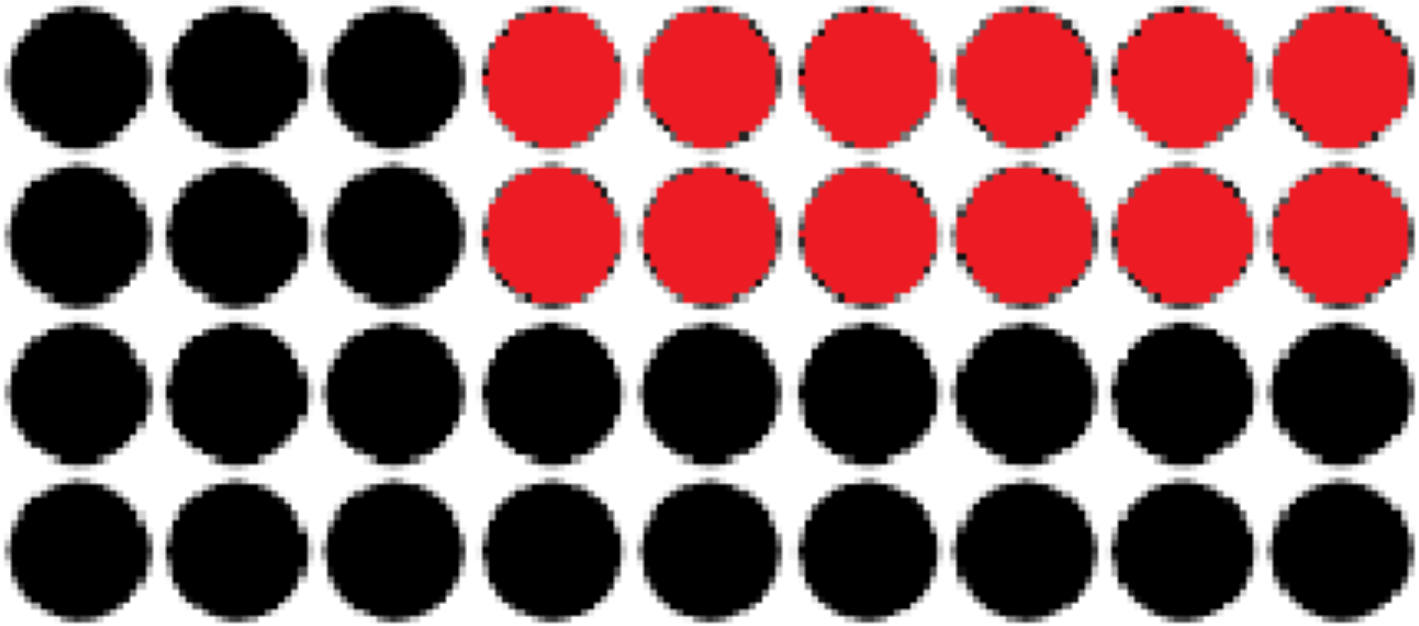


## Winning Strategy: Intuition from Dot Game

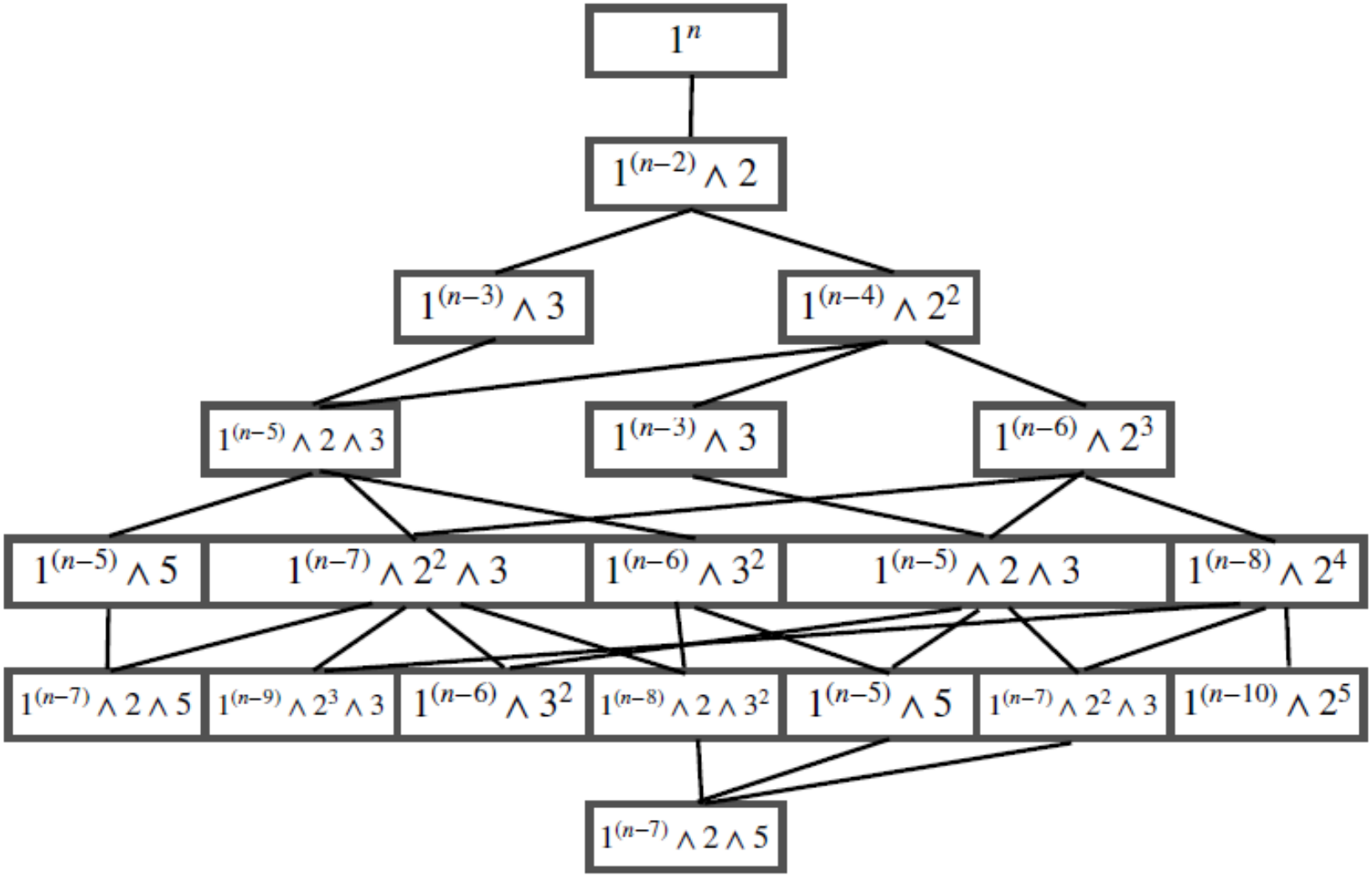
Two players, alternate. Turn is choosing a dot at $(i, j)$ and coloring every dot $(m, n)$ with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

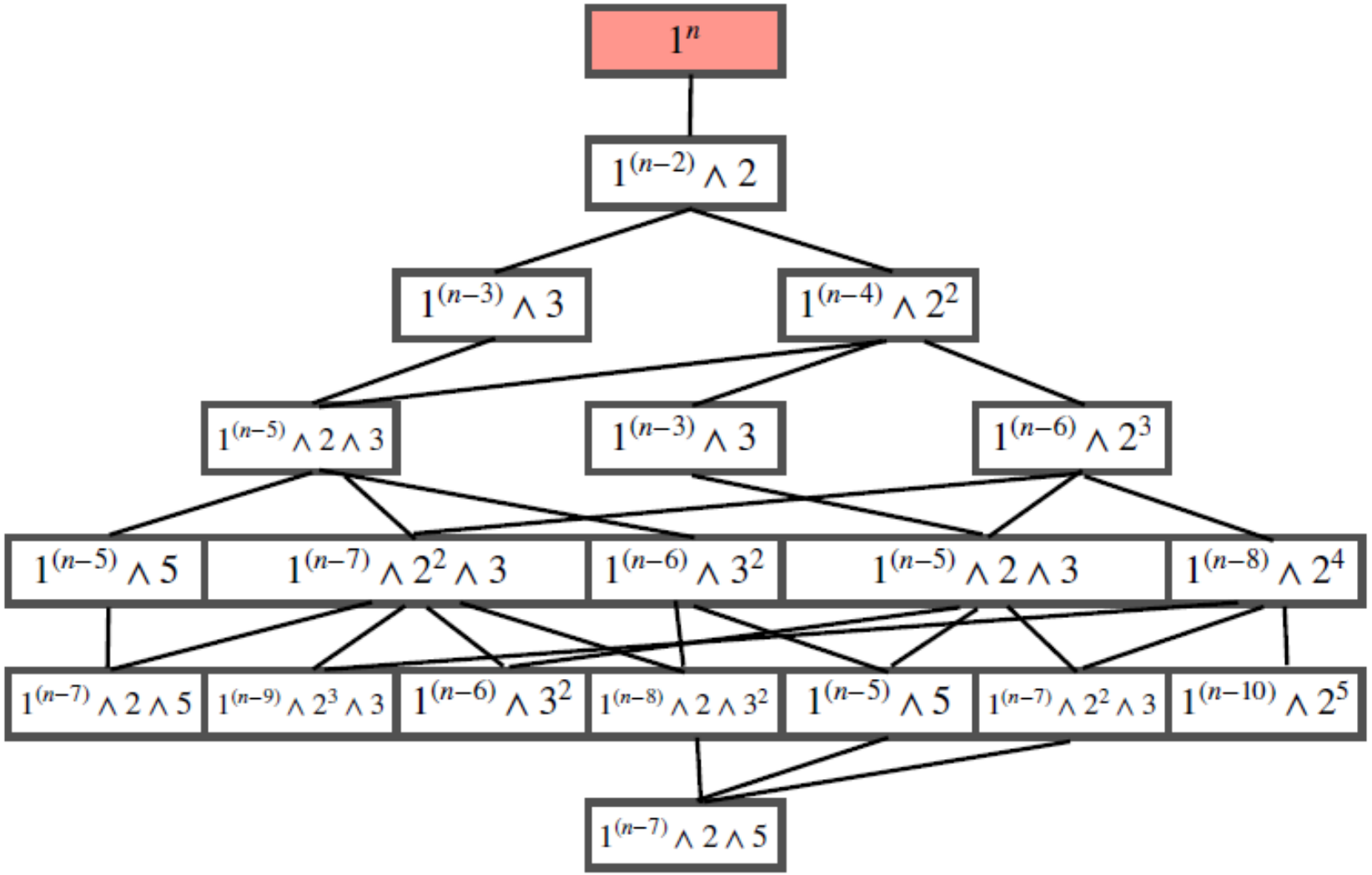
Proof Player 1 has a winning strategy. If have, play; if not, steal.



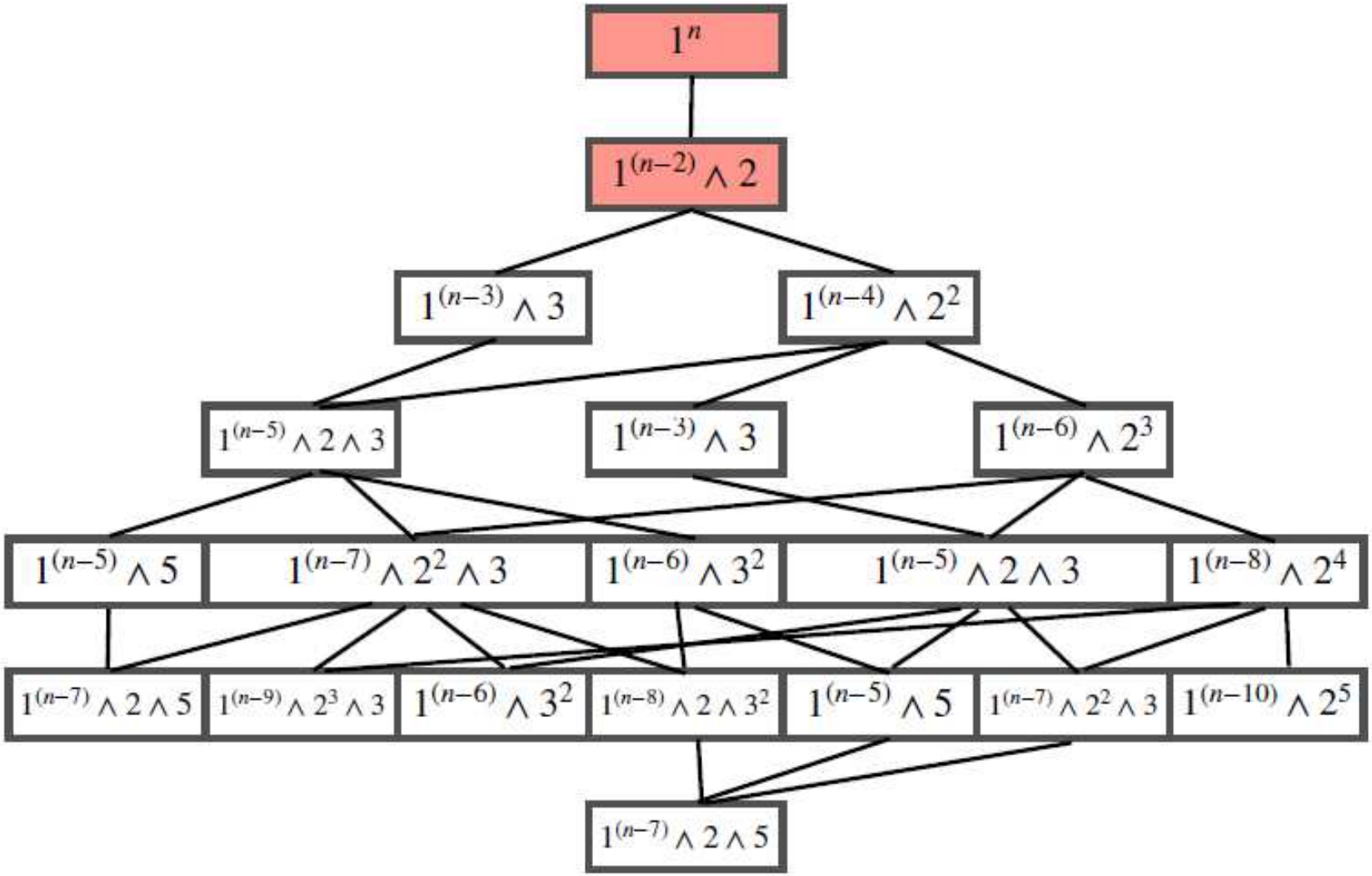
# Sketch of Proof for Player Two's Winning Strategy



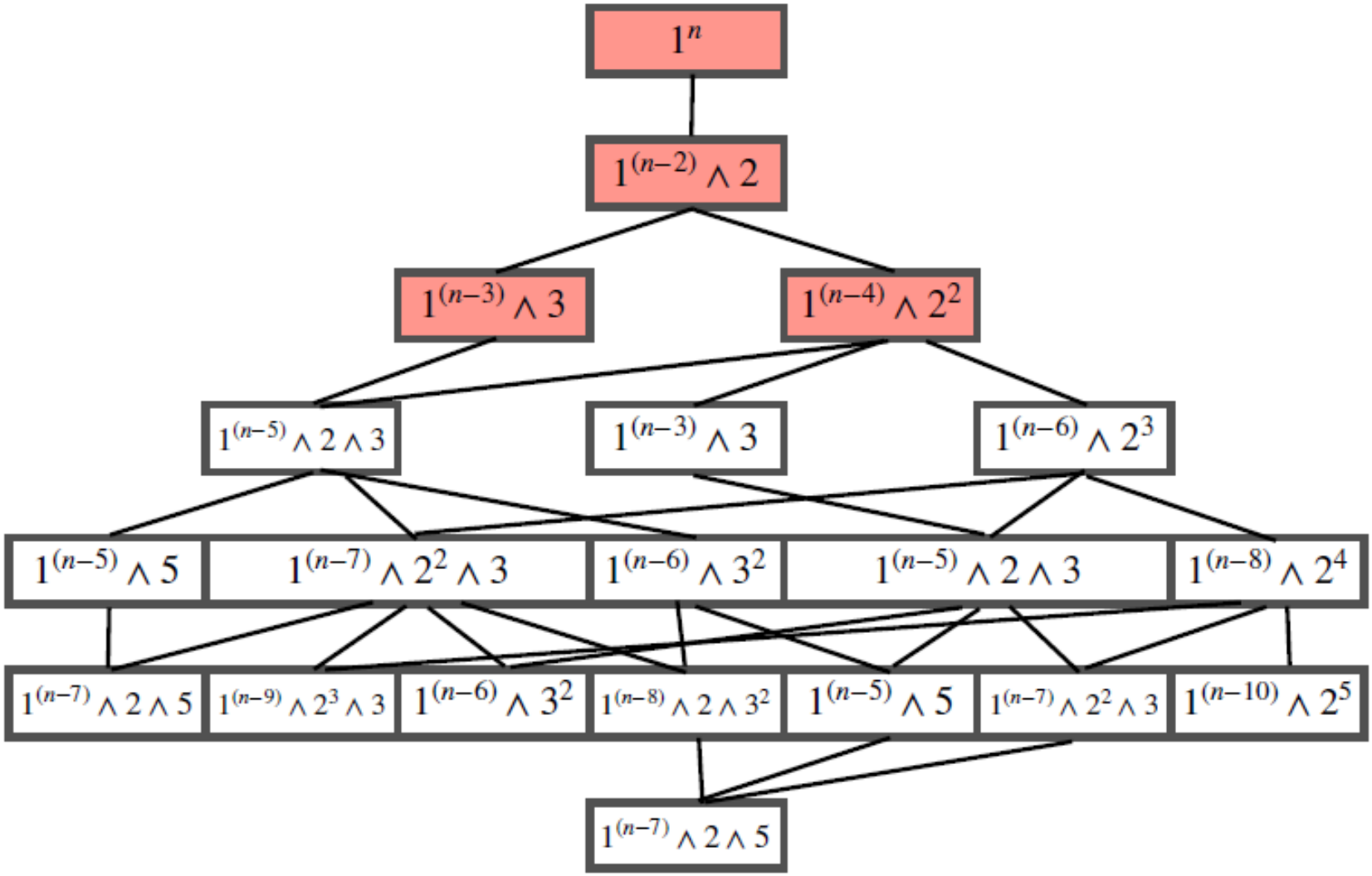
# Sketch of Proof for Player Two's Winning Strategy



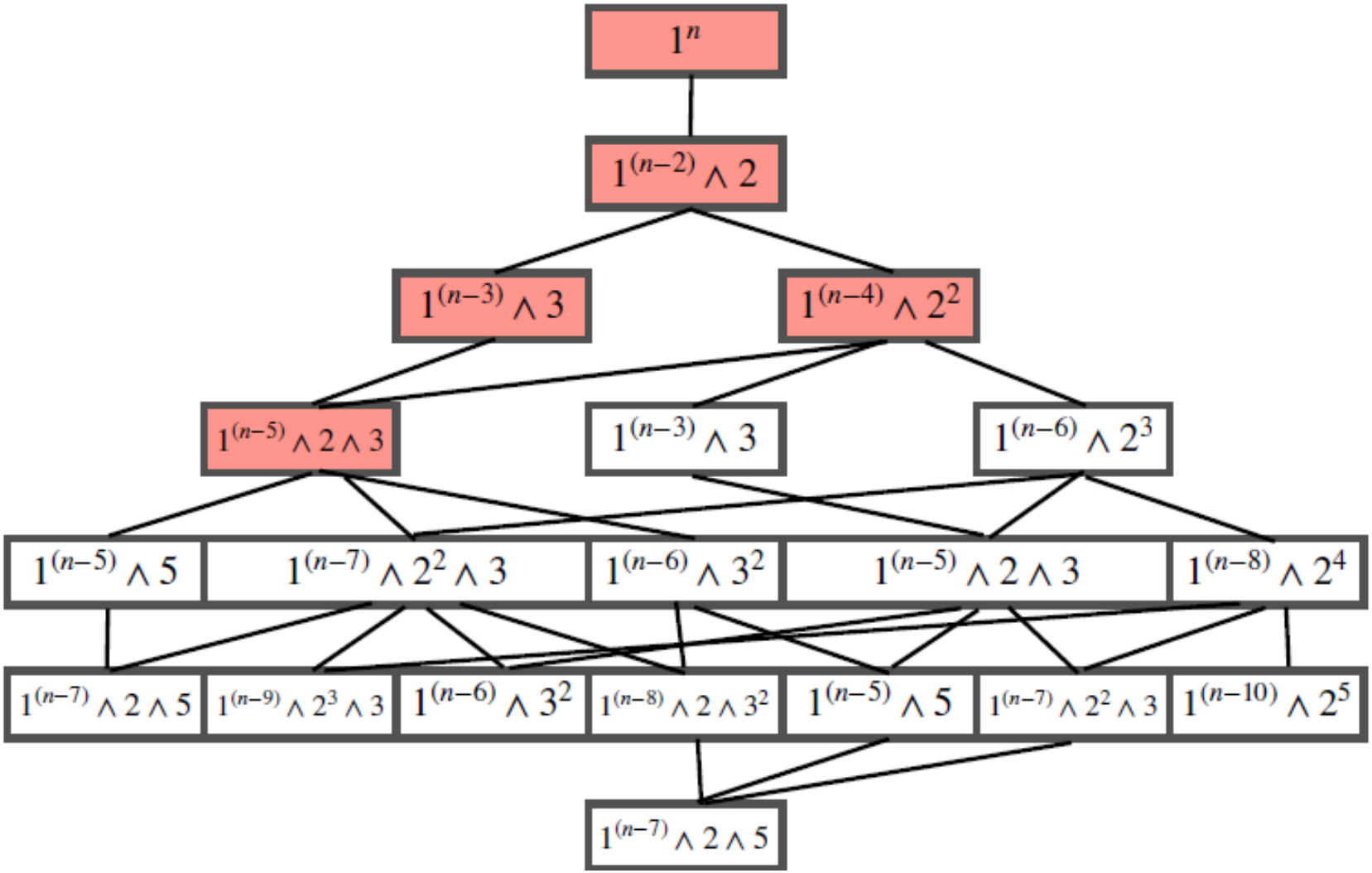
# Sketch of Proof for Player Two's Winning Strategy



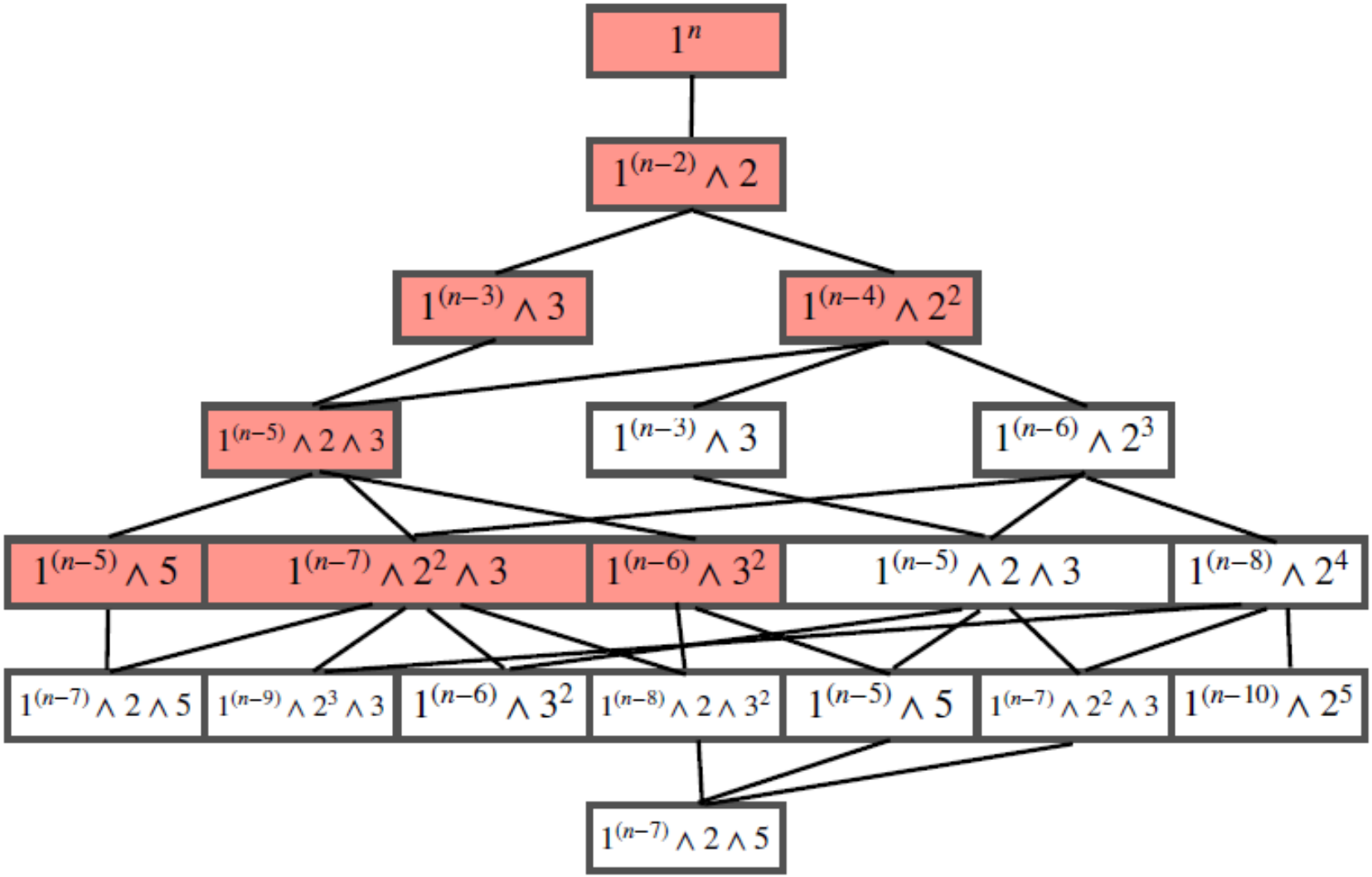
# Sketch of Proof for Player Two's Winning Strategy



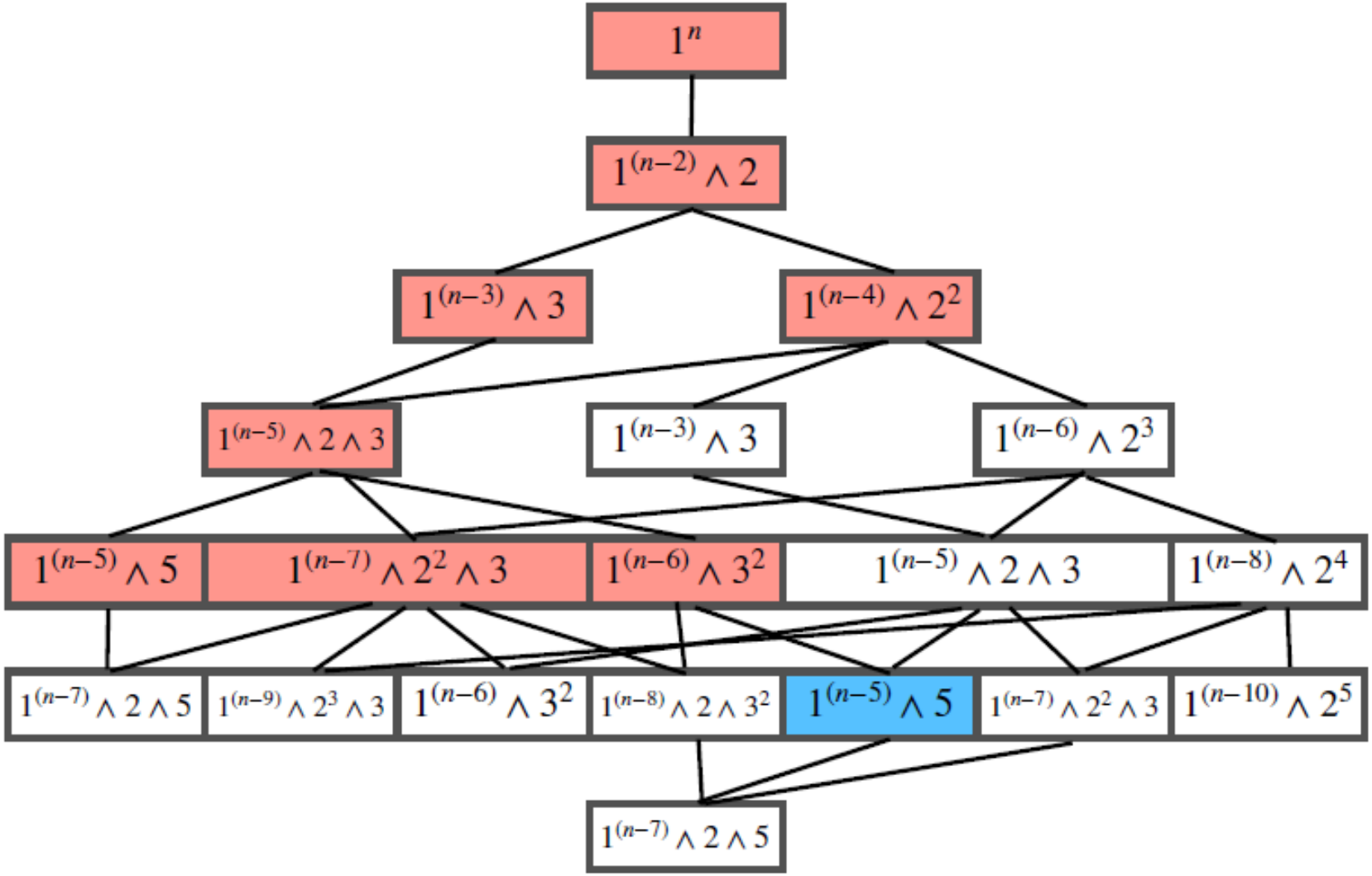
# Sketch of Proof for Player Two's Winning Strategy



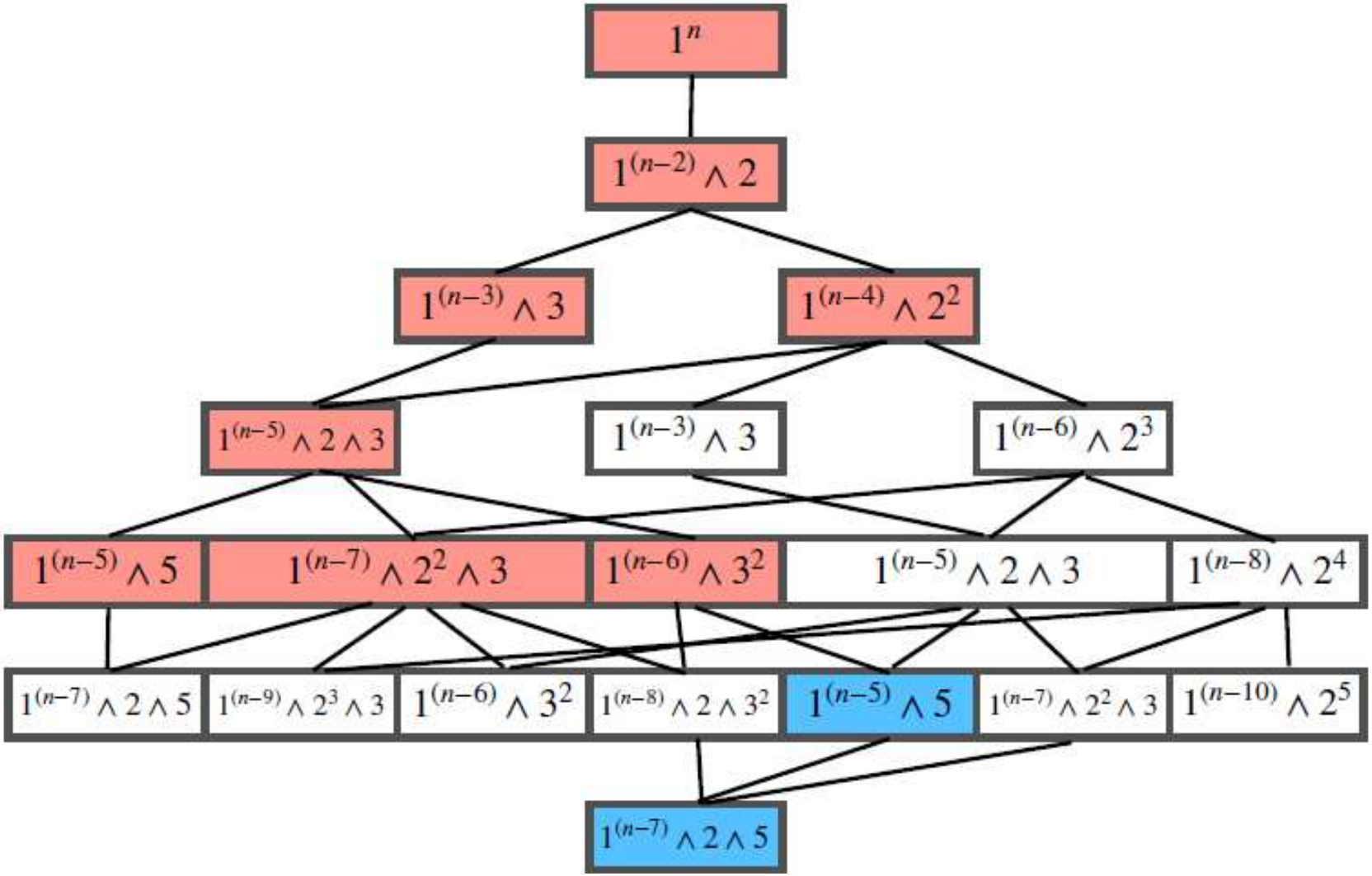
# Sketch of Proof for Player Two's Winning Strategy



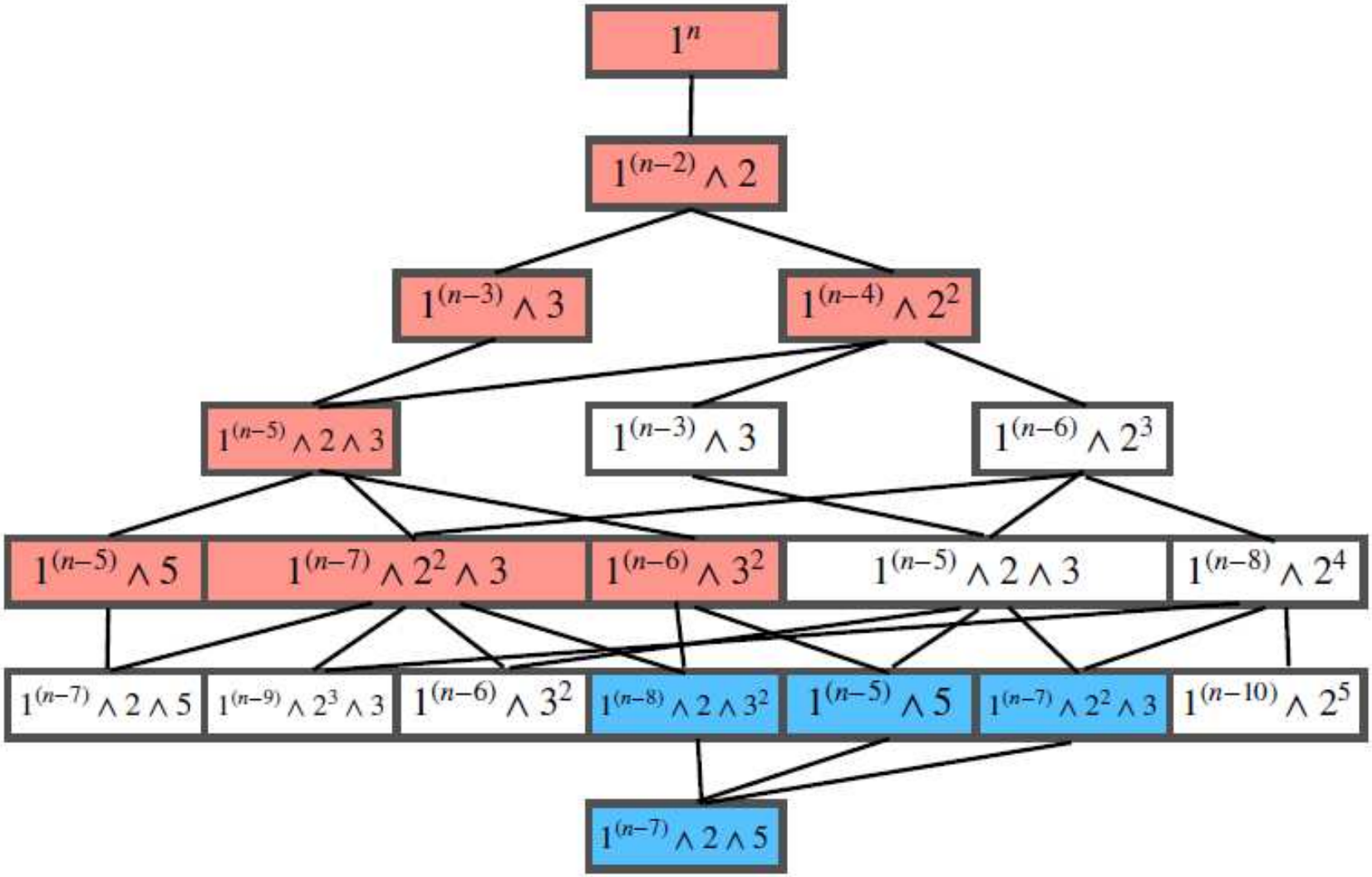
# Sketch of Proof for Player Two's Winning Strategy



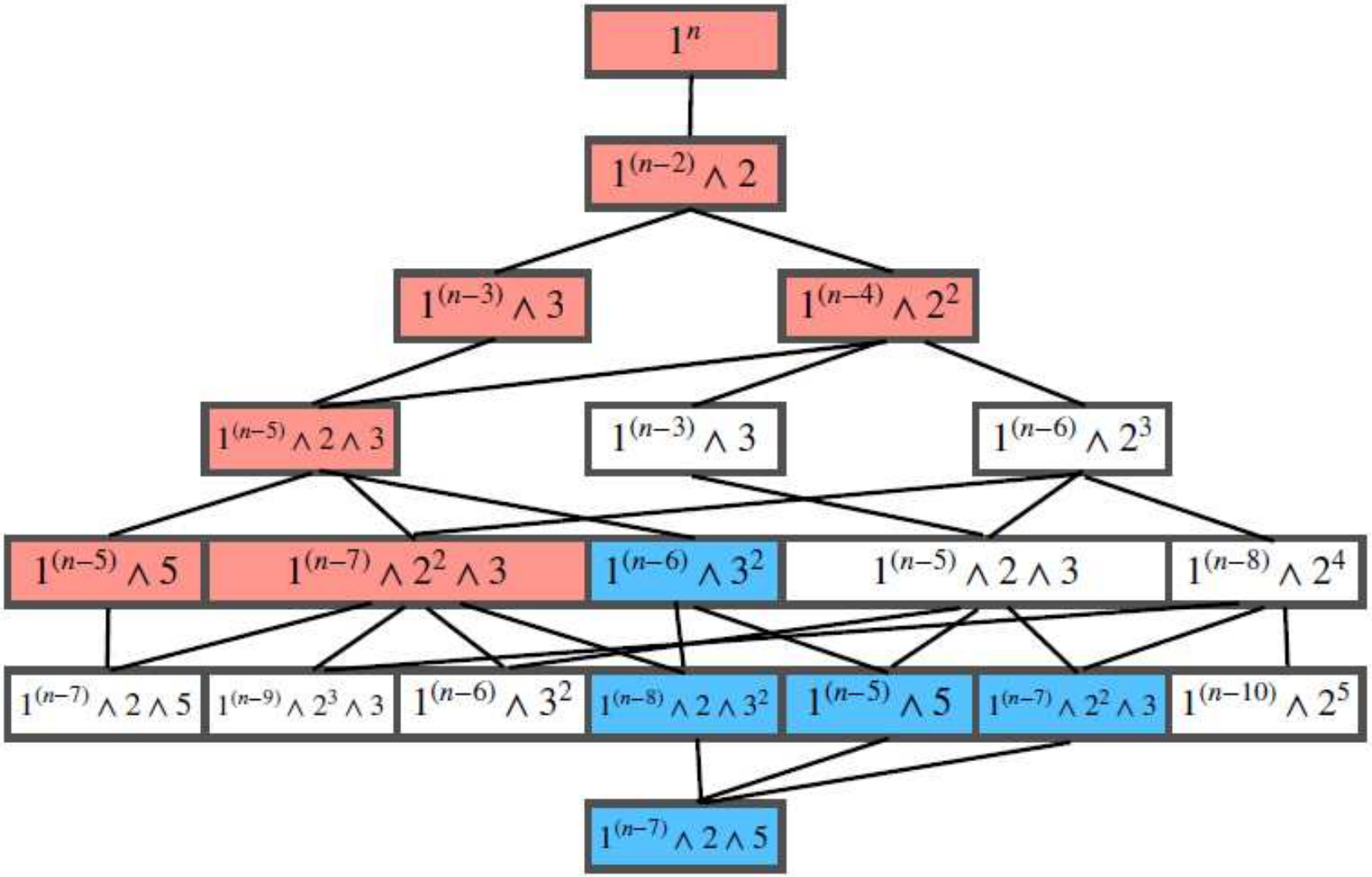
# Sketch of Proof for Player Two's Winning Strategy



# Sketch of Proof for Player Two's Winning Strategy



# Sketch of Proof for Player Two's Winning Strategy



## Future Work

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- Define $k$-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

# Games

## Games: Coins on a line

You have $2N$ coins of varying denominations (each is a non-negative real number) in a line. Players A and B take turns choosing one coin from either end. Does Player A or B have a winning strategy (i.e., a way to ensure they get at least as much as the other?) If yes, who has it and find it if possible!

## Games: Devilish Coins

You die and the devil comes out to meet you. In the middle of the room is a giant circular table and next to the walls are many sacks of coins. The devil speaks. *We'll take turns putting coins down flat on the table. I'll put down a coin and then you'll put down a coin, and so on. The coins cannot overlap and they cannot hang over the edge of the table. The last person to put down a coin wins, or equivalently, the last person who can no longer put a coin down on the table loses. You decide if you want to go first.*

Do you have a winning strategy for the game? If yes, what?

## Games: Prime Heaps

Alice and Bob play a game in which they take turns removing stones from a heap that initially has $n$ stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many such $n$ such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

# More Irrationals

# $\sqrt{3}$

Assume $\sqrt{3} = a/b$ with $b$ minimal.

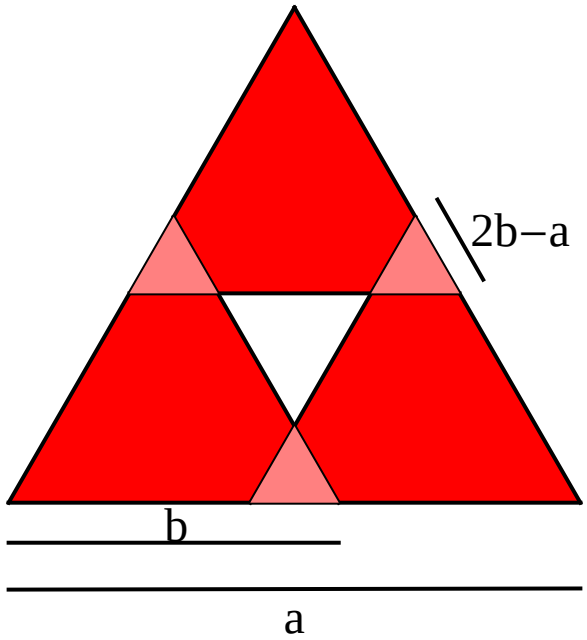



**Figure:** Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

Have $3(2b - a)^2 = (2a - 3b)^2$ so $\sqrt{3} = (2a - 3b)/(2b - a)$, note $2b - a < b$ (else $b \geq a$), violates minimality.

# $\sqrt{5}$

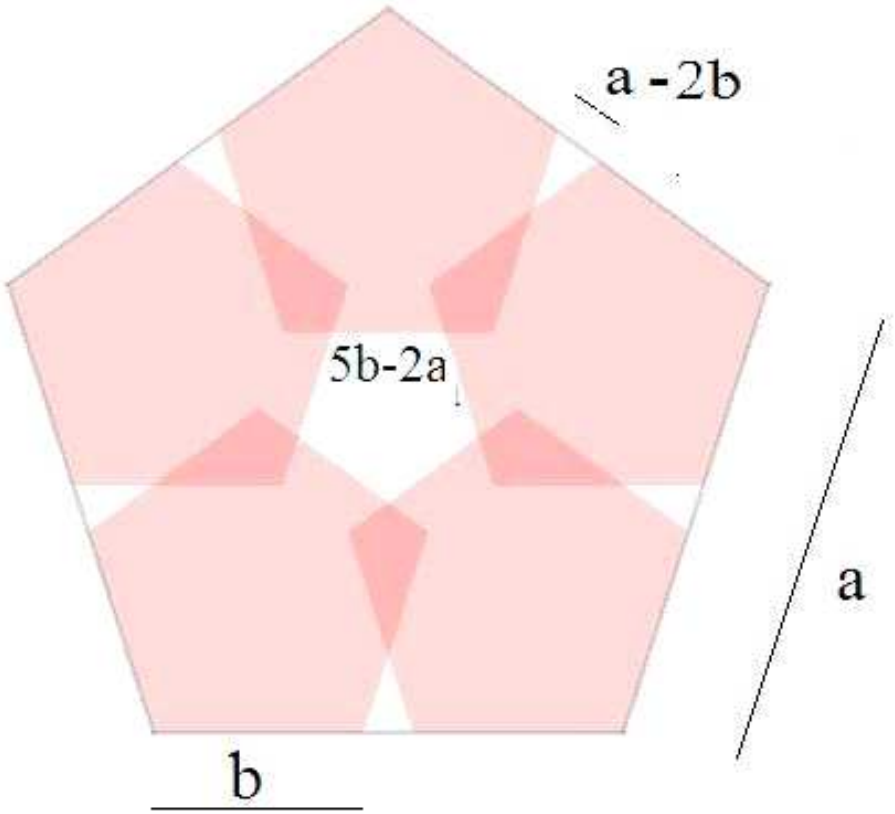
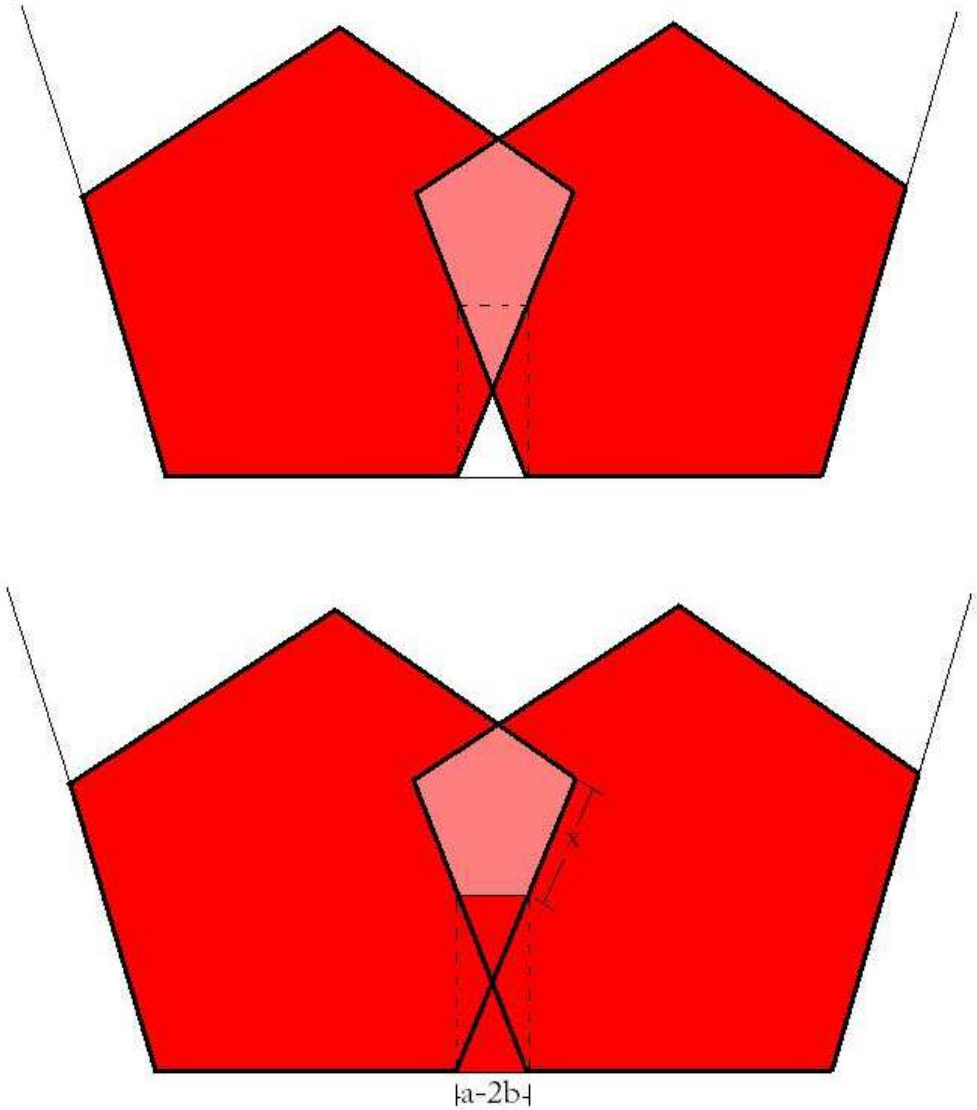



**Figure:** Geometric proof of the irrationality of $\sqrt{5}$.

# $\sqrt{5}$



## $\sqrt{5}$

A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length $b - 2(a - 2b) = 5b - 2a$.

We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a - 2b)^2 = (5b - 2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.

# $\sqrt{6}$

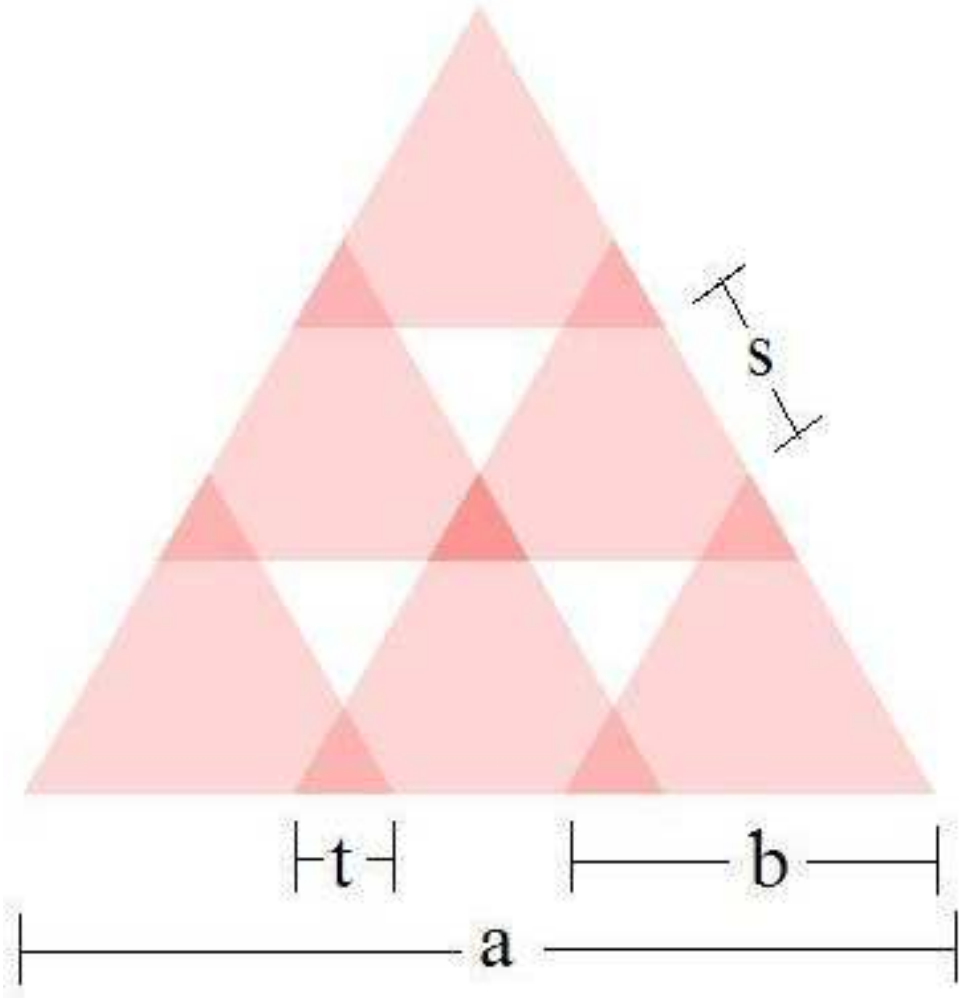



**Figure:** Geometric proof of the irrationality of $\sqrt{6}$.

## Closing Thoughts

Could try to do $\sqrt{10}$ but eventually must break down. Note $3, 6, 10$ are triangular numbers ($T_n = n(n+1)/2$).

$T_8 = 36$ and thus $\sqrt{T_8}$ is an integer!

Can you get a cube-root?

What other numbers?