Sums and Differences of Correlated Random Sets

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GSUMC 2014, Rowan University
April 5, 2014
Given $A \subset \mathbb{Z}$, let

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$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}.$$
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**Example:** For $A = \{1, 4, 5\}$:

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A + A = \{2, 5, 6, 8, 9, 10\} \\
A - A = \{0, \pm 1, \pm 3, \pm 4\}.
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Introduction

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A finite set $A$ is **sum dominated**, or more sum than difference (MSTD) if $|A + A| > |A - A|$.

The smallest example: $\{0; 2; 3; 4; 7; 11; 12; 14\}$. 
Previous Results

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**Hegarty-Miller (2009):** If the probability to pick a number from 1 to $n$ into set $A$ decays with $n$, then probability $A$ is MSTD converges to 0 as $n \to \infty$. 
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**Hegarty (2007):** The smallest size of a sum-dominated set is 8.
Correlated Random Pairs

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We investigate sums and differences of pairs of subsets \((A, B) \subset \{0, \ldots, n\}\). A pair is sum dominated or MSTD if

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|A + B| > |\pm (A - B)| = |(A - B) \cup (B - A)|.
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We select such pairs according to the dependent random process:

\[ P(k \in A) = p; \quad P(k \in B | k \in A) = \rho_1; \quad P(k \in B | k \notin A) = \rho_2. \]
Correlated Random Pairs

$(\rho_1, \rho_2) = (1, 0) \implies (A, A).$
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- \((\rho_1, \rho_2) = (0, 1) \implies (A, A^c)\).
- \(\rho_1 = \rho_2, \implies (A, B)\) independent.
Let $P(\tilde{\rho}, n)$ be the probability that a $\tilde{\rho}$-correlated pair $(A, B) \subset \{0, \ldots, n\}$ is MSTD.
Let $P(\vec{\rho}, n)$ be the probability that a $\vec{\rho}$-correlated pair $(A, B) \subset \{0, \ldots, n\}$ is MSTD.

**Theorem**

For any $\vec{\rho} \in [0, 1]^3$, the limit

$$\lim_{n \to \infty} P(\vec{\rho}, n) =: P(\vec{\rho})$$

exists. Moreover, as long as $p \notin \{0, 1\}$ and $(\rho_1, \rho_2) \neq (0, 0), (1, 1)$, then $P(\vec{\rho})$ is strictly positive.
Probability function

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Main idea of proof: same approach with Martin O’Bryant (2007) and Zhao (2010), construct an appropriate fringe (edge elements), then fill in the middle.
The function $P(\vec{\rho})$

Theorem

*The function $P(\vec{\rho})$ is continuous on $[0, 1]^3$.***
The function $P(\hat{\rho})$

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**Conjecture 1**: Function $P(p, \rho_1, \rho_2)$ is differentiable.
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**Conjecture 1:** Function $P(p, \rho_1, \rho_2)$ is differentiable.

**Conjecture 2:** $\max P=P(0, 1, 1/2) \approx 0.03$. 
Taking $p \to 0$

In previous section, we know that for any fixed $(p, \rho_1, \rho_2)$ there is a positive percentage of MSTD pairs.
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We get similar results to Hegarty-Miller (2009): if $\bar{\rho}$ decays with $n$ (either $p \to 0$ or $\rho_1 + \rho_2 \to 0$) then the probability a correlated pair $(A, B)$ in $\{1, \cdots, n\}$ is MSTD converges to 0 as $n \to \infty$. 
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The minimal MSTD pair

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Examples of minimal size MSTD pair:

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A = \{1, 2, 5, 7\}, \quad B = \{1, 3, 6, 7\}
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A = \{3, 4, 6\}, \quad B = \{1, 2, 5, 7, 8\}
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Idea of proof: different from Hegarty (2007), we use combinatorial approach. Show that if \((A, B)\) is MSTD then there must exist \(a_1, a_2, a_3 \in A\) and \(b_1, b_2, b_3 \in B\) such that \(a_1 + b_1 = a_2 + b_2 = a_3 + b_3\).
We prove results similar to previous research in a more general setting.
Summary of Results and Future Research

- We prove results similar to previous research in a more general setting.

- We show that the limit \( P(\tilde{\rho}) \) exists for each chosen \( p, \rho_1, \rho_2 \), and prove \( P \) is continuous.
Summary of Results and Future Research

- We prove results similar to previous research in a more general setting.

- We show that the limit $P(\hat{\rho})$ exists for each chosen $p, \rho_1, \rho_2$, and prove $P$ is continuous.

- In the future, we would like to prove our two conjectures, and find more analytic properties of $P$. 
We would like to thank our advisor, Steven J. Miller, our co-authors David Moon and Archit Kulkarni, the rest of the team at the Williams College SMALL REU 2013, and the National Science Foundation.

This research was funded by NSF grant DMS0850577.

Link to our paper:
http://arxiv.org/abs/1401.2588

P. Hegarty and S. Miller, When almost all sets are difference dominated, Random Structures and Algorithms 35 (2009), no. 1, 118-136.
