Classification of All Crescent Configurations on Four and Five Points

Rebecca F. Durst, Max Hlavacek, Chi Huynh

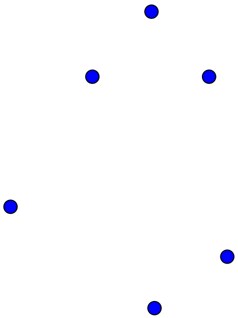
SMALL 2016

rfd1@williams.edu, mhlavacek@hmc.edu, nhuynh30@gatech.edu

Young Mathematicians Conference
Ohio State University
August 20th, 2016

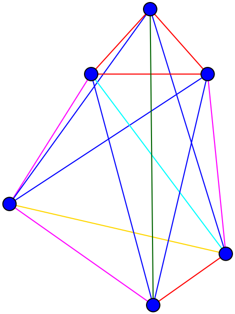
Motivation

Start with points: Given a set of points, how many distinct distances do I have?



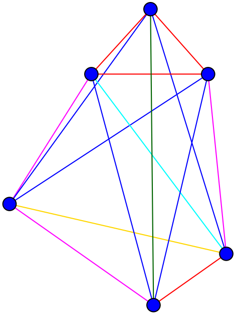
Motivation

Start with points: Given a set of points, how many distinct distances do I have?



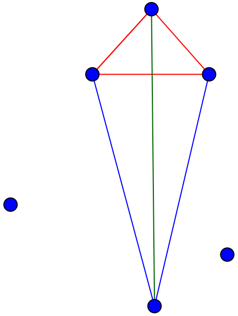
Motivation

Start with distances: Given a set of $n-1$ distinct distances, can I arrange n points such that for each $1 \leq i \leq n-1$ one of my $n-1$ distances shows up i times?



Motivation

Start with distances: Given a set of $n-1$ distinct distances, can I arrange n points such that for each $1 \leq i \leq n-1$ one of my $n-1$ distances shows up i times? Yes.



We have four points and three distinct distances with the required multiplicities.

We have four points and three distinct distances with the required multiplicities.

We have four points and three distinct distances with the required multiplicities.

We have four points and three distinct distances with the required multiplicities.

General Position in $\mathbb{R}^d$: No d+1 points on the same hyperplane and no d+2 points on the same hypersphere.

Crescent Configurations

Crescent Configuration(SMALL 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

- Erdős: **Conjecture:**(1989) There exists an N sufficiently large such that no crescent configuration exists on N points.

Crescent Configurations

Crescent Configuration(SMALL 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

- Erdős: **Conjecture:**(1989) There exists an N sufficiently large such that no crescent configuration exists on N points.
- Pomerance and Palásti: 1989,n=5, n=6, n=7,n=8.

Crescent Configurations

Crescent Configuration(SMALL 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

- Erdős: **Conjecture:**(1989) There exists an N sufficiently large such that no crescent configuration exists on N points.
- Pomerance and Palásti: 1989,n=5, n=6, n=7,n=8.

Crescent Configurations

Crescent Configuration(SMALL 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

- Erdős: **Conjecture:**(1989) There exists an N sufficiently large such that no crescent configuration exists on N points.
- Pomerance and Palásti: 1989,n=5, n=6, n=7,n=8.

Crescent Configurations

Crescent Configuration(SMALL 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

- Erdős: **Conjecture:**(1989) There exists an N sufficiently large such that no crescent configuration exists on N points.
- Pomerance and Palásti: 1989,n=5, n=6, n=7,n=8.



Crescent Configurations

Crescent Configuration(SMALL 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

- Erdős: **Conjecture:**(1989) There exists an N sufficiently large such that no crescent configuration exists on N points.
- Pomerance and Palásti: 1989,n=5, n=6, n=7,n=8.




Why Classify?

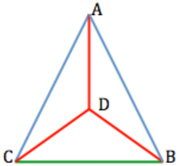
- **Distance Coordinate:** The distance coordinate, D_a of a point a is the set of all distances, counting multiplicity, between a and the other points in a set, $\mathcal{P}$.

Why Classify?

- **Distance Coordinate:** The distance coordinate, D_a of a point a is the set of all distances, counting multiplicity, between a and the other points in a set, $\mathcal{P}$.
- **Distance Set:** The distance set, $\mathcal{D}$, corresponding to a set of points, $\mathcal{P}$, is the set of the distance coordinates for each point in the $\mathcal{P}$.

Why Classify?

- **Distance Coordinate:** The distance coordinate, D_a of a point a is the set of all distances, counting multiplicity, between a and the other points in a set, $\mathcal{P}$.
- **Distance Set:** The distance set, $\mathcal{D}$, corresponding to a set of points, $\mathcal{P}$, is the set of the distance coordinates for each point in the $\mathcal{P}$.



(1)

Graph Isomorphism

Theorem (Durst-Hlavacek-Huynh 2016)

Let A and B be two crescent configurations on the same number of points n. If A and B have the same distance sets, then there exists a graph isomorphism $A \to B$.

Graph Isomorphism

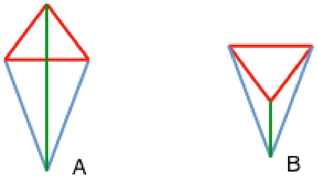
Theorem (Durst-Hlavacek-Huynh 2016)

Let A and B be two crescent configurations on the same number of points n. If A and B have the same distance sets, then there exists a graph isomorphism $A \to B$.

Graph Isomorphism (Gervasi)

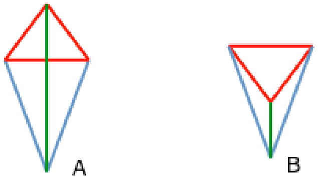
Graph A is isomorphic to graph B if and only if there exists a bijective function $f : V(A) \mapsto V(B)$, (where V(A) and V(B) are the vertex spaces) such that: 1. $\forall a_i \in A, l_A(a_i) = l_B(f(a_i))$, 2. $\forall a_i, a_j \in V, \ \{a_i, a_j\} \in E_A \leftrightarrow \{f(a_i), f(a_j)\} \in E_B$, and 3. $\forall \{a_i, a_j\} \in E_A, w_A(\{a_i, a_j\}) = w_B(f(\{a_i, a_j\}))$, where $\{l_A, l_B\}$ and $\{w_A, w_B\}$ are functions that define the labels of the vertices and edges of A and B respectively.

Graph Isomorphism



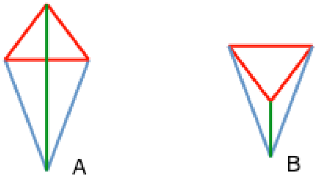
$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

Graph Isomorphism



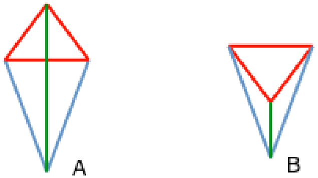
$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

Graph Isomorphism



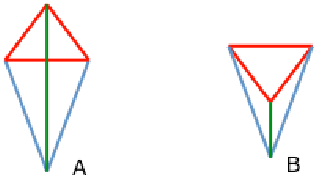
$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

Graph Isomorphism



$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

Graph Isomorphism



$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

All Configurations on Four and Five Points

Theorem (Durst-Hlavacek-Huynh 2016)

Given a set of three distinct distances, $\{d1, d2, d3\}$, on four points in crescent configuration, there are only three allowable crescent configurations up to graph isomorphism

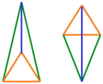
- We label these M-type, C-type, and R-type, respectively.

All Configurations on Four and Five Points

Theorem (Durst-Hlavacek-Huynh 2016)

Given a set of three distinct distances, $\{d1, d2, d3\}$, on four points in crescent configuration, there are only three allowable crescent configurations up to graph isomorphism

- We label these M-type, C-type, and R-type, respectively.



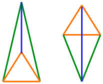
M

All Configurations on Four and Five Points

Theorem (Durst-Hlavacek-Huynh 2016)

Given a set of three distinct distances, $\{d1, d2, d3\}$, on four points in crescent configuration, there are only three allowable crescent configurations up to graph isomorphism

- We label these M-type, C-type, and R-type, respectively.



M

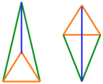
C

All Configurations on Four and Five Points

Theorem (Durst-Hlavacek-Huynh 2016)

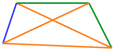
Given a set of three distinct distances, $\{d1, d2, d3\}$, on four points in crescent configuration, there are only three allowable crescent configurations up to graph isomorphism

- We label these M-type, C-type, and R-type, respectively.



M

C



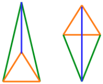
R

All Configurations on Four and Five Points

Theorem (Durst-Hlavacek-Huynh 2016)

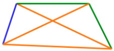
Given a set of three distinct distances, $\{d1, d2, d3\}$, on four points in crescent configuration, there are only three allowable crescent configurations up to graph isomorphism

- We label these M-type, C-type, and R-type, respectively.



M

C



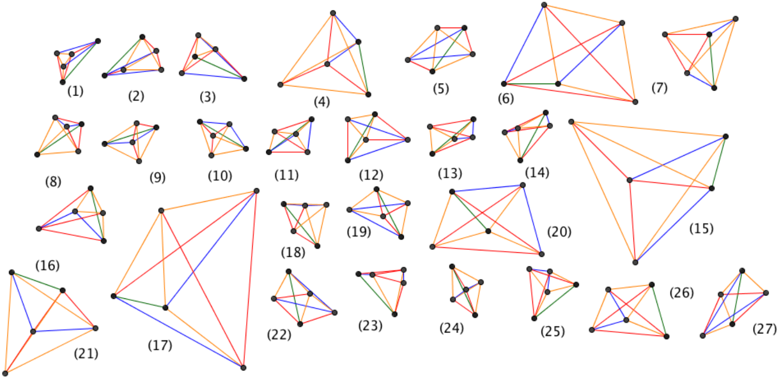
R

Theorem

Given a set of four distinct distances, $\{d1, d2, d3, d4\}$, on five points in crescent configuration, there are only 27 allowable crescent configurations up to graph isomorphism

-

Results



Remarks

Advantages

- New methods for approaching an Erdős problem.

Remarks

Advantages

- New methods for approaching an Erdős problem.
- Generates all possible distance sets.

Remarks

Advantages

- New methods for approaching an Erdős problem.
- Generates all possible distance sets.
- May be generalized to higher dimensions.

Remarks

Advantages

- New methods for approaching an Erdős problem.
- Generates all possible distance sets.
- May be generalized to higher dimensions.
- Permits the use of distance geometry to find new configurations.
  - All possible configurations on four and five points in $\mathbb{R}^2$.
  - Three new configurations on five points in $\mathbb{R}^3$.

Remarks

Advantages

- New methods for approaching an Erdős problem.
- Generates all possible distance sets.
- May be generalized to higher dimensions.
- Permits the use of distance geometry to find new configurations.
    - All possible configurations on four and five points in $\mathbb{R}^2$.
    - Three new configurations on five points in $\mathbb{R}^3$.

Remarks

Advantages

- New methods for approaching an Erdős problem.
- Generates all possible distance sets.
- May be generalized to higher dimensions.
- Permits the use of distance geometry to find new configurations.
  - All possible configurations on four and five points in $\mathbb{R}^2$.
  - Three new configurations on five points in $\mathbb{R}^3$.

Disadvantages

Remarks

Advantages

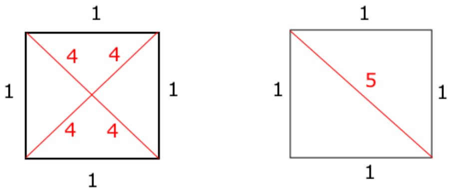
- New methods for approaching an Erdős problem.
- Generates all possible distance sets.
- May be generalized to higher dimensions.
- Permits the use of distance geometry to find new configurations.
    - All possible configurations on four and five points in $\mathbb{R}^2$.
    - Three new configurations on five points in $\mathbb{R}^3$.

Disadvantages

- Running time is $\mathcal{O}(n^n)$.

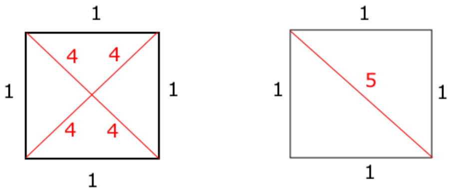
The Question of Geometric Realizability

- Given a distance set $\mathcal{D}$, can we find a set of points in a crescent configuration with $\mathcal{D}$ as its distance set in $\mathbb{R}^n$?



The Question of Geometric Realizability

- Given a distance set $\mathcal{D}$, can we find a set of points in a crescent configuration with $\mathcal{D}$ as its distance set in $\mathbb{R}^n$?
- **Distance Geometry Problem:** If we are given a set of distances between points, what can we find out about the positioning of these points?



Cayley-Menger Matrices

Cayley Menger Matrix: The Cayley Menger matrix for a set n points $\{P_1, P_2, \ldots P_n\}$ is an $(n+1) \times (n+1)$ matrix of the following form:

$$\begin{pmatrix} 0 & d_{1,2}^2 & \ldots & d_{1,n}^2 & 1 \\ d_{2,1}^2 & 0 & \ldots & d_{2,n}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n,1}^2 & d_{n,2}^2 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

where $d_{i,j}$ is the distance between P_i and P_j.

Cayley-Menger and Geometric realizability

Theorem (Sommerville 1958)

A distance set corresponding to 4 points is geometrically realizable in $\mathbb{R}^2$ *if and only if the Cayley-Menger matrix is not invertible.*

Cayley-Menger and Geometric realizability

Theorem (Sommerville 1958)

A distance set corresponding to 4 points is geometrically realizable in $\mathbb{R}^2$ if and only if the Cayley-Menger matrix is not invertible.

- We first use this to make sure the distances are realizable in the plane.

Cayley-Menger and Geometric realizability

Theorem (Sommerville 1958)

A distance set corresponding to 4 points is geometrically realizable in $\mathbb{R}^2$ *if and only if the Cayley-Menger matrix is not invertible.*

- We first use this to make sure the distances are realizable in the plane.
- We then use this to make sure no 3 points are on a line.

Cayley-Menger and Geometric realizability

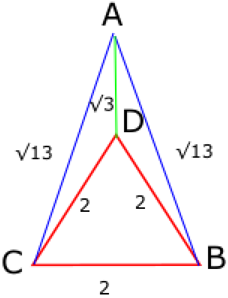
Theorem (Sommerville 1958)

A distance set corresponding to 4 points is geometrically realizable in $\mathbb{R}^2$ if and only if the Cayley-Menger matrix is not invertible.

- We first use this to make sure the distances are realizable in the plane.
- We then use this to make sure no 3 points are on a line.
- We can use similar techniques to make sure no 4 points are on a circle.

Example

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & \end{array} \\ \begin{pmatrix} 0 & 13 & 13 & 3 & 1 \\ 13 & 0 & 4 & 4 & 1 \\ 13 & 4 & 0 & 4 & 1 \\ 3 & 4 & 4 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



Solutions for a Given Crescent Configuration Type

- Suppose we are given a distance set with the multiplicities of the distances specified, but we are not given values for the distances.
- We can fix one of the unknown distances and use Cayley-Menger determinants to find a system of equations that yields geometrically realizable distances.

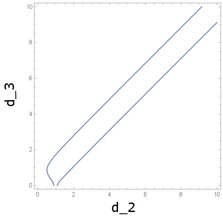


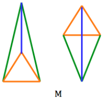
Figure: Possible values for d_2, d_3 for the M-type when $d_1 = 1$

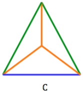
Results for $n = 4$

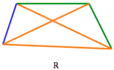
As expected, all 3 of our distance sets on 4 points are realizable in $\mathbb{R}^2$.

Results for $n = 4$

As expected, all 3 of our distance sets on 4 points are realizable in $\mathbb{R}^2$.





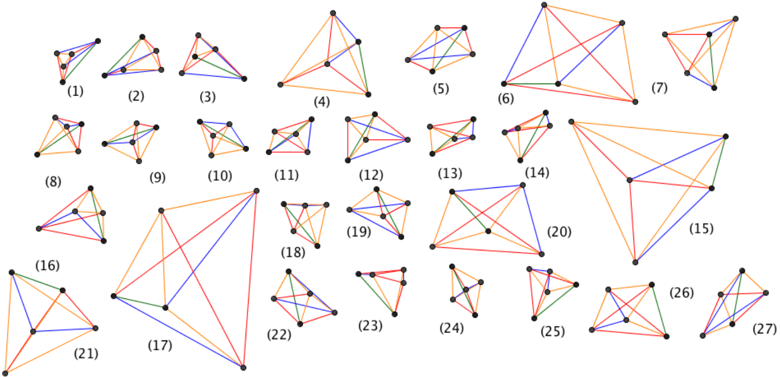


Results for $n = 5$

Exactly 27 of the 51 distance sets on 5 points are geometrically realizable.

Results for $n = 5$

Exactly 27 of the 51 distance sets on 5 points are geometrically realizable.



For a complete list of configurations, email rfd1@williams.edu.

Higher dimensions

- Cayley Menger Matrices can be used to determine whether the distances between $d + 2$ points are geometrically realizable in d-dimensional space.

Higher dimensions

- Cayley Menger Matrices can be used to determine whether the distances between $d+2$ points are geometrically realizable in d-dimensional space.

- Can some of the distance sets that are not geometrically realizable in $\mathbb{R}^2$ be realized in $\mathbb{R}^3$?

The Uniqueness Question

Given an appropriate set of $n-1$ distances, how many ways could we realize a crescent configuration on n points?

Inspiration from the Molecule Problem

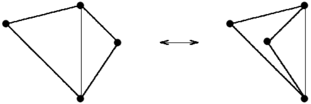


Figure: Two Realizations of a Flexible Graph[1]

- The Molecule Problem: given a set of distance measurements between points in Euclidean space, can we find the points in space? $\rightarrow$ NP-hard
- More generally: Graph realization (how many arrangements?) and rigidity (can we distort the arrangements?)

[1] B. Hendrickson. Conditions for Unique Graph Realization. SIAM Journal of Computing . 21(1). 64–84, Feb. 1992

Techniques and Terminologies

- Flexible Framework vs. Rigid Framework vs. Redundantly Rigid Framework

Techniques and Terminologies

- Flexible Framework vs. Rigid Framework vs. Redundantly Rigid Framework
- Gluck (1975): If a graph has a single rigid realization, then all its generic realizations are rigid.

Techniques and Terminologies

- Flexible Framework vs. Rigid Framework vs. Redundantly Rigid Framework
- Gluck (1975): If a graph has a single rigid realization, then all its generic realizations are rigid.
- The Rigidity Matrix
  Example: Complete graph K_3 with vertices mapped to $(0,1)$,$(-1,0)$ and $(1,0)$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 2 & 0 \end{bmatrix}$$

Theorem (Hendrickson 1992)

A framework $f(G)$ is rigid if and only if its rigidity matrix has rank exactly equal to $S(n,d)$ or the number of allowed motions, which equals $nd - d(d+1)/2$ for $n \geq d$ and $n(n-1)/2$ otherwise

A Realization for Type C

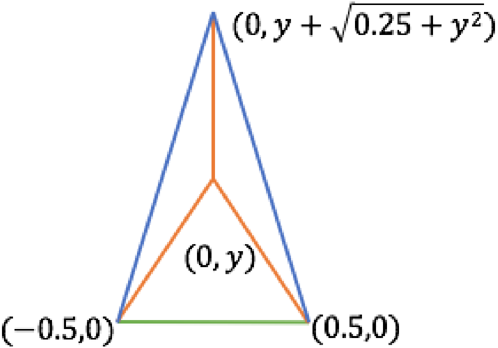


Figure: Realization obtained by fixing $d_1 = 1$

Rigidity Analysis for Type C

Rigidity Matrix A_C

$$\begin{bmatrix} \frac{1}{2} & y+\sqrt{\frac{1+4y^2}{4}} & -\frac{1}{2} & -y-\sqrt{\frac{1+4y^2}{4}} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & y+\sqrt{\frac{1+4y^2}{4}} & 0 & 0 & \frac{1}{2} & -y-\sqrt{\frac{1+4y^2}{4}} & 0 & 0 \\ 0 & \sqrt{\frac{1+4y^2}{4}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1+4y^2}{4}} \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -y & 0 & 0 & \frac{1}{2} & y \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -y & -\frac{1}{2} & y \end{bmatrix}$$

$\text{Rank}(A_C) = 5 = S(4,2) \rightarrow \text{rigid}$

Type R Realization

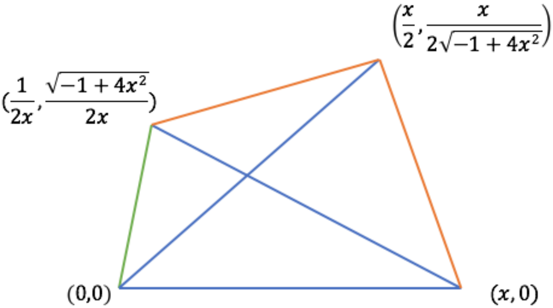


Figure: Realization obtained by fixing $d_1 = 1$

Rigidity Analysis for Type R

Letting $y = \sqrt{-1 + 4x^2}$, we get the rigidity matrix A_R:

$$\begin{bmatrix} -x & 0 & x & 0 & 0 & 0 & 0 & 0 \\ \frac{-x}{2} & \frac{-x}{y} & 0 & 0 & \frac{x}{2} & \frac{x}{y} & 0 & 0 \\ \frac{-1}{2x} & \frac{-y}{2x} & 0 & 0 & 0 & 0 & \frac{1}{2x} & \frac{y}{2x} \\ 0 & 0 & x - \frac{x}{2} & \frac{-x}{2y} & -x + \frac{x}{2} & \frac{x}{2y} & 0 & 0 \\ 0 & 0 & x - \frac{1}{2x} & \frac{-y}{2x} & 0 & 0 & -x + \frac{1}{2x} & \frac{y}{2x} \\ 0 & 0 & 0 & 0 & \frac{x}{2} - \frac{1}{2x} & \frac{x}{2y} - \frac{y}{2x} & \frac{-x}{2} + \frac{1}{2x} & \frac{-x}{2y} + \frac{y}{2x} \end{bmatrix}$$

$\text{Rank}(A_R) = 6 > S(4,2)$ but when removing any row, rank of remaining matrix is 5 $\rightarrow$ redundantly rigid

Type M Realizations

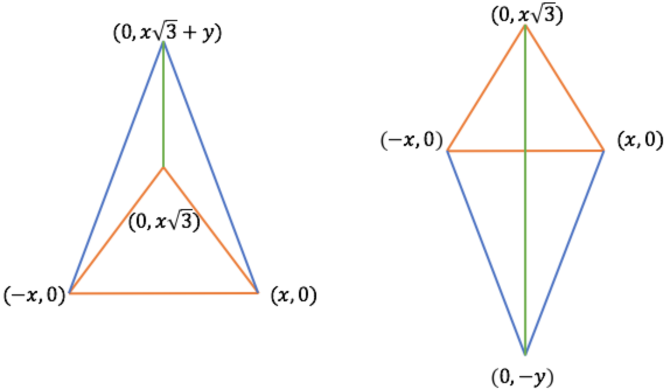


Figure: Two Realizations of Type M: M_1 and M_2

Rigidity Analysis for Type M

Rigidity matrix A_{M_1}

$$\begin{bmatrix} -2x & 0 & 2x & 0 & 0 & 0 & 0 & 0 \\ -x & -x\sqrt{3} & 0 & 0 & x & x\sqrt{3} & 0 & 0 \\ -x & -x\sqrt{3}-y & 0 & 0 & 0 & 0 & x & x\sqrt{3}+y \\ 0 & 0 & x & -x\sqrt{3} & -x & x\sqrt{3} & 0 & 0 \\ 0 & 0 & x & -x\sqrt{3}-y & 0 & 0 & -x & x\sqrt{3}+y \\ 0 & 0 & 0 & 0 & 0 & -y & 0 & y \end{bmatrix}$$

$\text{Rank}(A_{M_1}) = 5 = S(4,2) \rightarrow$ rigid

Same results for M_2

Questions to explore

- Find ways to speed up our techniques so we can find crescent configurations on a higher n?

Questions to explore

- Find ways to speed up our techniques so we can find crescent configurations on a higher n?
- Which distance sets can be realized in higher dimensions?

Questions to explore

- Find ways to speed up our techniques so we can find crescent configurations on a higher n?
- Which distance sets can be realized in higher dimensions?
- In addition to rigidity, which other properties of point configurations can we explore?

Acknowledgements

- Williams College Finnerty Fund
- Williams College and SMALL REU
- NSF Grants DMS1265673 and DMS1561945
- NSF Grant DMS1347804
- Prof. Steven J. Miller and Prof. Eyvi A. Palsson

References

Herman Gluck (1975)
Almost all simply connected closed surfaces are rigid
Geometric Toplogy, Lecture Notes in Mathematics 438(1), 225 – 239

Bruce A. Hendrickson (Sept. 1990)
The Molecule Problem: Determining Conformation from Pairwise Distances
Cornell University Computer Science Technical Reports, 90 – 1159

Bruce A. Hendrickson (Feb 1992)
Conditions for Unique Graph Realizations
SIAM Journal On Computing 21(1), 65 – 84.

M. Gavrilova, et al. (Eds.) (2006)
Computational Science and its Aplications– ICCSA 2006: International Conference, Glasgow, UK, May 8-11, 2006, Proceedings, Part 5, Springer.