On the density of low-lying zeros of a large family of automorphic L-functions

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Riemann Zeta Function

Definition (Riemann Zeta Function)

For $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1},$$

which has an analytic continuation to $\mathbb{C} \setminus \{1\}$ *and has no zeroes other than negative even integers and* s *with* $0 < \Re(s) < 1$.

Riemann Hypothesis

The nontrivial zeros of $\zeta(s)$ *have real part* $1/2$.

Duality Between Primes and the Zeros of the Riemann Zeta Function

Theorem (Riemann-von Mangoldt Explicit Formula)

For $X > 1$,

$$\sum_{p^j < X} \log p \;=\; X - \sum_{\rho} \frac{X^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}).$$

RH $\implies$ $[X, X + X^{1/2+\epsilon}]$ contains primes, so RH "knows" about patterns in primes.

L-functions

Definition

An L-function is a series

$$L(s,f) := \sum_n \frac{a_n}{n^s} = \prod_p (1-g(p)p^{-s})^{-1}$$

where $1-g(p)p^{-s}$ is the *Euler factor at p.*

Like $\zeta(s)$, the L-functions we study:

- can be meromorphically continued to $\mathbb{C}$,
- have zeroes only at negative reals and s with $0 \leq \Re(s) \leq 1$.

We are interested in the behavior of the zeros of L-functions.

Montgomery Pair Correlation Conjecture

Conjecture (Montgomery-Dyson 1973)

The zeros of the Riemann zeta function on the critical strip are distributed like the eigenvalues of random Hermitian matrices from the Gaussian Unitary Ensemble (GUE).

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Essentially, predicts that for f a Schwartz test function whose Fourier tranform has arbitrary compact support

$$\frac{1}{N(T)} \sum_{\substack{0\le\gamma,\gamma'\le T \\ \gamma\neq\gamma'}} f\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) \longrightarrow \int_{-\infty}^{\infty} f(x)W(x)\,dx, \quad T\to\infty.$$

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- Rudnick-Sarnak ('94, '96): introduced and extended n-level correlations to L-functions, showing universality for all automorphic cuspidal L-functions (agree with GUE).
- Also agree with classical compact groups $O(N)$, $SO(\text{even})$, $SO(\text{odd})$, $U(N)$, $Sp(2N)$.

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- Katz-Sarnak density conjecture: behavior of low-lying zeros of a family of L-functions governed by behavior of eigenvalues of a classical compact group.
- Low-lying zeros related to infinitude of primes, Chebyshev's bias, Birch and Swinnerton-Dyer conjecture, class number bounds.

Distribution of the Low-Lying Zeros of L-functions

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Montgomery pair correlation conjecture	$\rightsquigarrow$	Katz-Sarnak density conjecture

Density of low-lying zeros

Definition (1-level density)

Let Φ be a Schwartz function with $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$. Assume GRH and write $\rho_f = 1/2 + i\gamma_f$ for the non-trivial zeros of $L(s, f)$ counted with multiplicity. Then

$$\mathscr{OD}(f; \Phi) := \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right),$$

is the *1-level density*, where c_f is the analytic conductor of f.

- 1-level density captures density of the zeros within height $O(1/\log c_f)$ of $s = 1/2$.
- Cannot asymptotically evaluate $\mathscr{OD}(f; \Phi)$ for a single f, must perform averaging over the family ordered by analytic conductor.

Katz-Sarnak Density Conjecture

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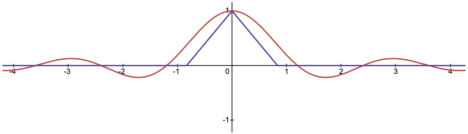
Let $\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f = Q\}$ or $\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f \leq Q\}$. Then for a Schwartz test function Φ whose Fourier transform has arbitrary compact support, we have that

$$\frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} \mathcal{OD}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(G)(x)\, dx \quad \text{as} \quad Q \to \infty,$$

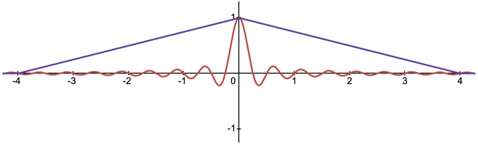
where $W(G)(x)$ is a distribution depending on the underlying symmetry group G.

Extending the Support

Taking the support of $\widehat{\Phi}$ (purple) to be bounded yet arbitrarily large corresponds to taking Φ (red) close to a Dirac delta function at $s = 1/2$.



Smaller support = less precise information



Larger support = more precise information

n-level density

Definition

In the setting as before, define the n*-level density* as

$$\mathscr{D}_n(f;\Phi) := \sum_{\substack{j_1,\ldots,j_n \\ j_i \neq \pm j_k}} \prod_{i=1}^{n} \Phi_i\left(\frac{\gamma_f(j_i)}{2\pi} \log c_f\right).$$

- Computing n-level density for $n > 2$ requires knowledge of distribution of signs of the functional equation of each $L(s,f)$, which is beyond current theory.
- Hughes-Rudnick (2003): introduced n-th centered moments.

1. Introduction
2. Automorphic L-functions
3. Prior Work
4. Main Results
5. Proof Sketch

Modular Forms

Definition (Modular form of trivial nebentypus)

We write $f \in M_k(q)$ and say f is a *modular form* of level q, even weight k, and trivial nebentypus if $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and

1. For each $\tau \in \Gamma_0(q) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}$ we have

$$f(\tau z) \; := \; f\left(\frac{az+b}{cz+d}\right) \; = \; (cz+d)^k f(z).$$

2. For $\tau \in \mathsf{SL}_2(\mathbb{Z})$, as $\Im(z) \to +\infty$ we have $(cz+d)^{-k} f(\tau z) \ll 1$.

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With $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $f(z) = f(z+1)$ so f is 1-periodic and thus has a Fourier expansion at ∞:
$$f(z) \;=\; \sum_{n=0}^{\infty} a_f(n) q^n, \quad q \;=\; e^{2\pi i z}.$$

Holomorphic Cuspforms

Definition (Cuspform)

If $f \in M_k(q)$ vanishes at all cusps of $\Gamma_0(q)$ we say f is a *cuspform* and denote by $\mathcal{S}_k(q) \subset M_k(q)$ the space of holomorphic cuspforms.

- By Atkin-Lehner Theory, we have the orthogonal decomposition

$$\mathcal{S}_k(q) \;=\; \mathcal{S}_k^{\text{old}}(q) \oplus \mathcal{S}_k^{\text{new}}(q).$$

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- A cuspform $f \in \mathcal{S}_k(q)$ is an eigenfunction of the Hecke operators T_n for $(n, q) = 1$ and $T_n f = \lambda_f(n) f$.

The Space of Cuspidal Newforms

Definition (Newform)

If f is an eigenform of *all* the Hecke operators and the Atkin-Lehner involutions $|_k W(q)$ and $|_k W(Q_p)$ for all the primes $p \mid q$, then we say that f is a *newform* and if, in addition, f is normalized so that $\psi_f(1) = 1$ we say that f is *primitive*.

- The space $\mathcal{S}_k^{\mathrm{new}}(q)$ of newforms has an orthogonal basis $\mathscr{H}_k(q)$ of primitive newforms.
- Trivial nebentypus $\implies$ T_n's are self-adjoint $\implies$ $\lambda_f(n) \in \mathbb{R}$ for all n.

L-functions Attached to Cuspidal Newforms

Fix $f \in \mathcal{S}_k^{\text{new}}(q)$. Then for $\Re(s) > 1$, we define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

where χ_0 is the principal character mod q. Note, $L(s,f)$ can be analytically continued to an entire function on $\mathbb{C}$. Moreover, $L(s,f) = L(s,\overline{f})$.

Katz-Sarnak Density Conjecture for Orthogonal Symmetry

The symmetry type of the family of automorphic L-functions attached to holomorphic cuspidal newforms is orthogonal. Thus, the Katz-Sarnak density conjecture predicts that for test functions Φ whose Fourier transform has arbitrary compact support,

$$\frac{1}{|\mathscr{H}_k(Q)|} \sum_{f \in \mathscr{H}_k(Q)} \mathscr{OD}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx \quad \text{as } Q \to \infty,$$

where O is the scaling limit of the group of square orthogonal matrices with density

$$W(O)(x) \;=\; 1 + \frac{1}{2}\delta_0(x),$$

where $\delta_0(x)$ denotes the Dirac delta function at $x = 0$.

Extending the Support

Theorem (Iwaniec-Luo-Sarnak '00)

Assume GRH. Then for Φ any even Schwartz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-2, 2)$, we have that

$$\lim_{\substack{q\to\infty \\ \square\text{free}}} \frac{1}{|\mathscr{H}_k(q)|} \sum_{f\in\mathscr{H}_k(q)} \mathscr{O}\mathscr{D}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx,$$

where O denotes the orthogonal type, showing agreement with the Katz-Sarnak philosophy predictions.

Recent Breakthrough

Theorem (Baluyot-Chandee-Li '23)

Assume GRH. Let Φ be an even Schwartz function such that $\mathrm{supp}(\widehat{\Phi}) \subset (-4, 4)$, and let Ψ be any smooth function compactly supported on $\mathbb{R}^+$ with $\widehat{\Psi}(0) \neq 0$. Then we have that

$$\langle \mathscr{OD}(f;\Phi)\rangle_* := \lim_{Q\to\infty} \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{f\in\mathscr{H}_k(q)}^h \mathscr{OD}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(O)(x)dx,$$

where $N(Q)$ is a normalizing factor, showing agreement with the Katz-Sarnak philosophy predictions.

The n-th Centered Moments of the 1-level Density

We study the n-th centered moments of the 1-level density averaged over levels $q \asymp Q$.

Definition (n-th centered moments of the 1-level density)

In the setting as above, define the *n-th centered moment of the 1-level density* to be

$$\left\langle \prod_{i=1}^{n} \left[\mathscr{OD}(f;\Phi_i) - \langle \mathscr{OD}(f;\Phi_i)\rangle_*\right] \right\rangle_* .$$

Main Results

Theorem (Cheek-Gilman-Jaber-Miller-Tomé '24)

Assume GRH. For Ψ non-negative and Φ_i even Schwartz functions with $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$ and $\sigma \leq \min\left\{\frac{3}{2(n-1)}, \frac{4}{2n-\mathbf{1}_{2\nmid n}}\right\}$ we have that

$$\left\langle \prod_{i=1}^{n} (\mathscr{O}\mathscr{D}(f;\Phi_i) - \langle \mathscr{O}\mathscr{D}(f;\Phi_i)\rangle_*) \right\rangle_* = \frac{\mathbf{1}_{2|n}}{(n/2)!} \sum_{\tau \in S_n} \prod_{i=1}^{n/2} \int_{-\infty}^{\infty} |u| \widehat{\Phi}_{\tau(2i-1)}(u) \widehat{\Phi}_{\tau(2i)}(u)\, du.$$

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Remark

Notably, for $n = 3$, we achieve $\sigma = \sigma_i = 3/4$, greater than currently best known $\sigma = \sigma_i = 2/3$.

Main results

Corollary (Cheek-Gilman-Jaber-Miller-Tomé '24)

Let $\sigma_1 = 3/2$ *and* $\sigma_2 = 5/6$. *Then the two-level density*

$$\left\langle \sum\sum_{j_1 \neq \pm j_2} \Phi_1\Big(\gamma_f(j_1)\Big)\Phi_2\Big(\gamma_f(j_2)\Big)\right\rangle_* = 2\int_{-\infty}^{\infty} |u|\widehat{\Phi}_1(u)\widehat{\Phi}_2(u)\,du + \prod_{i=1}^{2}\left(\frac{1}{2}\Phi_i(0) + \widehat{\Phi}_i(0)\right) - \Phi_1\Phi_2(0) - 2\widehat{\Phi_1\Phi_2}(0) + \mathscr{ODD}\ \Phi_1\Phi_2(0),$$

where $\mathscr{ODD} := \langle (1-\epsilon_f)/2\rangle_*$ *denotes the proportion of forms with odd functional equation.*

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where $\mathscr{ODD} := \langle (1-\epsilon_f)/2 \rangle_*$ denotes the proportion of forms with odd functional equation.

Remark

This is the first evidence of an interesting new phenomenon: only by taking different test functions are we able to extend the range in which the Katz-Sarnak density predictions hold. In particular, $\sigma_1 + \sigma_2 = 7/3 > 2$, where $\sigma_1 + \sigma_2 = 2$ was the previously best known.

Duality Between Primes and Zeros of L-functions

Using an explicit formula relating sums over zeros to sums of prime power coefficients of $L(s, f)$, we deduce that

$$\sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi}\log q\right) = \widehat{\Phi}(0) + \frac{1}{2}\Phi(0) - \frac{2}{\log q}\sum_{p \nmid q} \frac{\lambda_f(p)\log p}{\sqrt{p}}\widehat{\Phi}\left(\frac{\log p}{\log q}\right) + O\left(\frac{\log\log q}{\log q}\right).$$

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We use a combinatorial argument together with GRH for $L(s,\mathrm{sym}^2 f)$ to further reduce to studying sums over *distinct* primes:

$$\sum_{\substack{p_1,\ldots,p_n\nmid q\\ p_i\neq p_j}} \prod_{i=1}^{n} \frac{\lambda_f(p_i)\log p_i}{\sqrt{p_i}}\widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right).$$

Averaging Over the Extended Orthogonal Family

We average over $f \in \mathscr{H}_k(q)$ with $q \asymp Q$ and study

$$\frac{1}{N(Q)} \sum_{q} \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{f \in \mathscr{H}_k(q)}^{h} \sum_{\substack{p_1, \ldots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\lambda_f(p_i) \log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right)$$

$$= \frac{1}{N(Q)} \sum_{q} \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{\substack{p_1, \ldots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right) \sum_{f \in \mathscr{H}_k(q)}^{h} \lambda_f(1) \lambda_f\left(\prod_{i=1}^{n} p_i\right).$$

Trace formulae

- Ng's work allows us to convert sums over $\mathscr{H}_k(q)$ to a linear combination of sums over an orthogonal basis $\mathscr{B}_k(d)$ for the space $\mathcal{S}_k(d)$, $d \mid q$: Morally, if $(m,n,q)=1$ and for A a specific arithmetic function, then

$$\sum_{f\in\mathscr{H}_k(q)}^h \lambda_f(m)\lambda_f(n) = \sum_{\substack{q=L_1L_2d \\ L_1|q_1 \\ L_2|q_2 \\ q_2 \ \square\text{free}}} A(L_1,L_2,d) \sum_{e|L_2^\infty} \frac{1}{e} \sum_{f\in\mathscr{B}_k(d)}^h \lambda_f(e^2m)\lambda_f(n).$$

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- Petersson trace formula, a quasi-orthogonality relation for GL_2

$$\sum_{f\in\mathscr{B}_k(d)}{}^{h}\lambda_f(m)\lambda_f(n) \;=\; \delta(m,n)+\sum_{c\geq 1}\frac{S(m,n;cq)}{cq}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

The Kuznetsov Trace Formula

Let $x := \prod p_i$. We are essentially left to analyze

$$\sum_{c\geq 1} \sum_{\substack{p_1,\ldots,p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\log p_i}{\sqrt{p_i}} V\left(\frac{p_i}{P_i}\right) \mathrm{e}\left(v_i \frac{p_i}{P_i}\right) \sum_s \frac{S(e^2, x; cL_1 rds)}{cL_1 rds} h\left(\frac{4\pi\sqrt{e^2 x}}{cL_1 rds}\right)$$

where V is smooth and compactly supported and h is essentially a smooth truncation of J_{k-1}. We use the Kuznetsov trace formula to convert an average over $f \in \mathscr{B}_k(d)$ into spectral terms:

Holomorphic cuspforms + Maass cuspforms + Eisenstein series.

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