Machine Learning and Deterministic Zeckendorf Games

Vishal Muthuvel[1], Ren Watson[2]

Joint work with Pedro Espinosa, Michael Lucas, & Cameron White

[1]Columbia University

[2]University of Texas at Austin

SMALL REU 2025, Williams College
Advisor: Steven J. Miller

Young Mathematicians Conference 2025

Background

Zeckendorf's Theorem

The Fibonacci numbers are given by $F_1 = 1, F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$.

Theorem (Zeckendorf, 1972)

Every positive integer can be written uniquely as a sum of nonconsecutive Fibonacci numbers.

Greedy algorithm: Given a positive integer N,

- find largest Fibonacci number less than or equal to N:

$$F_k \leq N < F_{k+1};$$

- and repeat the algorithm with $N - F_k$, a smaller integer.

The Zeckendorf Game

In 2018, Baird-Smith, Epstein, Flint, and Miller introduced a 2-player game based on the Zeckendorf decomposition of a positive integer $N \geq 2$.

- The game is played on a board of bins labeled with the Fibonacci numbers and begins with N tokens in the F_1 bin.

F_1	F_2	F_3	...	F_n
N	0	0	...	0

- Players alternate making moves (*Combine* or *Split*) based on the Fibonacci recurrence.
- The last player to move wins.

Moves in the Zeckendorf Game

Combine Moves:

- $C_1 : F_1 \wedge F_1 \mapsto F_2$
- $C_k : F_{k-1} \wedge F_k \mapsto F_{k+1} \quad (k > 1)$

Split Moves:

- $S_2 : F_2 \wedge F_2 \mapsto F_3 \wedge F_1$
- $S_k : F_k \wedge F_k \mapsto F_{k-2} \wedge F_{k+1} \quad (k > 2)$

Moves in the Zeckendorf Game

Combine Moves:

- $C_1 : F_1 \wedge F_1 \mapsto F_2$
- $C_k : F_{k-1} \wedge F_k \mapsto F_{k+1} \quad (k > 1)$

Split Moves:

- $S_2 : F_2 \wedge F_2 \mapsto F_3 \wedge F_1$
- $S_k : F_k \wedge F_k \mapsto F_{k-2} \wedge F_{k+1} \quad (k > 2)$

Note that

- The total weighted sum of tokens remains constant.
- The total number of tokens is a monovariant.

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0

Sample Zeckendorf Game

$$N = 7$$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0
2	0	0	1

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0
2	0	0	1
0	1	0	1

Sample Zeckendorf Game

$N = 7$

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0
2	0	0	1
0	1	0	1

Game Termination and Winning Strategy

Theorem (Baird-Smith et al., 2018)

The Zeckendorf Game terminates in a finite number of moves at the Zeckendorf decomposition of N.

Theorem (Baird-Smith et al., 2018)

For all $N > 2$, Player 2 has a winning strategy for the Zeckendorf Game.

A constructive winning strategy for Player 2 remains unknown for all N.

Game Tree

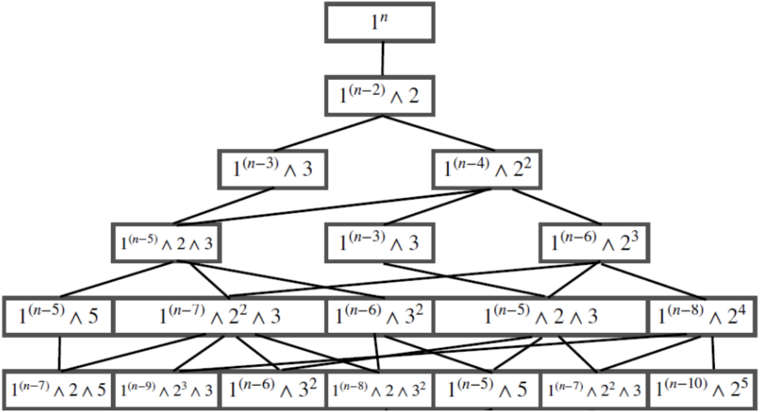
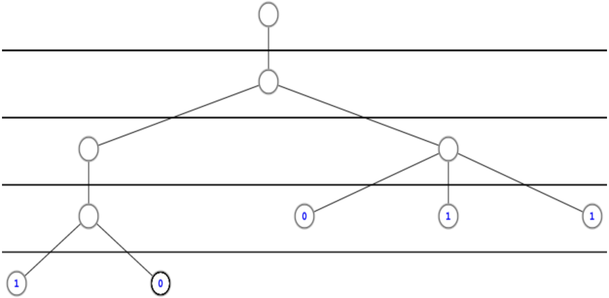


Figure: Game tree given by the first several moves of the Zeckendorf Game (image credit: Baird-Smith et al.)

Machine Learning for Constructive Proofs

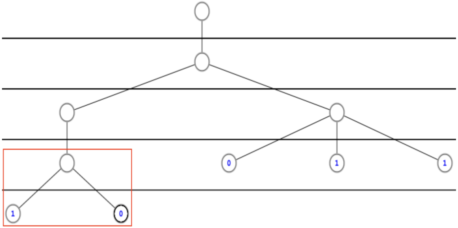
Minimax Algorithm

- Naive approach: Traverse the entire game tree and determine which states are winning and losing for "small" N.



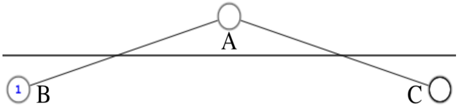
Minimax Algorithm

- Naive Approach: Traverse the (entire) game tree and determine which states are winning and losing for "small" N.
- Alpha-Beta Pruning: Ignore branches that cannot supply additional information.



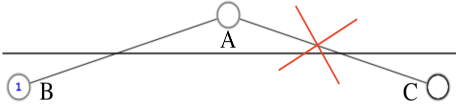
Minimax Algorithm

- Naive Approach: Traverse the (entire) game tree and determine which states are winning and losing for "small" N.
- Alpha-Beta Pruning: Ignore branches that cannot supply additional information.



Minimax Algorithm

- Naive Approach: Traverse the (entire) game tree and determine which states are winning and losing for “small” N.
- Alpha-Beta Pruning: Ignore branches that cannot supply additional information.



Value-Policy Neural Network

- Implement the minimax algorithm with alpha-beta pruning for "small" N.
- Feed this minimax data to a neural network and train it to predict the quality of games.

$$\text{State} \mapsto (\text{Value}, \text{Policy})$$

Monte Carlo Tree Search

- Implement the minimax algorithm for “small" N.
- Train neural network with minimax data.
- Perform a Monte Carlo Tree Search guided by this value-policy network.

Machine Learning for a Constructive Proof

- Implement the minimax algorithm for “small" N.
- Train neural network with minimax data.
- Perform a Monte Carlo Tree Search (MCTS) guided by this value-policy network.
- Train and test the neural network with the MCTS data repeatedly.

Deterministic Zeckendorf Games

Deterministic Zeckendorf Games

Li et al. began the study of Deterministic Zeckendorf Games, in which available moves must be made in a prescribed order. They introduced four variants:

- *Combine Largest (CL):* Apply C_k moves from largest to smallest, apply S_k moves from largest to smallest

Deterministic Zeckendorf Games

Li et al. began the study of Deterministic Zeckendorf Games, in which available moves must be made in a prescribed order. They introduced four variants:

- *Combine Largest (CL):* Apply C_k moves from largest to smallest, apply S_k moves from largest to smallest
- *Combine Smallest (CS):* Apply C_k moves from smallest to largest, apply S_k moves from smallest to largest

Deterministic Zeckendorf Games

Li et al. began the study of Deterministic Zeckendorf Games, in which available moves must be made in a prescribed order. They introduced four variants:

- *Combine Largest (CL):* Apply C_k moves from largest to smallest, apply S_k moves from largest to smallest
- *Combine Smallest (CS):* Apply C_k moves from smallest to largest, apply S_k moves from smallest to largest
- *Split Largest (SL):* Apply S_k moves from largest to smallest, apply C_k moves from largest to smallest

Deterministic Zeckendorf Games

Li et al. began the study of Deterministic Zeckendorf Games, in which available moves must be made in a prescribed order. They introduced four variants:

- *Combine Largest (CL):* Apply C_k moves from largest to smallest, apply S_k moves from largest to smallest
- *Combine Smallest (CS):* Apply C_k moves from smallest to largest, apply S_k moves from smallest to largest
- *Split Largest (SL):* Apply S_k moves from largest to smallest, apply C_k moves from largest to smallest
- *Split Smallest (SS):* Apply S_k moves from smallest to largest, apply C_k moves from smallest to largest

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0
2	0	0	1

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0
2	0	0	1
0	1	0	1

Sample Deterministic Game

$N = 7$; Player 1 plays Split Largest and Player 2 plays Split Smallest

F_1	F_2	F_3	F_4
7	0	0	0
5	1	0	0
3	2	0	0
4	0	1	0
2	1	1	0
2	0	0	1
0	1	0	1

Strategy in Deterministic Zeckendorf Games

Question: Given deterministic variants (A,B), what can be determined about the Deterministic Zeckendorf Game where Player 1 plays variant A and Player 2 variant B?

Several of our games mirror each other in structure.

Theorem (SMALL 2025)

For $N \geq 4$, Player 1 wins the SL-SS game on N if and only if Player 2 wins the SS-SL game on N.

Theorem (SMALL 2025)

For $N \geq 4$, Player 1 wins the SL-CS game on N if and only if Player 2 wins the CS-SL game on N.

Minimal Length of Zeckendorf Games

Let $Z(N)$ be the number of terms in the Zeckendorf decomposition of N.

Theorem (Baird-Smith et al., 2018)

The shortest Zeckendorf Game on N arrives at the Zeckendorf decomposition in $N - Z(N)$ moves.

This game is achieved by making the largest possible combine move at each turn.

Deterministic Zeckendorf Games of Minimal Length

Theorem (Li et al., 2020)

The CL-CL and SL-SL games achieve the shortest Zeckendorf Game of $N - Z(N)$ moves for all $N \geq 2$.

Theorem (SMALL 2025)

The SL-CL and CL-SL games achieve the shortest Zeckendorf Game for all $N \geq 2$. No other of our deterministic games besides these, CL-CL, and SL-SL achieve the shortest game for all N.

Parity of Minimal Length Games

We can observe a recursive structure in the parity of $N - Z(N)$.

N	2
$N - Z(N) \pmod 2$	1

N	3	4
$N - Z(N) \pmod 2$	0	0

N	5	6	7
$N - Z(N) \pmod 2$	1	0	0

Parity of Minimal Length Games

N	3	4
$N - Z(N) \pmod 2$	0	0

N	5	6	7
$N - Z(N) \pmod 2$	1	0	0

N	8	9	10	11	12
$N - Z(N) \pmod 2$	1	1	0	1	1

Theorem (SMALL 2025)

For $N := F_i + a \in [F_i, F_i + F_{i-2})$, $N - Z(N) \equiv F_{i-1} + a - Z(F_{i-1} + a) \pmod 2$ if and only if $i \equiv 1 \pmod 3$. For $N := F_i + b \in [F_i + F_{i-2}, F_{i+1})$, $N - Z(N) \equiv F_{i-2} + b - Z(F_{i-2} + b) \pmod 2$ if and only if $i \equiv 0, 1 \pmod 3$.

Winning Proportion in Minimal Length Games

Utilizing the recursive structure of the parity of $N - Z(N)$, we can determine the following.

Theorem (SMALL 2025)

As $i \to \infty$, the proportions of shortest games won by Player 1 and by Player 2 for $N \in [F_i, F_{i+1})$ both converge to 0.5.

Theorem (SMALL 2025)

As $i \to \infty$, the proportion of shortest games won by Player 1 and by Player 2 for $N < F_i$ both converge to 0.5.

Acknowledgments

We are grateful to Professor Steven J. Miller for his mentorship.

This research was supported by NSF Grant DMS2241623, Williams College, the Finnerty Fund, the Dr. Herchel Smith Fellowship Fund, and Columbia University.

References

Paul Baird-Smith, Alyssa Epstein, Kristen Flint, and Steven J Miller.
The Zeckendorf Game.
In *Combinatorial and Additive Number Theory, New York Number Theory Seminar*, pages 25–38. Springer, 2018.

Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Micah McClatchey, Steven J Miller, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, and Xiaoyan Zheng.
Bounds on Zeckendorf Games.
arXiv preprint arXiv:2009.09510, 2020.

Ruoci Li, Xiaonan Li, Steven J Miller, Clayton Mizgerd, Chenyang Sun, Dong Xia, and Zhyi Zhou.
Deterministic Zeckendorf Games.
The Fibonacci Quarterly, 58(5):152–160, 2020.