# On a Pair of Diophantine Equations

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# Motivation

For relatively prime $a, b \in \mathbb{N}$, consider

$$ax + by = \frac{(a-1)(b-1)}{2}, \tag{1}$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2}. \tag{2}$$

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Exactly one of the equations has a nonnegative solution $(x, y)$. The solution is unique.

**Goal**

Investigate which equation is used by two consecutive terms of a given sequence.

# Notation

$\Gamma(a, b)$ tells which Diophantine equation is used by $(a/d)$ and $(b/d)$, for $d = \gcd(a, b)$.

$\Gamma(5, 15) = \Gamma(1, 3) = 1.$

$\Gamma(12, 18) = \Gamma(2, 3) = 2.$

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Given a sequence $(a_n)_n$,

$$\Delta((a_n)_n) = (\Gamma(a_n, a_{n+1}))_n.$$

$\Delta(\mathbb{N}) = 1, 2, 1, 2, 1, 2, \ldots.$

# Definition of $\Theta$

**Definition**

Let $a, b \in \mathbb{N}$ and $d = \gcd(a, b)$. Then $\Theta(a, b)$ is defined as the multiplicative inverse of $a/d$ under modulo $b/d$.

# Pairwise theorem

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Let $a, b \in \mathbb{N}$ and $d = \gcd(a, b)$.

If $a$ divides $b$ or $b$ divides $a$, then $\Gamma(a, b) = 1$.

Otherwise,

if $a/d$ is odd, then $\Gamma(a, b) = 1 \iff \Theta(b, a)$ is odd;

if $a/d$ is even, then $\Gamma(a, b) = 1 \iff \Theta(a, b)$ is odd.

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Compute $\Gamma(25, 110)$ :
$d = \gcd(25, 110) = 5$

$$\left.\begin{array}{l} 25/5 = 5 \text{ is odd} \\ \Theta(110, 25) = 3 \text{ is odd} \end{array}\right\} \implies \Gamma(25, 110) = 1.$$

# Period $k$

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$$1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, \ldots.$$

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$$1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, \ldots .$$

Given $k \in \mathbb{N}$, can we find a sequence $(a_n)_{n\geq 1}$ with

$$\Delta((a_n)_n) = \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \ldots ?$$

# Period $k$

**Theorem**

Fix $k \in \mathbb{N}$. Then $\Delta\left(\left(\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil\right)_n\right)$ is

$$\underbrace{1,\ldots,1}_{k},\underbrace{2,\ldots,2}_{k},\underbrace{1,\ldots,1}_{k},\underbrace{2,\ldots,2}_{k},\ldots.$$

# Step 1 of proof: construct a desired sequence

**Lemma**

Let $a \in \mathbb{N}$. Then the pair $(a, 2a)$ uses the first equation. If $a \geq 2$, the pair $(a, 2a-1)$ uses the second equation.

For all $a \in \mathbb{N}$, $a \mid 2a \implies \Gamma(a, 2a) = 1$.

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For all $a \geq 2$, $a \not| \, 2a-1$ and $d = \gcd(a, 2a-1) = 1$.

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For all $a \geq 2$, $a \nmid 2a - 1$ and $d = \gcd(a, 2a-1) = 1$.

$$a \text{ odd} \implies \Theta(2a-1, a) = a - 1 \implies \Gamma(a, 2a-1) = 2;$$

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For all $a \geq 2$, $a \nmid 2a-1$ and $d = \gcd(a, 2a-1) = 1$.

$$a \text{ odd} \implies \Theta(2a-1, a) = a-1 \implies \Gamma(a, 2a-1) = 2;$$
$$a \text{ even} \implies \Theta(a, 2a-1) = 2 \implies \Gamma(a, 2a-1) = 2.$$

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Let $a \in \mathbb{N}$. Then the pair $(a, 2a)$ uses the first equation. If $a \geq 2$, the pair $(a, 2a-1)$ uses the second equation.

Fix $k \in \mathbb{N}$. Construct $(a_n)_{n \geq 1}$ as follows: let $a_1 := 1$, and for all integer $n \geq 1$,

$$a_{n+1} := \begin{cases} 2a_n & \text{if } n \equiv 1, \ldots, \text{or } k \mod 2k \\ 2a_n - 1 & \text{if } n \equiv k+1, \ldots, \text{or } 2k \mod 2k. \end{cases}$$

$$\Delta((a_n)_n) = \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \ldots.$$

# Step 2 of proof: finding the closed formula

It remains to show that for all $n \geq 1$, $a_n = \left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil$.

**Lemma**

Let $k, n \in \mathbb{N}$ and suppose $n \equiv m \mod 2k$ $(1 \leq m \leq 2k)$. Then

$$2^{n+k-1} \equiv \begin{cases} 2^k - 2^{m-1} + 1 \mod 2^k + 1 & \text{if } 1 \leq m \leq k, \\ 2^{m-k-1} \mod 2^k + 1 & \text{if } k+1 \leq m \leq 2k. \end{cases}$$

# Arithmetic sequences

Given $a, d \in \mathbb{N}$, consider the sequence $x_n = a + (n-1)d$.

**Theorem**

Consecutive terms of $x_n$ use the 2 equations alternatively.

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Examples

- For $x_n = 3 + 6(n-1)$, $\Delta((x_n)_n) = 1, 2, 1, 2, 1, 2, \ldots$.
- For $x_n = 9 + 6(n-1)$, $\Delta((x_n)_n) = 2, 1, 2, 1, 2, 1, \ldots$.

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Consecutive terms of $x_n$ use the 2 equations alternatively.

We want to show that for all $n \in \mathbb{N}$, $\Gamma(x_n, x_{n+1}) \neq \Gamma(x_{n+1}, x_{n+2})$.

**Observation:** $a_n \nmid a_{n+1}$ for all $n \geq 2$.

# Geometric sequences

Let $y_n = ar^{n-1}$ for $a, r, n \in \mathbb{N}$.

$$\Gamma(ar^{n-1},\ ar^n) = \Gamma(1,\ r) = 1.$$

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$\Delta((y_n)_n) = 1, 1, 1, 1, 1, 1, \ldots.$

What about the shifted geometric sequence $x_n = ar^{n-1} + 1$?

# Shifted geometric sequences

**Theorem**

If $\gcd(a+1,\ r-1)$ is odd, then $\Delta((x_n)_{n\geq 2})$ is constant.
If $\gcd(a+1,\ r-1)$ is even, $\Delta((x_n)_{n\geq 2})$ alternates between 1 and 2.

**Lemma**

$\gcd(ar^{n-1}+1,\ ar^n+1) \ = \ \gcd(a+1,\ r-1)$

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**Lemma**

$\gcd(ar^{n-1}+1,\ ar^n+1)\ =\ \gcd(a+1,\ r-1)$

**Examples**

- $x_n\ =\ 4\cdot 11^{n-1}\ +\ 1$; $\gcd(a+1,\ r-1)\ =\ 5$.
  $\Delta((x_n)_n)\ =\ 1,2,2,2,2,2,\ldots$
- $x_n\ =\ 3\cdot 9^{n-1}\ +\ 1$; $\gcd(a+1,\ r-1)\ =\ 4$.
  $\Delta((x_n)_n)\ =\ 1,1,2,1,2,1,2,\ldots$

# The sequence $(n^k)_n$

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$\Delta((n)_n) = 1, 2, 1, 2, 1, 2, \ldots$
$\Delta((n^2)_n) = 1, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$
$\Delta((n^3)_n) = 1, 1, 1, 2, 1, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$
$\Delta((n^4)_n) = 1, 2, 1, 1, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, \ldots$

$\ldots$

# General theorem

**Theorem**

Fix $k \in \mathbb{N}$. Then $\Delta((n^k)_n)$ is eventually $1, 2, 1, 2, 1, 2, \ldots$.

# Bound for when the pattern starts

Define

$$g(x) = \left(\sum_{i=1}^{k} x^{i-1}\right)^{k} \mod x^k.$$

# Find $M_k$

If $k$ is odd, let $M_k$ be the smallest positive, even integer such that for all $n \geq M_k$,

$$0 < n^k - g(n) < n^k \quad \text{and} \quad 0 < g(-n) < n^k;$$

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$$0 \ < \ n^k + g(-n) \ < \ n^k \quad \text{and} \quad 0 \ < \ g(n) \ < \ n^k.$$

Then the sequence $(\Gamma(n^k, (n+1)^k))_{n \geq 1}$ starts to be $1, 2, 1, 2, 1, 2, \ldots$ at $n \leq M_k + 1$.

# Other sequences we investigated

$$\Delta((a_n)_n),\ a_n = ka_{n-1} + a_{n-2}$$

$$(\Gamma(k,n)_n)_{n\geq 1}$$

# Limiting density

$$G(x) := \frac{\#\{(a,b) \in \mathbb{N}^2 : 1 \le a \le b \le x, \gcd(a,b) = 1, \Gamma(a,b) = 1\}}{\#\{(a,b) \in \mathbb{N}^2 : 1 \le a \le b \le x\}}$$

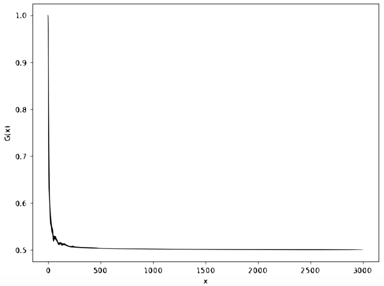


Figure: Plots of $G(x)$ for $1 \le x \le 3000$ with 2000 sample points. In particular, $G(3000) \approx 0.50051166$.

# Limiting density

$$H(x) := \frac{\#\{(a,b) \in \mathbb{N}^2 : 1 \leq a \leq b \leq x, \Gamma(a,b) = 1\}}{\#\{(a,b) \in \mathbb{N}^2 : 1 \leq a \leq b \leq x\}}$$

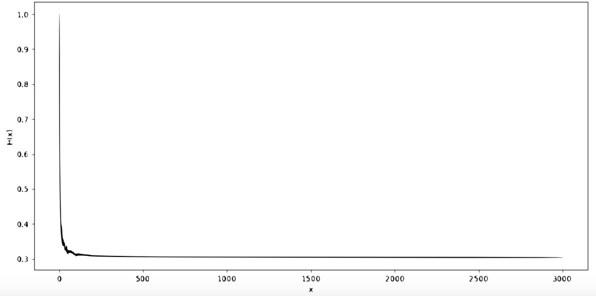


Figure: Plots of $H(x)$ for $1 \leq x \leq 3000$ with 2000 sample points. In particular, $H(3000) \approx 0.30423059$.

*Thank you for listening*