The Katz-Sarnak Density Conjecture and Bounding Central Point Vanishing of L-Functions

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Second Int'l Webinar: Recent Developments in Number Theory
School of Applied Sciences (Mathematics)
Kalinga Institute of Industrial Technology University
Bhubaneswar, India, October 3, 2021

Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Goals

- Determine correct scale and statistics to study zeros of *L*-functions.
- See similar behavior in different systems (random matrix theory).
- Discuss the tools and techniques needed to prove the results.
- Highlight calculations for Dirichlet *L*-functions (simplest case).
- New world records for bounding vanishing at central point.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha \bmod 1$.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

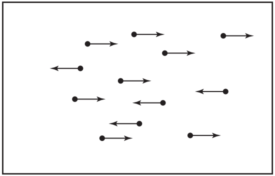
$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Classical Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

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$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Not most general case, gives flavor.

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \sim N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2$$

Gives $N\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ji}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \quad \cdot \prod_{\substack{(k,l)\neq(i,j)\\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} \; = \; 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Main term $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did have Gaussian, see in Real Symmetric Palindromic Toeplitz matrices; interesting results for circulant ensembles (joint with Gene Kopp, Murat Kologlu).

Introduction to *L*-Functions

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty}\frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s)>1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s)=\frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

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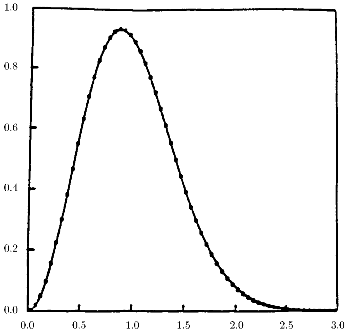
Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{d}{ds}\log\zeta(s) \;=\; -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

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$$= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}\\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right)\\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)e^{-\sigma\log p}e^{-it\log p}ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Riemann Zeta Function: Let $\sum_\rho$ denote the sum over the zeros of $\zeta(s)$ in the critical strip, g an even Schwartz function of compact support and $\phi(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du$. Then

$$\sum_\rho \phi(\gamma_\rho) \;=\; 2\phi\left(\frac{i}{2}\right) - \sum_p \sum_{k=1}^{\infty} \frac{2\log p}{p^{k/2}} g\left(k \log p\right) + \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{iy - \frac{1}{2}} + \frac{\Gamma'(\frac{iy}{2} + \frac{5}{4})}{\Gamma(\frac{iy}{2} + \frac{5}{4}))} - \frac{1}{2}\log \pi \right) \phi(y)\, dy.$$

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$$\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} h(y)dy$$

$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}}$$

$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\left(\frac{1}{\log m}\right).$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned} \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) \\ &\quad + O\left(\frac{\log\log R}{\log R}\right) \\ P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \end{aligned}$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{ \left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k \right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: *n*-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. *n*-level correlations for all automorphic cupsidal *L*-functions (Rudnick-Sarnak).
4. *n*-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: *n*-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
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Katz-Sarnak Conjecture

For a 'nice' family of *L*-functions, the *n*-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\begin{aligned}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) &= \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)\\ &\to \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx.\end{aligned}$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Correspondences

Similarities between *L*-Functions and Nuclei:

$$\text{Zeros} \longleftrightarrow \text{Energy Levels}$$

$$\text{Schwartz test function} \longrightarrow \text{Neutron}$$

$$\text{Support of test function} \longleftrightarrow \text{Neutron Energy.}$$

Some Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak, Ricotta-Royer: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).
- Symplectic: Rubinstein, Gao, Levinson-Miller, and Entin, Roddity-Gershon and Rudnick: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.
- Unitary: Fiorilli-Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.
- Orthogonal: Miller, Young: One and two-parameter families of elliptic curves.

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.
2. Explicit Formula: Relates sums over zeros to sums over primes.
3. Averaging Formulas: Petersson formula in Iwaniec-Luo-Sarnak, Orthogonality of characters in Fiorilli-Miller, Gao, Hughes-Rudnick, Levinson-Miller, Rubinstein.

Applications of *n*-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \le\ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Applications of *n*-level density

One application: bounding the order of vanishing at the central point.

Average rank $\cdot\ \phi(0) \ \le\ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \ge 5$.

Example:
Dirichlet L-functions

Dirichlet Characters (m prime)

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g. Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & (k,m) = 1 \\ 0 & (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each l, $1 \le l \le m-2$ we have

$$\chi_l(k) = \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a(m) \\ 0 & (k,m) > 0 \end{cases}$$

Dirichlet L-Functions

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}.$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = \ \ 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

Explicit Formula

Let ϕ be an even Schwartz function with compact support $(-\sigma,\sigma)$, let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right)$$

$$= \int_{-\infty}^{\infty} \phi(y)dy$$

$$-\sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p)+\bar{\chi}(p)]p^{-\frac{1}{2}}$$

$$-\sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p)+\bar{\chi}^2(p)]p^{-1}$$

$$+O\left(\frac{1}{\log m}\right).$$

Expansion

$\{\chi_0\} \cup \{\chi_l\}_{1 \le l \le m-2}$ are all the characters mod m.
Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned} & \int_{-\infty}^{\infty} \phi(y)dy \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}} \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1} \\ + & O\left(\frac{1}{\log m}\right). \end{aligned}$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise.} \end{cases}$$

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For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} -1 + m - 1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

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$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise.} \end{cases}$$

For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \;=\; \begin{cases} -1 + m - 1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$\ll\quad \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}+\sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$
\begin{aligned}
& \frac{-2}{m-2} \sum_{p}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \\
+ \quad & 2\frac{m-1}{m-2} \sum_{p\equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}} \ll \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} + \sum_{\substack{k\equiv 1(m) \\ k\geq m+1}}^{m^\sigma} k^{-\frac{1}{2}}
\end{aligned}
$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac12}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac12}$$

$$\ll\quad \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac12}+\sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac12}\ll\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac12}+\sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac12}$$

$$\ll\quad \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac12}+\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac12}\ll\frac{1}{m}m^{\sigma/2}.$$

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)+\bar{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\bar{\chi}^2(p)] = \begin{cases} 2(m-2) & p\equiv\pm 1(m) \\ -2 & p\not\equiv\pm 1(m)\end{cases}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\ll \frac{1}{m-2}\sum_p^{m^{\sigma/2}} p^{-1} + \frac{2m-2}{m-2}\sum_{p\equiv\pm 1(m)}^{m^{\sigma/2}} p^{-1}$$

$$\ll \frac{1}{m-2}\sum_k^{m^{\sigma/2}} k^{-1} + \sum_{k\equiv 1(m)}^{m^{\sigma/2}} k^{-1} + \sum_{k\equiv -1(m)}^{m^{\sigma/2}} k^{-1}$$

Cuspidal Newforms
Hughes-Miller

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.
- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case and talk about extensions (joint with Chris Hughes).

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \quad f(\gamma z) = (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dxdy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) \ = \ \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) \quad = \quad 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right) + \ \delta(m,n).$$

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Modular Form Preliminaries:Fourier Coefficient Review

$$\begin{aligned} \lambda_f(n) &= a_f(n) n^{\frac{k-1}{2}} \\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d\mid(m,n) \\ (d,M)=1}} \lambda_f\left(\frac{mn}{d}\right). \end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k \mu(N) \lambda_f(N) \sqrt{N}.$$

The above will allow us to split into even and odd families: $1 \pm \epsilon_f$.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) \;=\; \sum_{a \bmod q}^{*} e(an/q) \;=\; \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ *be an even Schwartz function with* $\mathrm{supp}(\widehat{\Psi}) \subset (-2,2)$. *Then*

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{dy}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$ *and* φ *is Euler's totient function.*

Limited Support ($\sigma < 1$): Sketch of proof

- Estimate Kloosterman-Bessel terms trivially.
  $\diamond$ Kloosterman sum: $d\overline{d} \equiv 1 \bmod q$, $\tau(q)$ is the number of divisors of q,

$$S(m,n;q) \;=\; \sum_{d \bmod q}^{*} e\left(\frac{md}{q} + \frac{n\overline{d}}{q}\right)$$

$$|S(m,n;q)| \;\leq\; (m,n,q)\sqrt{\min\left\{\frac{q}{(m,q)}, \frac{q}{(n,q)}\right\}}\;\tau(q).$$

  $\diamond$ Bessel function: integer $k \geq 2$, $J_{k-1}(x) \ll \min\left(x, x^{k-1}, x^{-1/2}\right)$.

- Use Fourier Coefficients to split by sign: N fixed: $\pm\sum_f \lambda_f(N) * (\cdots)$.

Increasing Support ($\sigma < 2$): Sketch of the proof

- Using Dirichlet Characters, handle Kloosterman terms.
- Have terms like

$$\int_0^\infty J_{k-1}\left(4\pi\frac{\sqrt{m^2yN}}{c}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{\sqrt{y}}$$

  with arithmetic factors to sum outside.
- Works for support up to $(-2, 2)$.

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Stating in greater generality for later use.

Gauss sum: χ a character modulo q: $|G_\chi(n)| \le \sqrt{q}$ with

$$G_\chi(n) \ = \ \sum_{a \bmod q} \chi(a) \exp(2\pi i a n/q).$$

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Kloosterman expansion:

$$S(m^2, p_1 \cdots p_n N; Nb) = \frac{-1}{\varphi(b)} \sum_{\chi (\bmod b)} \chi(N) G_\chi(m^2) G_\chi(1) \overline{\chi}(p_1 \cdots p_n).$$

Lemma: Assuming GRH for Dirichlet L-functions, $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, non-principal characters negligible.
Proof: use $J_{k-1}(x) \ll x$ and see

$$\ll \frac{1}{\sqrt{N}} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b < N^{2006}}} \frac{1}{b} \frac{1}{\varphi(b)} \sum_{\substack{\chi (\bmod b) \\ \chi \neq \chi_0}} \left| G_\chi(m^2) G_\chi(1) \right|$$

$$\times \frac{m}{b\sqrt{N}} \prod_{j=1}^{n} \left| \sum_{p_j \neq N} \overline{\chi}(p_j) \log p_j \cdot \frac{1}{\log R} \widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \right|.$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma} \widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right) J_{k-1}\left(4\pi\frac{\sqrt{m^2 x_1 x_2 N}}{c}\right)\frac{dx_1 dx_2}{\sqrt{x_1 x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1 x_2 & \quad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix} = \frac{1}{u_1}.$$

Left with

$$\int\int \widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\frac{du_1 du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

3-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\int_{x_3=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)\widehat{\phi}\left(\frac{\log x_3}{\log R}\right)$$

$$* \quad J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2x_3N}}{c}\right)\frac{dx_1dx_2dx_3}{\sqrt{x_1x_2x_3}}$$

Change variables as below and get Jacobian:

$$\begin{array}{rclcrcl} u_3 & = & x_1x_2x_3 & \quad & x_3 & = & \frac{u_3}{u_2} \\ u_2 & = & x_1x_2 & & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} & 0 \\ 0 & -\frac{u_3}{u_2^2} & \frac{1}{u_2} \end{vmatrix} = \frac{1}{u_1u_2}.$$

n-Level Density: Determinant Expansions from RMT

- U(N), $U_k(N)$: $\det\left(K_0(x_j, x_k)\right)_{1\le j,k\le n}$
- USp(N): $\det\left(K_{-1}(x_j, x_k)\right)_{1\le j,k\le n}$
- SO(even): $\det\left(K_1(x_j, x_k)\right)_{1\le j,k\le n}$
- SO(odd): $\det\left(K_{-1}(x_j, x_k)\right)_{1\le j,k\le n} + \sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j, x_k)\right)_{1\le j,k\ne\nu\le n}$

where

$$K_\epsilon(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \epsilon\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

Support for n-Level Density

Careful book-keeping gives (originally just had $\frac{1}{n-1/2}$)

$$\sigma_n \;<\; \frac{1}{n-1}.$$

n-Level Density is trivial for $\sigma_n < \frac{1}{n}$, non-trivial up to $\frac{1}{n-1}$.

Expected $\frac{2}{n}$. Obstruction from partial summation on primes.

Support Problems: 2-Level Density

Partial Summation on p_1 first, looks like

$$\sum_{\substack{p_1 \\ p_1 \neq p_2}} S(m^2, p_1 p_2 N, c) \frac{2 \log p_1}{\sqrt{p_1} \log R} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) J_{k-1}\left(4\pi \frac{\sqrt{m^2 p_1 p_2 N}}{c}\right)$$

Similar to ILS, obtain ($c = bN$):

$$\sum_{\substack{p_1 \leq x_1 \\ p_1 \nmid b}} S(m^2, p_1 p_2 N, c) \frac{\log p}{\sqrt{p}} = \frac{2\mu(N)}{\phi(b)} \widetilde{R}(m^2, b, p_2) x_1^{\frac{1}{2}} + O\left(b(bx_1 N)^{\epsilon}\right)$$

$\sum_{p_1}$ to $\int_{x_1}$, error $\ll\ b(bN)^{\epsilon} m \sqrt{p_2 N} N^{\sigma_2/2}/bN$, yields

$$\sqrt{N} \sum_{m \leq N^{\epsilon}} \frac{1}{m} \sum_{b \leq N^5} \frac{1}{bN} \sum_{p_2 \leq N_2^{\sigma}} \frac{1}{\sqrt{p_2}} \frac{b(bN)^{\epsilon} m \sqrt{p_2 N} N^{\frac{\sigma_2}{2}}}{bN}$$

$$\ll \quad N^{\frac{1}{2}+\epsilon'+\sigma_2+\frac{1}{2}+\frac{\sigma_2}{2}-2} \ll N^{\frac{3}{2}\sigma_2 - 1 + \epsilon'}$$

Support Problems: n-Level Density: Why is $\sigma_2 < 1$?

- If no $\sum_{p_2}$, have above *without* the N^σ which arose from $\sum_{p_2}$, giving

$$\ll\ N^{\frac{1}{2}+\epsilon'+\frac{1}{2}+\frac{\sigma_1}{2}-2}\ =\ N^{\frac{1}{2}\sigma_1-1+\epsilon'}.$$

- Fine for $\sigma_1 < 2$. For 3-Level, have two sums over primes giving N^{σ_3}, giving

$$\ll\ N^{\frac{1}{2}+\epsilon'+2\sigma_3+\frac{1}{2}+\frac{\sigma_3}{2}-2}\ =\ N^{\frac{5}{2}\sigma_3-1+\epsilon'}$$

- n-Level, have an additional $(n-1)$ prime sums, each giving N^{σ_n}, yields

$$\ll\ N^{\frac{1}{2}+\epsilon'+(n-1)\sigma_n+\frac{1}{2}+\frac{\sigma_n}{2}-2}\ =\ N^{\frac{(2n-1)}{2}\sigma_n-1+\epsilon'}$$

Summary

- More support for RMT Conjectures.
- Control of Conductors.
- Averaging Formulas.

Theorem (Hughes-Miller 2007)

n-level densities of weight k cuspidal newforms of prime level N, $N \to \infty$, agree with orthogonal in non-trivial range (with or without splitting by sign).

Extending Support and Bounding Vanishing

With Elżbieta Bołdyriew, Fangu Chen, Charles Devlin VI, Justine Dell, Simran Khunger, Stella Li, Alexander E. Shashkov, Alicia G. Smith Reina, Stephen D. Willis, Yingzi Yang, Jason Zhao

Methods: the term of interest

- Conditional on GRH, we want to show that for $a \leq n/2$, if $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-a}, \frac{1}{n-a})$, then the n^{th} centered moments of $D(f;\phi)$ averaged over $f \in H_k^\sigma$ satisfy the Density Conjecture.

- To compute the n^{th} centered moments of $D(f;\phi)$, it suffices to convert the following sum over primes into an integral:

$$S_2^{(n)} := \sum_{p_1 \nmid N, \dots, p_n \nmid N} \prod_{j=1}^{n} \left(\hat{\phi}\left(\frac{\log p_j}{\log R}\right) \frac{2 \log p_j}{\sqrt{p_j} \log R} \right) \left\langle \lambda_f(N) \prod_{j=1}^{n} \lambda_f(p_i) \right\rangle_* ,$$

$\lambda_f(p_i)$ are the normalized Fourier coefficients of f.

Extending to Higher Support

- Expanding $S_2^{(n)}$ yields many terms. The smaller the support, the more of these terms vanish as $N \to \infty$. As we increase support, the terms we have to handle become more complex and more numerous.

- Our work fully generalized the number theoretic techniques needed to convert these sums over primes into sums over integers.

- We are currently generalizing the work of Iwaniec, Luo, and Sarnak to convert these terms to integrals in order to show agreement with random matrix theory.

Methods: removing negligible subterms

Expand $S_2^{(n)}$ into subterms, each of which is a sum over distinct primes

↓

Apply inclusion-exclusion to convert to sums over non-distinct primes

↓

Expand Fourier coefficients into exponential (*Bessel-Kloosterman*) sums

↓

Expand exponential sums into terms involving Dirichlet characters

↓

Use Mellin transforms to convert to integrals

- We extended and generalized to centered moments with arbitrary test functions, which introduces combinatorial challenges that we surmount through careful analysis, in particular constructing new ancillary functions to assist in the estimation, and using inclusion / exclusion.

- Needed to construct a new ancillary test function $\widehat{\psi}_\epsilon(x)$ that is point-wise larger than any test function $\widehat{\phi}_j$ to use in our error analysis.

- Usually after first few papers (Iwaniec-Luo Sarnak, Freeman-Miller, Bołdyriew-Chen-Devlin-Miller-Zhao) progress is small, in deep digits, but we see remarkable improvements.

Comparison of order of vanishing bounds for SO(even) functions from various approaches.

Order vanish	1-level	2-level	4^{th} centered moment*
6	0.144090	0.0157687	0.008538410
8	0.108067	0.0157687	0.000813368
10	0.086454	0.0047306	0.000186846

- These are upper bounds for vanishing at least r (number in order vanishing column).
- For the 1-level column, used the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2, 2)$.
- For the 2-level column, used the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions is $(-1, 1)$.
- For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test function used is $(-1/3, 1/3)$.

Comparison of order of vanishing bounds for SO(odd) forms from various approaches.

Order vanishing	1-level	2-level	4^{th} centered moment*
5	0.222908	0.0674429	0.06580440
7	0.159220	0.0299746	0.00221997
9	0.123838	0.0168607	0.00036405

- These are upper bounds for vanishing at least r (number in order vanishing column).
- For the 1-level column, using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2, 2)$.
- For the 2-level column, using the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions is $(-1, 1)$.
- For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test function used is $(-1/3, 1/3)$.

Comparison of order of vanishing bounds for SO(odd) forms from various approaches.

Order vanishing	1-level	2-level	4th centered moment*	4th centered moment**
500	0.00172908	$1.51987 * 10^{-6}$	$5.48858 * 10^{-12}$	$4.94335 * 10^{-12}$
600	0.00144090	$1.05476 * 10^{-6}$	$2.63448 * 10^{-12}$	$2.36440 * 10^{-12}$
700	0.00123506	$7.74556 * 10^{-7}$	$1.41727 * 10^{-12}$	$1.26878 * 10^{-12}$

- These are upper bounds for vanishing at least r (order vanishing column).
- For the 1-level column, using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test functions is $(-2, 2)$.
- For the 2-level column, using the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions is $(-1, 1)$.
- For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test functions is $(-1/3, 1/3)$.
- For the 4^{th} centered moment** column, the** signifies that we used 2 copies of ϕ generated by $g(x) = \sin x^2$ for $x < 1/8$ and 2 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test functions is $(-1/4, 1/4)$.

Upper bounds for vanishing at least r (number in order vanishing column).

Order vanishing	1-level	2-level	4th centered moment*	4th centered moment**
499	0.00223355	$4.35109*10^{-6}$	$5.52644*10^{-12}$	$4.98359*10^{-12}$
599	0.00186067	$3.01755*10^{-6}$	$2.64907*10^{-12}$	$2.3804*10^{-12}$
699	0.00159448	$2.21486*10^{-6}$	$1.42374*10^{-12}$	$1.27612*10^{-12}$

- For the 1-level column, using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test functions is $(-2, 2)$.
- For the 2-level column, using the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions is $(-1, 1)$.
- For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test functions is $(-1/3, 1/3)$.
- For the 4^{th} centered moment** column, the ** signifies that we used 2 copies of ϕ generated by $g(x) = \sin x^2$ for $x < 1/8$ and 2 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test functions is $(-1/4, 1/4)$.