Optimal point sets determining few distinct triangles

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Introduction

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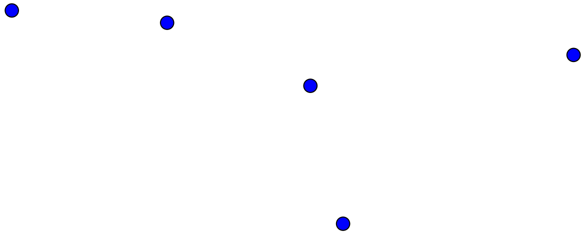

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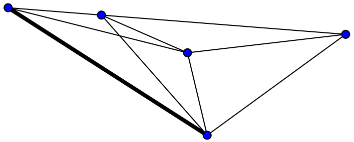

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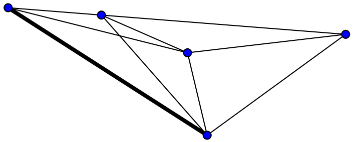

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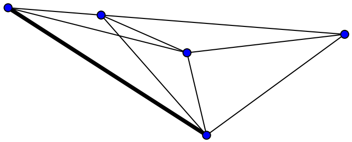

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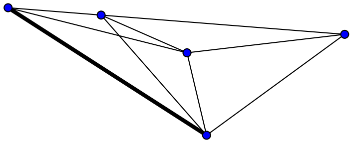

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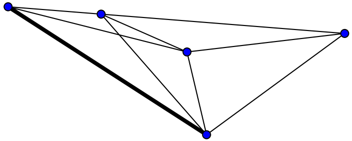

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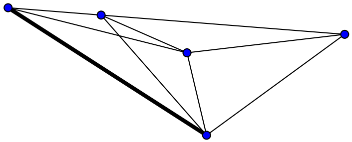

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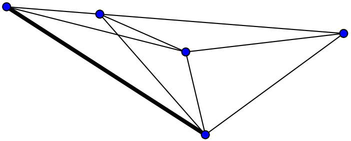
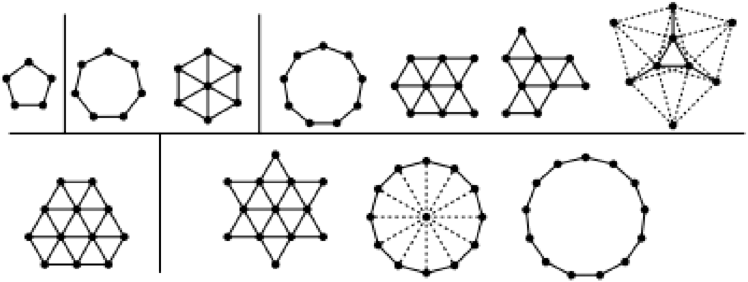
- Optimal configurations for few distances:
  - Characterized for small numbers (≤ 6) of distinct distances. [Erdős and Fishburn]





SETS WITH k DISTINCT DISTANCES, $2 \leq k \leq 6$,
AND MAXIMUM NUMBER OF POINTS

[Brass, Moser, and Pach]

Triangles

2 points $\Rightarrow$ distance.
3 points $\Rightarrow$ triangle.

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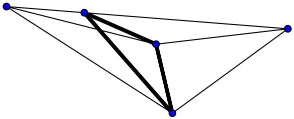

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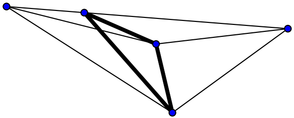

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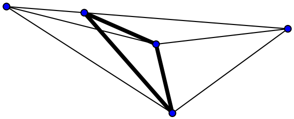
Can ask analogous questions for triangles:

- Minimum number of distinct triangles?
- Maximum number of a given triangle?
- Optimal configurations for few triangles?



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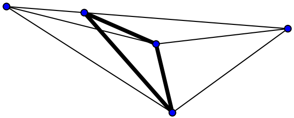
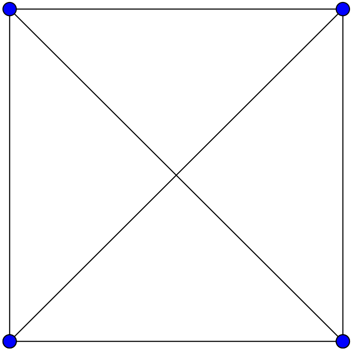


Some motivation

4 vertices of a square $\Rightarrow$ 1 distinct triangle.



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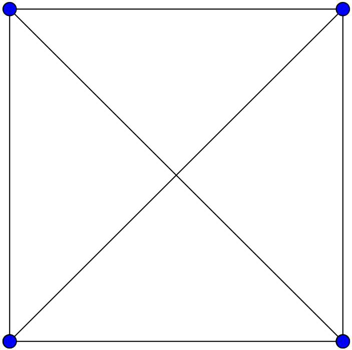

Can we do this with 5 points?

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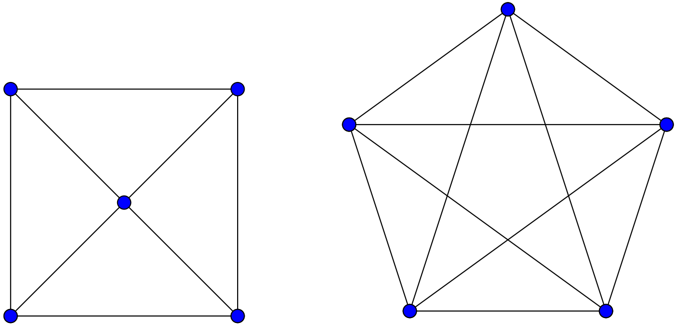
Square plus center $\Rightarrow$ 2 distinct triangles.
Regular pentagon $\Rightarrow$ 2 distinct triangles.



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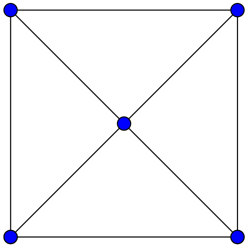

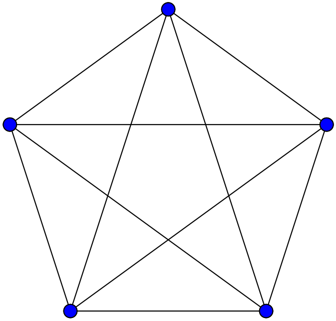

Can we do this with 6 points?

Main results

Theorem (Epstein, Lott, Miller, and Palsson)

Let $F(t) =$ maximum # of points that can be placed in the plane to determine exactly t distinct triangles.

- $F(1) \ = \ 4$
- $F(2) \ = \ 5$
- $F(t) \ < \ 24(t+1)$

Proof sketch for $F(1)$

- Enough to show that a rectangle is the *only* 4-point configuration determining a single triangle.

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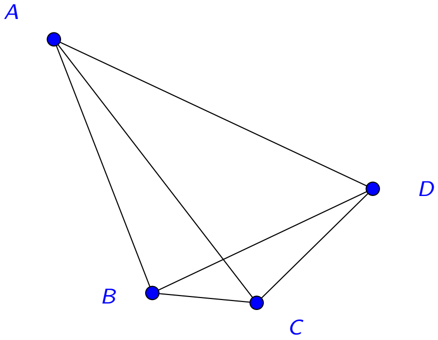
- Idea: brute force by cases.

Example case

4 points form a quadrilateral with all side lengths distinct:



$\triangle ABC$ and $\triangle ADC$ are distinct.

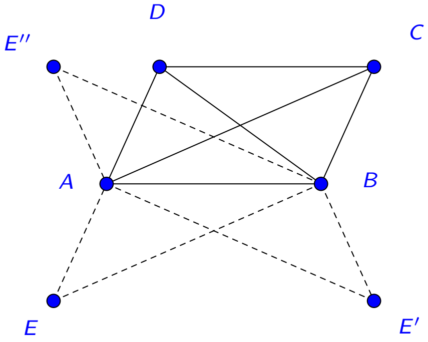
Proof sketch for $F(2)$

- Enough to show that the square with its center and the regular pentagon are the *only* 5-point configurations determining two triangles.

- Idea: most 4-point configurations determine three triangles. For the ones that don't, show that the addition of a fifth point generates a third.

Example case

4 points form a non-rhombus parallelogram:



Any choice of E will generate a third distinct triangle.

Proof sketch for upper bound

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  - Bound from above the number of times a given triangle can appear.
  - Bound from below the number of noncollinear triples that generate a triangle.
  - # of distinct triangles $\geq$ ratio of these two quantities.

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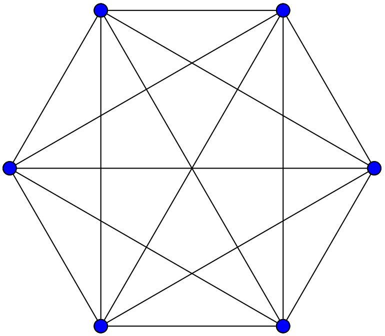
- $\Rightarrow$ # distinct triangles $\geq \frac{n(n-1)(n-k)}{12(n-k+1)(n-k)+24(k-1)}$.
- Minimize over $k \in [2, n-1] \Rightarrow n/24$.

3-triangle sets



3 distinct triangles.
Can we do this with 7 points?

3-triangle sets

Conjecture

$F(3) = 6$, i.e. any set of 7 points determines at least 4 triangles.

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- Vertices of a regular polygon?
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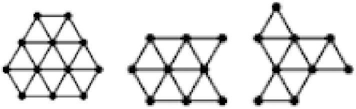
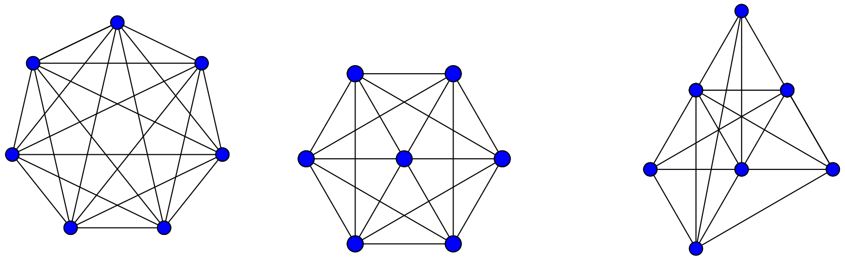
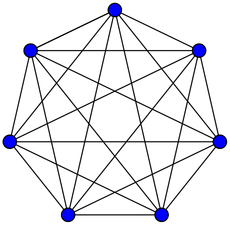
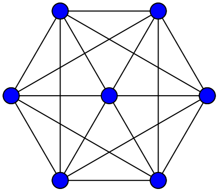
- Subsets of the triangular lattice?
  - Erdős conjectured that optimal distance sets come from the triangular lattice.







- 4 triangles



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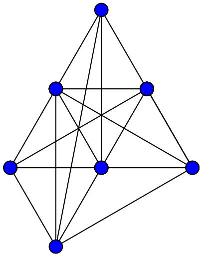

- ≥ 5 triangles

General structure of optimal configurations

Conjecture

The regular n-gon minimizes (not necessarily uniquely) the number of distinct triangles determined by an n-point set.

Application of conjecture

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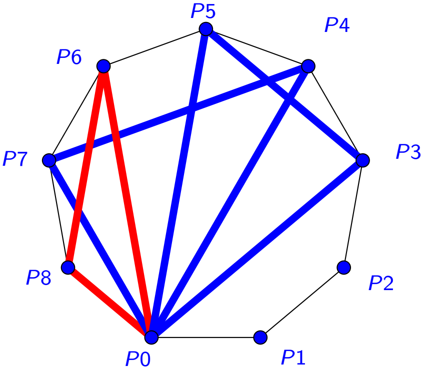
Explaining the bijection



$n = 9$

Thanks

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- Eyvindur Palsson

References

P. Erdős and P. Fishburn, "Maximum planar sets that determine k distances", *Discrete Mathematics*, 160:1-3 (1996), 115-125.

L. Guth and N. H. Katz, "On the Erdős distinct distances problem in the plane", *Annals of Mathematics*, 181:1 (2015), 155-190.

L.A. Szekely, "Crossing numbers and hard Erdős problems in discrete geometry", *Combinatorics, Probability, and Computing*, 11 (1993), 1-10.

P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer (2005).