Sato-Tate Groups, Elliptic Curves, and Second Moment Distributions

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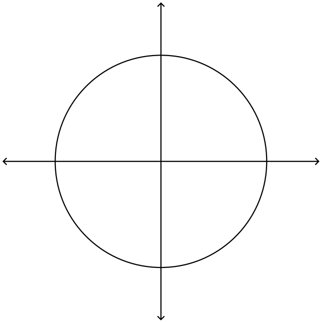
Algebraic Curves

A **curve** is a zero set of some polynomial over a field. For example:

$$C\colon x^2 + y^2 = 1 \text{ over } \mathbb{R}.$$



Algebraic Curves (and Arithmetic!)

Solutions over $\mathbb{Q}$ correspond in some sense to solutions over $\mathbb{Z}$ (e.g. clear denominators), so it encodes arithmetic properties.

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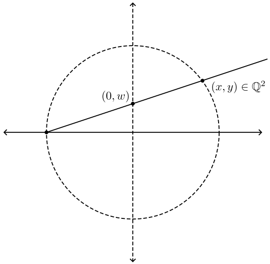

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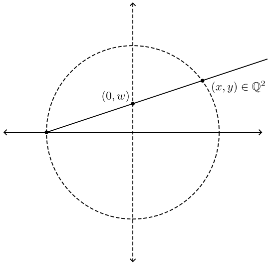

The solution set to $x^2 + y^2 = 1$ in $\mathbb{Q}$ encodes the information of pythagorean triples $x^2 + y^2 = z^2$ in $\mathbb{Z}$.

An important invariant: genus

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- **Genus** is a topological invariant of surfaces. It counts the number of holes. This is an invariant of curves as well.

What's an Elliptic Curve?

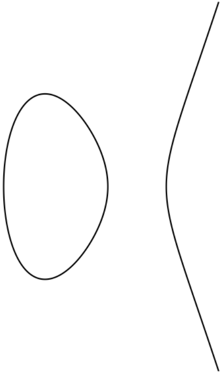
Formally, smooth curve of genus 1. Easier over $\mathbb{Q}$: an elliptic curve E is given by

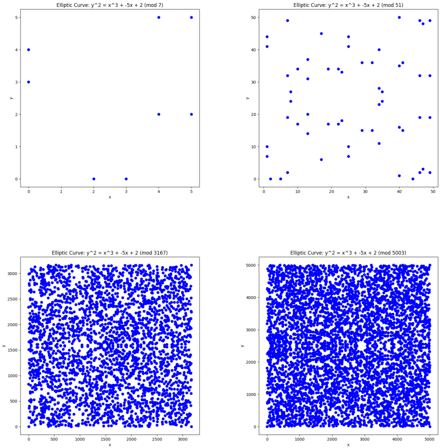
$$E\colon y^2 = x^3 + ax + b$$

with $a, b \in \mathbb{Z}$, $4a^3 + 27b^2 \neq 0$.

- They are of study throughout number theory.
- They are one of the main objects used in the proof of Fermat's Last Theorem.

What's an Elliptic Curve?





Counting Points

The mod p reductions looked very chaotic, we need tools to help understand them.

Define the Legendre symbol: for $a \in \mathbb{N}$ and p a prime,

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a non-zero square mod p} \\ -1 & \text{if a is not a non-zero square mod p} \\ 0 & \text{if } a \equiv 0 \mod p \end{cases}$$

Important observation: $1 + \left(\frac{a}{p}\right)$ is the number of solutions to $y^2 = a \mod p$.

Sato-Tate Distributions over $\mathbb{Q}$

That is, $\#E(\mathbb{F}_p) = \sum_{x=0}^{p-1}\left[1 + \left(\frac{x^3+ax+b}{p}\right)\right] + \text{point at } \infty = p + 1 + a(p)$.

We wish to understand how $a(p)$ varies with p.

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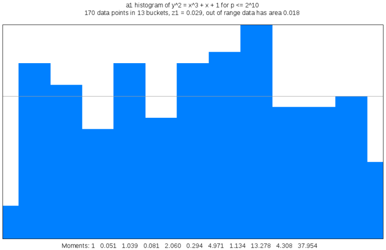
Theorem (Sato-Tate Conjecture, 2011)

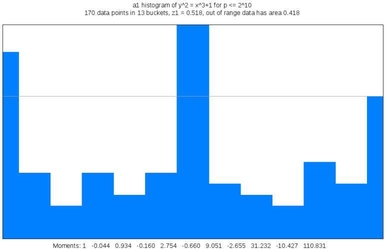
Let E be an elliptic curve without complex multiplication. As $p \to \infty$, the sequence $\{\frac{a(p)}{\sqrt{p}}\}_p$ becomes equidistributed according to the density

$$\mathrm{d}\mu_{ST} = \frac{\sqrt{4-x^2}}{2\pi}\mathrm{d}x$$

which is compactly supported on $[-2, 2]$.

Sato-Tate Distributions over $\mathbb{Q}$

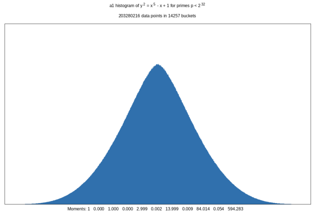


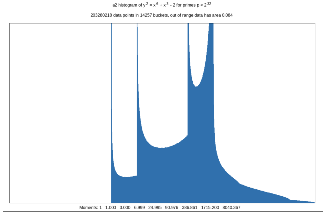


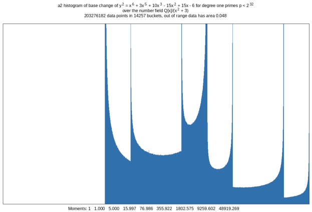
https://math.mit.edu/~drew/g1_D1_a1f.gif
https://math.mit.edu/~drew/g1_D2_a1f.gif

Genus 2

The above come from the Haar measure on certain subgroups of the compact group USp(2). To classify Sato-Tate distributions of genus 2 curves, use USp(4). Exactly 34 possibilities over $\mathbb{Q}$.







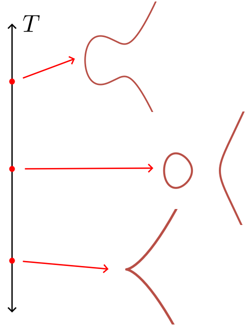
Families of curves

Idea: Allow a free parameter T to vary. One elliptic curve for each T:

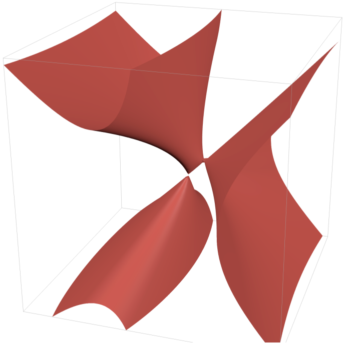
$$\mathcal{E}_T\colon y^2 = x^3 + A(T)x + B(T),$$

where $A(T), B(T) \in \mathbb{Q}(T)$.

Families of curves



(a) Elliptic curve fibers



(b) Plot of $\mathcal{E}_T$ in $\mathbb{R}^3$

Figure: $t \in \mathbb{Q}$ gives an elliptic curve $y^2 = x^3 + A(t)x + B(t)$.

The Bias Conjecture

Definition

The second moment of the bias is defined as

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The lower order terms are $p^{3/2}$, p, $p^{1/2}$ and 1.

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One way to disprove the bias conjecture: $\mathcal{E}_T$ such that

$$\mathcal{B}_{2,\mathcal{E}}(p) := \frac{\mathcal{A}_{2,\mathcal{E}}(p) - p^2}{p^{3/2}}$$

averages to a negative value.

Why Do We Care?

- "Rank" of elliptic surface (family of elliptic curves) related to first moment
- Applications to Katz-Sarnak conjecture
- Order of vanishing of L-functions associated to the family
- Higher moments show family-specific behavior (rate of convergence)

Pencils of cubics

Given a family $y^2 = x^3 + A(T)x + B(T)$, we can think of it as a surface $\pi \colon \mathcal{E}_T \to \mathbb{P}^1$ fibered in elliptic curves.

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Let's specialize to the case

$$\mathcal{E}_T\colon y^2 = P(x)T + Q(x)$$

where $\deg P(x), \deg Q(x) \leq 3$. This is a **pencil of cubics.**

Some Algebraic Geometry

Proposition (SMALL 2025)

Let

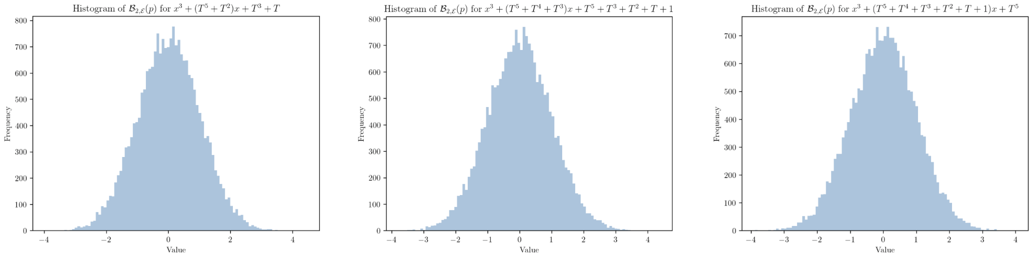
$$\mathcal{E}_T : y^2 = P(x)T + Q(x)$$

be a pencil of cubics. If the curve is "generic"[a] *all moments of* $\mathcal{B}_{2,\mathcal{E}}$ *are integers.*

[a] and the associated pair $(\widetilde{\Delta}, \widetilde{C})$ to $\mathcal{E}$ is K*-typical*,

Previous numerical investigations/motivation

Motivation from last year's SMALL [CGJ$^+$24]:



Variance = 1.0151610262691482 | Variance = 1.0046817588446215 | Variance = 0.9891139950619033

Figure: SMALL 2024's numerical evidence suggests the second moment converges to an integer.

Some Algebraic Geometry

The following approach is taken from Professors Bartosz Naskręcki and Matija Kazalicki's paper [KN25].

Proof. To the elliptic surface $\pi : \mathcal{E}_T \to \mathbb{P}^1$ associate a threefold

$$M \subset \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{P}^1 \times \mathbb{A}^1$$

whereby they show in [Theorem 2.2] that

$$\#M(\mathbb{F}_p) = p^3 + p^2 + \widetilde{A}_{2,\mathcal{E}}(p).$$

This allows us to attack the problem through arithmetic geometry.

Computing the distribution of $\mathcal{B}_2(p)$

It follows from their main results, [Prop 3.15, Cor 3.19] that in our special case we have the explicit formula

$$\mathcal{A}_{2,\mathcal{E}}(p) = p^2 - p \cdot d_p - p \cdot \#S(\mathbb{F}_p) - a_\infty^2(p)$$

where

1. d_p is the trace on some curve $\overline{D}$,
2. S is a polynomial,
3. $a_\infty^2(p)$ is the contribution of the fiber at infinity.

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$$\mathcal{B}_{2,\mathcal{E}}(p) = \frac{\mathcal{A}_{2,\mathcal{E}}(p) - p^2}{p^{3/2}} = -\frac{d_p}{\sqrt{p}} - \frac{\#S(\mathbb{F}_p)}{\sqrt{p}} - \frac{a_\infty^2}{p^{3/2}}.$$

Average over p to compute moments. We find that $B_2(p)$ is distributed exactly as $d_p/\sqrt{p}$.

Distribution of $d_p/\sqrt{p}$

$\mathcal{B}_{2,\mathcal{E}}(p) \sim d_p/\sqrt{p}$, so it suffices to show all moments are integers.

- In the "typical" case considered in their paper, $\overline{D}$ is genus 2.

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- In general, assuming the generalized Sato-Tate conjecture allows us to compute the distribution. ∎

Our Other Work

- Again, assuming the generalized Sato-Tate conjecture, we prove this result for more families of elliptic curves.

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- Again, assuming the generalized Sato-Tate conjecture, we prove this result for more families of elliptic curves.
- We have expanded data from SMALL 2024: $p \leq 250\,000 \to p \leq 1\,000\,000$. This allows for deeper numerical investigations.

Acknowledgements

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References I

- Timothy Cheek, Pico Gilman, Kareem Jaber, Steven J. Miller, Vismay Sharan, and Marie-Hélène Tomé, *Lower order biases in moment expansions of one parameter families of elliptic curves*, 2024.
- Francesc Fité, Kiran S. Kedlaya, Víctor Rotger, and Andrew V. Sutherland, *Sato-tate distributions and galois endomorphism modules in genus 2*, Compositio Mathematica **148** (2012), no. 5, 1390–1442.
- Matija Kazalicki and Bartosz Naskręcki, *Second moments and the bias conjecture for the family of cubic pencils*, 2025.
- Philippe Michel, *Rang moyen de familles de courbes elliptiques et lois de sato-tate*, Monatshefte für Mathematik **120** (1995), no. 2, 127–136.