# Eigenvalues of matrix-valued matrices

Arman Rysmakhanov (ar21@williams.edu)

Advised by Steven J. Miller (sjm1@williams.edu)

SMALL REU 2025

Yale, July 23, 2025

Eigenvalues are important!

### Example

- PCA
- Spectral graph theory
- Differential equations
- Markov chains
- Control theory
- $\cdots$

## Extensions

### Example

- **Multiparameter**: Solve $\sum_{j=0}^{k} \lambda_j A_{ij} x = 0$ for $i = 1, \ldots, k$ simultaneously. [Atk72]
- **Quaternions**: Solve $Ax = x\lambda$ where $A \in \mathbb{H}^{n \times n}$ and $\lambda \in \mathbb{H}$. [Lee48]
- **Tensors**: Solve $\mathcal{A}(I_n, x, \ldots, x) = \lambda \phi_{p-1}(x)$. [Lim06]

What if the eigenvalues are matrices themselves?

## Definitions

**Definition (Nested matrix)**

An $n \times n$ nested matrix of order $k$ over $\mathbb{C}$ is an $n \times n$ array $\mathscr{A}$ where each entry $\mathscr{A}_{i,j}$ is a $k \times k$ matrix over $\mathbb{C}$. The set of all such nested matrices is denoted $MM_{n,k}(\mathbb{C}) := M_n(M_k(\mathbb{C}))$.

## Definitions

**Definition (Nested matrix)**

An $n \times n$ nested matrix of order $k$ over $\mathbb{C}$ is an $n \times n$ array $\mathscr{A}$ where each entry $\mathscr{A}_{i,j}$ is a $k \times k$ matrix over $\mathbb{C}$. The set of all such nested matrices is denoted $MM_{n,k}(\mathbb{C}) := M_n(M_k(\mathbb{C}))$.

**Remark**

*$MM_{n,k}(\mathbb{C})$ forms a ring of matrices over a non-commutative ring.*

## Generalizing “scalar” multiplication

**Definition (Scalar multiplication on nested matrices)**

For $z \in \mathbb{C}$ and $\mathscr{A} \in MM_{n,k}(\mathbb{C})$, we define:
**Left matrix multiplication:** The map $L_X : MM_{n,k}(\mathbb{C}) \to MM_{n,k}(\mathbb{C})$ given by

$$(L_X(\mathscr{A}))_{i,j} := z\mathscr{A}_{i,j} \quad \text{for all } 1 \leq i, j \leq n$$

We similarly define the **right scalar multiplication** $\mathscr{A}_{i,j} z$. It is easy to show that they are the same.

## Generalizing "scalar" multiplication

**Definition (Matrix multiplication on nested matrices)**

For $X \in M_k(\mathbb{C})$ and $\mathscr{A} \in MM_{n,k}(\mathbb{C})$, we define:
**Left matrix multiplication:** The map $L_X : MM_{n,k}(\mathbb{C}) \to MM_{n,k}(\mathbb{C})$ given by

$$(L_X(\mathscr{A}))_{i,j} := X \cdot \mathscr{A}_{i,j} \quad \text{for all } 1 \leq i, j \leq n,$$

where $\cdot$ denotes matrix multiplication in $M_k(\mathbb{C})$. We similarly define the **right matrix multiplication** $\mathscr{A}_{i,j} \cdot X$.

Isn't this the same as block matrices?

Isn't this the same as block matrices?

Almost.

## Flattening Map

### Definition

The flattening map is the function

$$\Phi : MM_{n,k}(\mathbb{C}) \to M_{nk}(\mathbb{C})$$

$$A_{i,j} \mapsto \left(\mathscr{A}_{\lfloor (i-1)/k \rfloor + 1, \lfloor (j-1)/k \rfloor + 1}\right)_{((i-1) \bmod k)+1, ((j-1) \bmod k)+1}$$

## Flattening Map

### Definition

The flattening map is the function

$$\Phi : MM_{n,k}(\mathbb{C}) \to M_{nk}(\mathbb{C})$$

$$A_{i,j} \mapsto \left(\mathscr{A}_{\lfloor (i-1)/k \rfloor + 1, \lfloor (j-1)/k \rfloor + 1}\right)_{((i-1) \bmod k)+1, ((j-1) \bmod k)+1}$$

### Remark

*$\Phi$ can be understood as removing the "internal boundaries" between the $k \times k$ matrix blocks.*

## Flattening Map

### Definition

The flattening map is the function

$$\Phi : MM_{n,k}(\mathbb{C}) \to M_{nk}(\mathbb{C})$$

$$A_{i,j} \mapsto \left(\mathscr{A}_{\lfloor (i-1)/k \rfloor+1, \lfloor (j-1)/k \rfloor +1}\right)_{((i-1) \bmod k)+1, ((j-1) \bmod k)+1}$$

### Remark

*$\Phi$ can be understood as removing the "internal boundaries" between the $k \times k$ matrix blocks.*

### Proposition

*$\Phi$ is a ring isomorphism.*

**Definition (Inverse of a nested matrix)**

A nested matrix $\mathscr{A} \in MM_{n,k}(\mathbb{C})$ is *invertible* if there exists $\mathscr{B} \in MM_{n,k}(\mathbb{C})$ such that

$$\mathscr{A}\mathscr{B} = \mathscr{B}\mathscr{A} = \mathscr{I}_{n,k}.$$

## Existence of inverses

For all rings, even non-commutative, if an element has both a left and a right inverse, then the inverses are the same.

**Proposition**

*In $MM_{n,k}$, the left inverse exists if and only if the right inverse exists.*

## Invertibility via flattening

**Theorem**

*A nested matrix* $\mathscr{A} \in MM_{n,k}(\mathbb{C})$ *is invertible if and only if* $\Phi(\mathscr{A}) \in M_{nk}(\mathbb{C})$ *is invertible.*

## General Linear Group

**Definition (General linear group of nested matrices)**

The **general linear group of nested matrices** is

$$\mathscr{GL}_{n,k}(\mathbb{C}) := \{\mathscr{A} \in MM_{n,k}(\mathbb{C}) : \mathscr{A} \text{ is invertible}\}$$

**Theorem**

$$\mathscr{GL}_{n,k}(\mathbb{C}) \approx \mathrm{GL}_{nk}(\mathbb{C})$$

## Eigenvalues and eigenvectors

### Definition

Given $\mathscr{A} \in MM_{n,k}(\mathbb{C})$, a matrix $\Lambda \in M_k(\mathbb{C})$ is called a **left eigenvalue** of $\mathscr{A}$ if there exists a nonzero $\vec{X} \in M_k^n(\mathbb{C})$ such that

$$\mathscr{A}\vec{X} = \Lambda\vec{X}$$

Similarly, $\Lambda \in M_k(\mathbb{C})$ is called a **right eigenvalue** of $\mathscr{A}$ if there exists a nonzero $\vec{X} \in M_k^n(\mathbb{C})$ such that

$$\mathscr{A}\vec{X} = \vec{X}\Lambda$$

In both cases, such $\vec{X}$ is called an **eigenvector** corresponding to $\Lambda$.

## Alternative definitions

What about $\vec{X}\mathscr{A} = \Lambda\vec{X}$ and $\vec{X}\mathscr{A} = \vec{X}\Lambda$?

## Alternative definitions

What about $\vec{X}\mathscr{A} = \Lambda\vec{X}$ and $\vec{X}\mathscr{A} = \vec{X}\Lambda$?

### Proposition

$$(\vec{X}\mathscr{A})^H = (\Lambda\vec{X})^H \iff \mathscr{A}^H\vec{X}^H = \vec{X}^H\Lambda^H$$
$$(\vec{X}\mathscr{A})^H = (\vec{X}\Lambda)^H \iff \mathscr{A}^H\vec{X}^H = \Lambda^H\vec{X}^H$$

## Notation

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ be matrices. The **Kronecker product** $A \otimes B$ is defined as the $mp \times nq$ block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \tag{1}$$

## Trivial eigenvalues

**Definition (Trivial eigenvalues)**

If $\lambda$ is an eigenvalue of $A \in M_k(\mathbb{C})$, then $\lambda I_k$ is called a trivial eigenvalue. An eigenvalue is non-trivial if it is not trivial.

## Trivial eigenvalues

**Definition (Trivial eigenvalues)**

If $\lambda$ is an eigenvalue of $A \in M_k(\mathbb{C})$, then $\lambda I_k$ is called a trivial eigenvalue. An eigenvalue is non-trivial if it is not trivial.

**Theorem**

*A trivial eigenvalue $\lambda I_k$, where $\lambda$ is an eigenvalue of $A \in M_k(\mathbb{C})$, is both a left and a right eigenvalue of $\Phi^{-1}(A) \in MM_{n,k}$ for all $n, k$ such that $n \bmod k = 0$.*

## How do we compute eigenvalues?

Let $\mathscr{A} \in MM_{n,k}(\mathbb{C})$, with $\Lambda \in M_k(\mathbb{C})$ and $\vec{X} \in M_k^n(\mathbb{C})$ as unknowns.

### Theorem

*$\mathscr{A}\vec{X} = \Lambda\vec{X}$ has non-zero solutions if and only if $\Phi(\mathscr{S}) \in M_{nk^2}(\mathbb{C})$ is singular, where $\mathscr{S}_{i,j} = I_k \otimes \mathscr{A}_{i,j}$ if $i \neq j$ and $I_k \otimes (\mathscr{A}_{i,j} - \Lambda)$ otherwise.*

### Theorem

*$\mathscr{A}\vec{X} = \vec{X}\Lambda$ has non-zero solutions if and only if $\Phi(\mathscr{S}) \in M_{nk^2}(\mathbb{C})$ is singular, where $\mathscr{S}_{i,j} = I_k \otimes \mathscr{A}_{i,j}$ if $i \neq j$ and $I_k \otimes \mathscr{A}_{i,j} - \Lambda^T \otimes I_k$ otherwise.*

## Example

Let's look at an example of matrix-valued eigenvalues.

$$\mathscr{A} = \begin{bmatrix} \begin{bmatrix} 16 & 13 \\ 13 & 10 \end{bmatrix} & \begin{bmatrix} 4 & 7 \\ 7 & 0 \end{bmatrix} \\ \begin{bmatrix} 4 & 7 \\ 7 & 0 \end{bmatrix} & \begin{bmatrix} 16 & 13 \\ 13 & 10 \end{bmatrix} \end{bmatrix}$$

## Example

Let's look at an example of matrix-valued eigenvalues.

$$\mathscr{A} = \begin{bmatrix} \begin{bmatrix} 16 & 13 \\ 13 & 10 \end{bmatrix} & \begin{bmatrix} 4 & 7 \\ 7 & 0 \end{bmatrix} \\ \begin{bmatrix} 4 & 7 \\ 7 & 0 \end{bmatrix} & \begin{bmatrix} 16 & 13 \\ 13 & 10 \end{bmatrix} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 12 & 6 \\ \frac{4x+100}{7} & x \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} z & \frac{1}{49}w(4x+9) + \frac{1}{7}u(x-10) \\ y & -\frac{1}{49}u(4x+9) - \frac{8}{343}w(2x+29) \\ \frac{1}{7}(x-10)y + \frac{1}{49}(4x+9)z & w \\ -\frac{1}{49}(4x+9)y - \frac{8}{343}(2x+29)z & u \end{bmatrix}$$

## Properties of right eigenvalues

Let $\Lambda \in M_k(\mathbb{C})$ is a right eigenvalue of $\mathscr{A} \in MM_{n,k}(\mathbb{C})$.

**Theorem (Right eigenvalues form conjugacy classes)**

*$P^{-1}\Lambda P$ is a right eigenvalue for any $P \in GL_k(\mathbb{C})$.*

## Properties of right eigenvalues

Let $\Lambda \in M_k(\mathbb{C})$ is a right eigenvalue of $\mathscr{A} \in MM_{n,k}(\mathbb{C})$.

**Theorem (Right eigenvalues form conjugacy classes)**

*$P^{-1}\Lambda P$ is a right eigenvalue for any $P \in GL_k(\mathbb{C})$.*

**Theorem**

*If $\lambda \in \mathbb{C}$ is an eigenvalue of $\Lambda$ and a flattened eigenvector corresponding to $\Lambda$ has full rank, then $\lambda$ is also a eigenvalue of $\Phi(\mathscr{A})$. [JEDJ71]*

## Extensions

Current work is on the spectral properties of matrices in the left spectrum of “symmetric” nested matrices.

## Goal of the project

1. Formalize the algebraic structure of nested matrices
2. Analyze the spectral properties of nested matrices

## Applications for Matrix Polynomials

- Matrix polynomial in matrix variable: $M(X) = X^m + A_{m-1}X^{m-1} + \cdots + A_0$, with $X, A_i \in \mathbb{C}^{n \times n}$
- $X$ is a block eigenvalue of block companion matrix $C$ if $CV = VX$ for full-rank $V$ [JEDJ71]
- Arise in control theory, queueing theory, and other fields

## Acknowledgements

This project is a part of SMALL REU under the guidance of Professor Steven J. Miller. This research was supported by the National Science Foundation, under NSF Grant DMS2241623, Williams College, and the Finnerty Fund.

## References I

Frederick Valentine Atkinson, *Multiparameter eigenvalue problems*, Mathematics in Science and Engineering, vol. 82, Academic Press, New York, 1972.

R. P. Weber J. E. Dennis Jr., J. F. Traub, *On the matrix polynomial, lambda-matrix, and block eigenvalue problems*, Tech. report, Defense Technical Information Center, 1971.

H. C. Lee, *Eigenvalues and canonical forms of matrices with quaternion coefficients*, Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences **52** (1948), 253–260.

## References II

Lek-Heng Lim, *Singular values and eigenvalues of tensors: A variational approach*, 2006.