Erdős Distinct Angle Problems

Henry Fleischmann[1] and Ethan Pesikoff[2]

Joint work with Faye Jackson, Hongyi Hu, Sergey Konyagin, Eyvindur A. Palsson, Steven J. Miller, and Charles Wolf.

January 7, 2023

[1]henryfl@umich.edu, University of Michigan
[2]ethan.pesikoff@yale.edu, Yale University

Erdős Distinct Distance Problem

Question (Erdős Distance Problem)

What is the minimum number of distinct distances between n points in the plane?

- The $\sqrt{n} \times \sqrt{n}$ integer lattice provides upper bound $O(n/\sqrt{\log n})$ (Erdős 1946).
- Guth and Katz gave an almost matching lower bound of $\Omega(n/\log(n))$ in 2015.

Variants of the Distance Problem

1. What is the minimal number of distinct distances among sets of n points in "general position?"
2. What is the largest number such that every set of n points admits a subset of that size with all distinct distances?

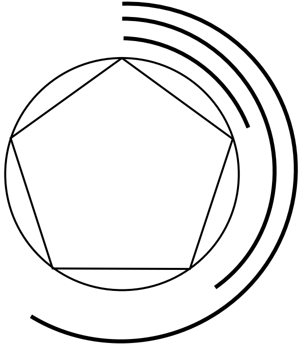
There are many, many more. See Adam Sheffer's survey.

The Erdős Distinct Angle Problem

Question (Erdős Distinct Angle Problem)

What is the minimum number of distinct angles, $A(n)$, in $(0, \pi)$ formed by n non-collinear points in the plane?

- Introduced by Erdős and Purdy in 1995.
- They conjectured that regular n-gons are optimal ($n - 2$ distinct angles):



General Lower Bound on the Erdős Angle Problem

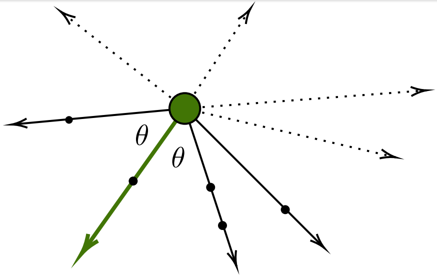
Conjecture (Weak Dirac Conjecture)

Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil n/2 \rceil$ lines between points in $\mathcal{P}$.

The best current bound of $\left\lceil \frac{n}{3} \right\rceil + 1$ was proven by Han in 2017.

Corollary

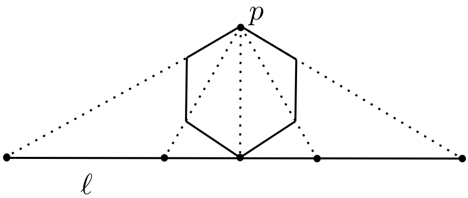
$A(n) \geq \frac{n}{6}$, $A_{no3l}(n) \geq \frac{n-2}{2}$.

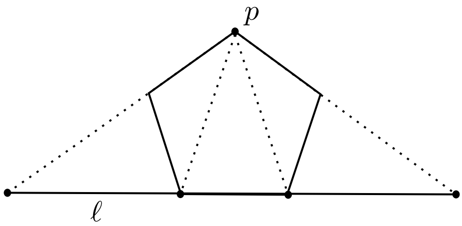


Projected Polygon

Question (Distance angle problem with non-cocircular points)

What is the the minimum number of distinct angles, $A_{\text{no4c}}(n)$, among n points with no 4 cocircular?





General Position Bounds

Question (Distance angle problem in general position)

What is the the minimum number of distinct angles, $A_{\text{gen}}(n)$, among n points with no 4 cocircular and no 3 collinear?

Theorem (FHJMPPW 2022)

$A_{gen}(n) = O(n^{\log_2(7)})$.

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

We place the points on a small arc of a logarithmic spiral, spaced at equal angles. □

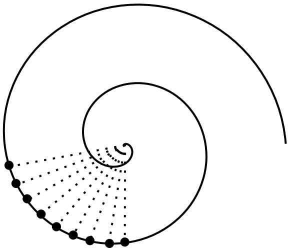
Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

We place the points on a small arc of a logarithmic spiral, spaced at equal angles.



Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

By the self-similarity of the logarithmic spiral, all triangles formed by the points are similar to one including the first point on the spiral. □

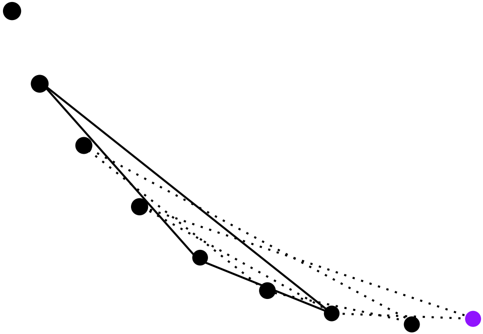
Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

By the self-similarity of the logarithmic spiral, all triangles formed by the points are similar to one including the first point on the spiral.



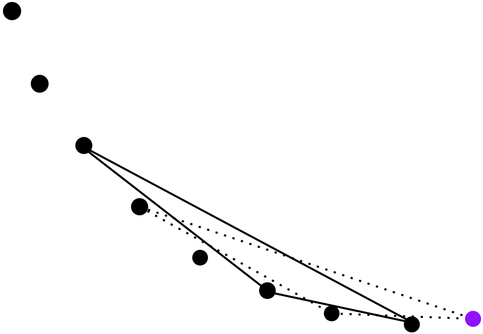
Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

By the self-similarity of the logarithmic spiral, all triangles formed by the points are similar to one including the first point on the spiral.



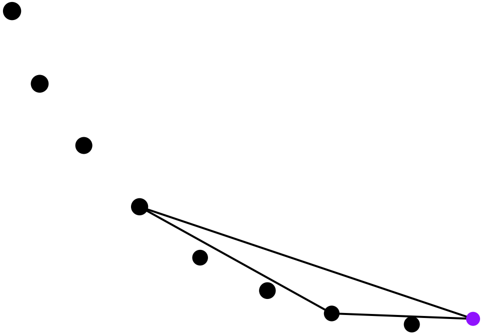
Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

By the self-similarity of the logarithmic spiral, all triangles formed by the points are similar to one including the first point on the spiral.



Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

- Hence, there are are $O(n^2)$ non-similar triangles formed by the points on the spiral and $O(n^2)$ distinct angles.

□

Logarithmic spiral construction

Theorem (FKMPPW 2022)

$A_{gen}(n) = O(n^2)$.

Sketch of the Proof.

- Hence, there are are $O(n^2)$ non-similar triangles formed by the points on the spiral and $O(n^2)$ distinct angles.
- The points are in general position by the curvature of the spiral and the fact that the points are on a small arc of the spiral.

□

General Position Bounds #2

Question

What if we defined general position more strictly, to remove the case of many points on a logarithmic spiral (or any other class of curve)?

General Position Bounds #2

Question

What if we defined general position more strictly, to remove the case of many points on a logarithmic spiral (or any other class of curve)?

Theorem (FKMPPW 2022)

We have $A_{gen}(n) = n^2 2^{O(\sqrt{\log n})}$.

General Position Bounds #2

Theorem (FKMPPW 2022)

We have $A_{gen}(n) = n^2 2^{O(\sqrt{\log n})}$.

Proof.

This bound arises from projecting the points at the intersection of a high-dimensional sphere and grid onto a generic plane. □

Maximal subsets with all distinct angles

Question

What is the largest number $R(n)$ such that every set of n points in the plane admits a subset of $R(n)$ points inducing no repeated angles?

Maximal subsets with all distinct angles

Question

What is the largest number $R(n)$ such that every set of n points in the plane admits a subset of $R(n)$ points inducing no repeated angles?

Lemma

Let $\mathcal{P}$ be a point configuration such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points. Then, $R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

Proof.

Maximal subsets with all distinct angles

Question

What is the largest number $R(n)$ such that every set of n points in the plane admits a subset of $R(n)$ points inducing no repeated angles?

Lemma

Let $\mathcal{P}$ be a point configuration such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points. Then, $R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

Proof.

- $S \subseteq \mathcal{P}$ admits at most $A(\mathcal{P})$ distinct angles.

Maximal subsets with all distinct angles

Question

What is the largest number $R(n)$ such that every set of n points in the plane admits a subset of $R(n)$ points inducing no repeated angles?

Lemma

Let $\mathcal{P}$ be a point configuration such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points. Then, $R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

Proof.

- $S \subseteq \mathcal{P}$ admits at most $A(\mathcal{P})$ distinct angles.
- Moreover, if $3\binom{|S|}{3} > A(\mathcal{P})$, there are repeated angles in S.

□

Maximal subsets with all distinct angles

Question

What is the largest number $R(n)$ such that every set of n points in the plane admits a subset of $R(n)$ points inducing no repeated angles?

Lemma

Let $\mathcal{P}$ be a point configuration such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points. Then, $R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

Proof.

- $S \subseteq \mathcal{P}$ admits at most $A(\mathcal{P})$ distinct angles.
- Moreover, if $3\binom{|S|}{3} > A(\mathcal{P})$, there are repeated angles in S.

□

- $\implies R(n), R_{\text{no3l}}(n) = O(n^{1/3})$

Maximal subsets with all distinct angles

Question

What is the largest number $R(n)$ such that every set of n points in the plane admits a subset of $R(n)$ points inducing no repeated angles?

Lemma

Let $\mathcal{P}$ be a point configuration such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points. Then, $R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

Proof.

- $S \subseteq \mathcal{P}$ admits at most $A(\mathcal{P})$ distinct angles.
- Moreover, if $3\binom{|S|}{3} > A(\mathcal{P})$, there are repeated angles in S.

□

- $\implies R(n), R_{\text{no3l}}(n) = O(n^{1/3})$
- Moreover, $R_{\text{no4c}}(n), R_{\text{gen}}(n) = O(n^{2/3})$.

A better upper bound in general position

Theorem (**F**KMP**P**W 2022)

$R_{no4c}(n), R_{gen}(n) = O(\sqrt{n})$.

Proof.

□

A better upper bound in general position

Theorem (FKMPPW 2022)

$R_{no4c}(n), R_{gen}(n) = O(\sqrt{n})$.

Proof.

- Let S be a subset of the logarithmic spiral configuration.

□

A better upper bound in general position

Theorem (FKMPPW 2022)

$R_{no4c}(n), R_{gen}(n) = O(\sqrt{n})$.

Proof.

- Think of each point in S as a number in $\{0, 1, \ldots, n-1\}$ characterizing the number of equiangular rotations around the spiral required to reach that point.

□

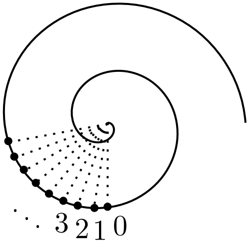
A better upper bound in general position

Theorem (FKMPPW 2022)

$R_{no4c}(n), R_{gen}(n) = O(\sqrt{n})$.

Proof.

- Think of each point in S as a number in $\{0, 1, \ldots, n-1\}$ characterizing the number of equiangular rotations around the spiral required to reach that point.



A better upper bound in general position

Theorem (FKMPPW 2022)

$R_{gen}(n) = O(\sqrt{n})$.

Proof.

- For any pair of points, there are $n-1$ possible non-negative differences.

A better upper bound in general position

Theorem (FKMPPW 2022)

$R_{gen}(n) = O(\sqrt{n})$.

Proof.

- For any pair of points, there are $n-1$ possible non-negative differences.
- Hence, if $\binom{|S|}{2} \geq 2n-1 = (n-1)+(n-1)+1$, there must be three pairs each with the same difference. This yields a pair of equivalent triples and a repeated angle.

□

Lower bound in general position

Theorem (FHJMPPW 2022)

$R_{gen}(n) = \Omega(n^{1/5})$.

- Let $\mathcal{P}$ be a point configuration in general position with n points.

Lower bound in general position

Theorem (FHJMPPW 2022)

$R_{gen}(n) = \Omega(n^{1/5})$.

- Let $\mathcal{P}$ be a point configuration in general position with n points.
- Let $\mathcal{Q} \subseteq \mathcal{P}$ with each element chosen with probability p.

Lower bound in general position

Theorem (FHJMPPW 2022)

$R_{gen}(n) = \Omega(n^{1/5})$.

- Let $\mathcal{P}$ be a point configuration in general position with n points.
- Let $\mathcal{Q} \subseteq \mathcal{P}$ with each element chosen with probability p.
- Let $q_i(n)$ be the number of pairs of equal angles on i total points (in $\mathcal{P}$).

Lower bound in general position

Theorem (FHJMPPW 2022)

$R_{gen}(n) = \Omega(n^{1/5})$.

- Let $\mathcal{P}$ be a point configuration in general position with n points.
- Let $\mathcal{Q} \subseteq \mathcal{P}$ with each element chosen with probability p.
- Let $q_i(n)$ be the number of pairs of equal angles on i total points (in $\mathcal{P}$).
- Remove an element from $\mathcal{Q}$ in each of the pairs in the q_i-sets to form $\mathcal{Q}'$.

Lower bound in general position

Theorem (FHJMPPW 2022)

$R_{gen}(n) = \Omega(n^{1/5})$.

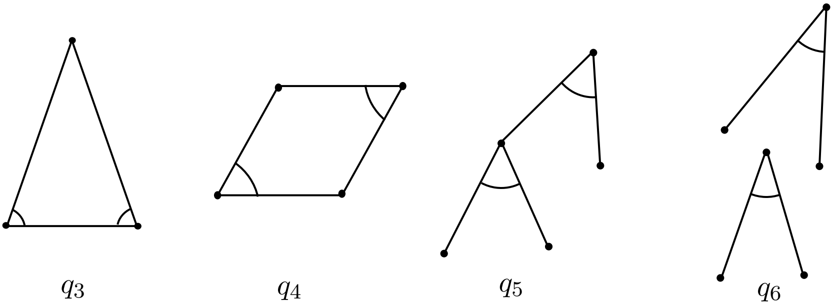
- Let $\mathcal{P}$ be a point configuration in general position with n points.
- Let $\mathcal{Q} \subseteq \mathcal{P}$ with each element chosen with probability p.
- Let $q_i(n)$ be the number of pairs of equal angles on i total points (in $\mathcal{P}$).
- Remove an element from $\mathcal{Q}$ in each of the pairs in the q_i-sets to form $\mathcal{Q}'$.
- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.

Lower bound in general position

- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.

Lower bound in general position

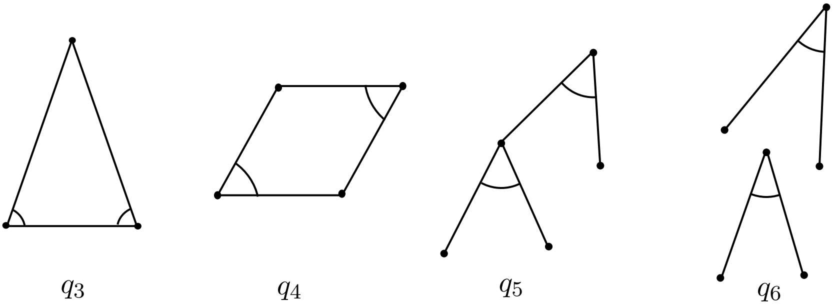
- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.
- $q_3(n) = O(n^{7/3}), q_4(n) = O(n^3), q_5(n) = O(n^4), q_6(n) = O(n^5)$.



Example configurations of $q_3(n), q_4(n), q_5(n), q_6(n)$.

Lower bound in general position

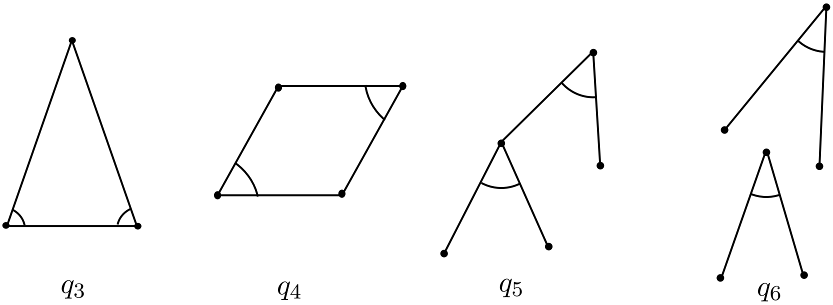
- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.
- $q_3(n) = O(n^{7/3}), q_4(n) = O(n^3), q_5(n) = O(n^4), q_6(n) = O(n^5)$.



Example configurations of $q_3(n), q_4(n), q_5(n), q_6(n)$.

Lower bound in general position

- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.
- $q_3(n) = O(n^{7/3}), q_4(n) = O(n^3), q_5(n) = O(n^4), q_6(n) = O(n^5)$.



Example configurations of $q_3(n), q_4(n), q_5(n), q_6(n)$.

- Let $p = cn^{-4/5}$ for some carefully chosen constant c, and conclude the result!

Acknowledgements

Most of this research was conducted as part of the SMALL REU program and was funded by NSF grant number 1947438. The collaboration with Sergei Konyagin occurred after CANT 2022.

Special thanks to our co-authors Faye Jackson, Hongyi Hu, Sergey Konyagin, Eyvindur A. Palsson, Steven J. Miller, and Charles Wolf.

References

- P. Erdős, D. Hickerson, and J. Pach, A problem of Leo Moser about repeated distances on the sphere, *American Mathematical Monthly* **96** (1989), 569-575.
- P. Erdős, G. Purdy, Extremal problems in combinatorial geometry, *Handbook of Combinatorics*, Vol. 1, R.L. Graham et al., eds., Elsevier (1995), 809–874.
- P. Erdős, On Sets of Distances of n Points, *The American Mathematical Monthly* **53(5)** (1946), 248-250.
- H. L. Fleischmann, H. B. Hu, F. Jackson, S. J. Miller, E. A. Palsson, E. Pesikoff, and C. Wolf, Distinct Angle Problems and Variants, *Discrete & Computational Geometry* (forthcoming).
- H. L. Fleischmann, S. Konyagin, S. J. Miller, E. A. Palsson, E. Pesikoff, and C. Wolf, Distinct Angles in General Position, *Discrete Mathematics* **346(4)** (2022), 113283.
- L. Guth and N. Katz, On the Erdős distinct distances problem in the plane, *Annals of Mathematics* **181(1)** (2015), 155-190.
- Z. Han, A Note on the Weak Dirac Conjecture, *Electronic Journal of Combinatorics* **24(1)** (2017), P1.63.
- A. Sheffer, Distinct Distances: Open Problems and Current Bounds, preprint (2018). `https://arxiv.org/abs/1406.1949`.