Erdős Distinct Angle Problems

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[1] Joint work with Faye Jackson, Hongyi Hu, Steven J. Miller, Eyvindur A. Palsson, and Charles Wolf.

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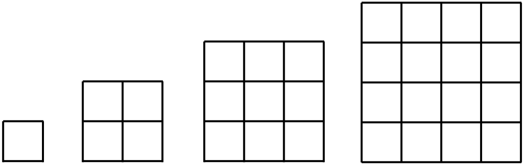
- The integer lattice provides upper bound $O(n/\sqrt{\log n})$.
  - The number of positive integers smaller than n that are the sum of two squares is $\Theta(n/\sqrt{\log n})$ (Landau-Ramanujan).
- Only finally resolved in 2015 by Guth and Katz with a lower bound of $\Omega(n/\log(n))$.



Variants of the Distance Problem

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There are many, many more. See Adam Sheffer's survey.

The Erdős Distinct Angle Problem

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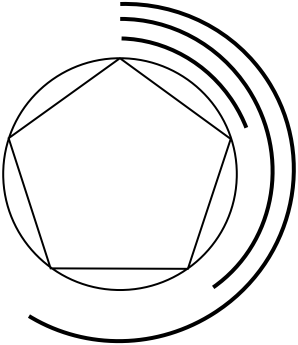
- They conjectured that regular n-gons are optimal ($n - 2$ distinct angles):



General Lower Bound on the Erdős Angle Problem

Conjecture (Weak Dirac Conjecture)

Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil n/2 \rceil$ lines between points in $\mathcal{P}$.

The best current bound of $\left\lceil \frac{n}{3} \right\rceil + 1$ was proven by Han in 2017.

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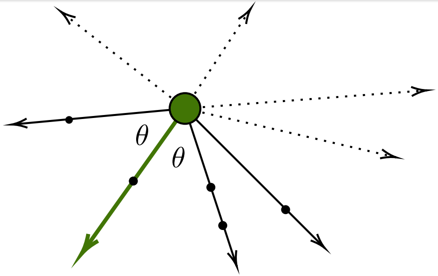
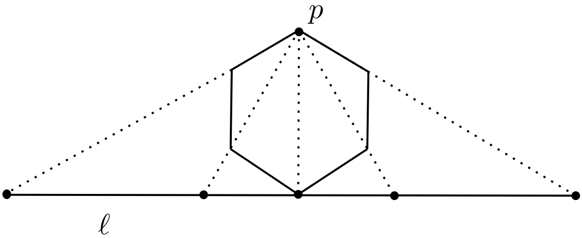
Corollary

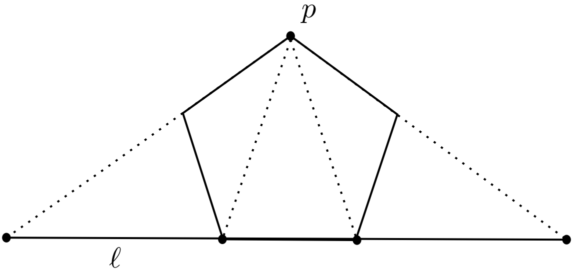
$A(n) \geq \frac{n}{6}$, $A_{no3l}(n) \geq \frac{n-2}{2}$.



Projected Polygon

What is the optimal configuration for $A_{no4c}(n)$, the minimum number of distinct angles among n points with no 4 cocircular?





General Position Bounds

Theorem

$A_{\text{gen}}(n) = O(n^{\log_2(7)})$.

Proof sketch:

- Project the points of a d-dimensional hypercube onto a plane chosen such that the orthogonal projection is injective and the projected points are in general position. Call the projection T.

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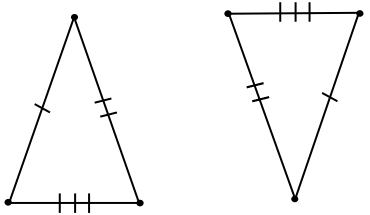

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- **This means triangles congruent up to edge translation are congruent under the projection.**
- The number of equivalence classes of edge translation equivalent triangles in a d-dimensional cube is

$$\frac{7^d - 3^{d+1} + 2}{12}.$$

Low Angle Constructions

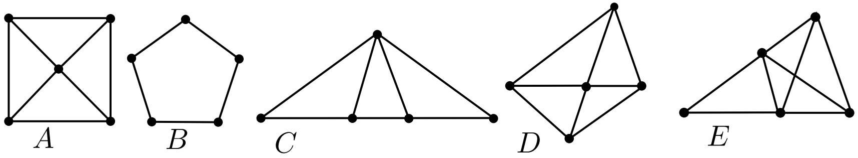
Definition

Let $P(k)$ be the maximum number of points in a planar point configuration admitting at most k distinct angles.

Lemma

$k + 2 \leq P(k) \leq 6k.$

- $P(2) = 5$. The unique optimal configuration is A.
- $P(3) = 5$. There are 5 unique optimal configurations.



What is the minimum maximum size of a subset of n points with no repeated angles, $R(n)$?

Lemma

Let $\mathcal{P} \subseteq \mathbb{R}^2$ such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points.
$R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

- $S \subseteq \mathcal{P}$ admits at most $A(\mathcal{P})$ distinct angles.
- Moreover, if $3\binom{|S|}{3} > A(\mathcal{P})$, there are repeated angles in S.
- $\implies R(n), R_{\text{no3l}}(n) = O(n^{1/3})$
- Moreover, $R_{\text{no4c}}(n), R_{\text{gen}}(n) = O(n^{\log_2(7)/3})$.

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Theorem

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- Let $q_i(n)$ be the number of pairs of equal angles on i total points.
- Remove an element from $\mathcal{Q}$ in each of the pairs in the q_i-sets to form $\mathcal{Q}'$.
- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.

Lower bound in general position #2

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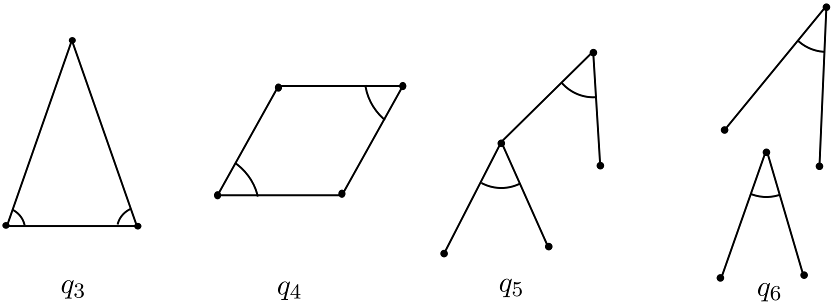
- $q_3(n) = O(n^{7/3}), q_4(n) = O(n^3), q_5(n) = O(n^4), q_6(n) = O(n^5)$.



Example configurations of $q_3(n), q_4(n), q_5(n), q_6(n)$.

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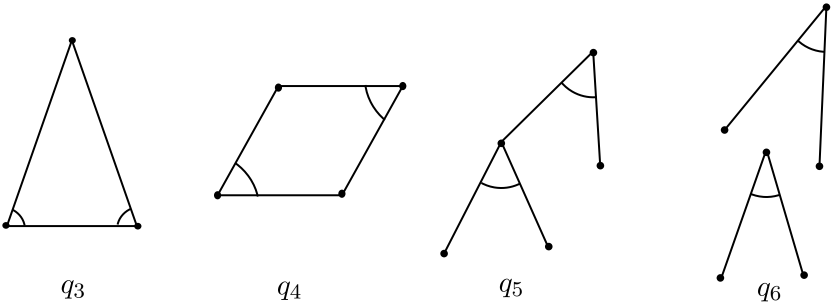

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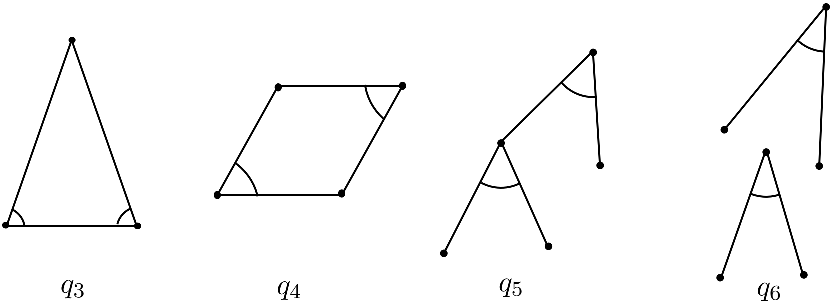

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- Let $p = cn^{-4/5}$ for some carefully chosen constant c, and conclude the result!

Acknowledgements

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