Inrtoduwtion to Erorr Dwtetcion and Erorr Czrrectmon

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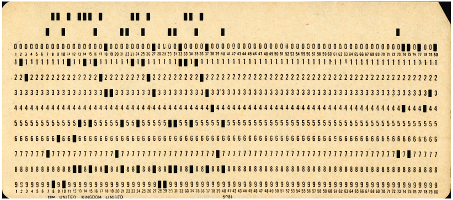

MothCulb, Univrseity of Mihciagn, Otcober 1, 2012

Introduction to Error Detection and Error Correction

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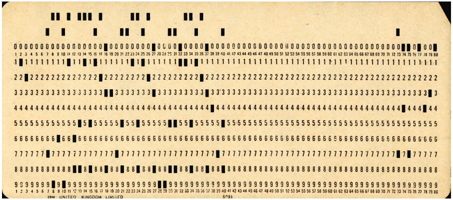

Mathclub, University of Michigan, October 1, 2021

Introduction

Cryptography Basics

Enough to send 0's and 1's:

◇ A = 00000, B = 00001, C = 00010, . . .
Z = 11010, 0 = 11011, 1 = 11100, ...

Two major issues:

◇ Transmit message so only desired recipient can read.

◇ Ensure correct message received.

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Bit Error Dangers: RSA

If receive wrong bit in RSA, message completely different.

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Secret: $p = 15217$, $q = 17569$, $d = 80998505$.
Public: $N = pq = 267347473$, $e = 3141593$.
Note: $ed = 1 \bmod (p-1)(q-1)$.
Message: $M = 195632041$, send $M^e \bmod N$ or
$X = 121209473$.
Decrypt: $X^d \bmod N$ or 195632041.

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Imagine receive $\widetilde{X} = 121209483$.
Message 195632041
Decrypts 121141028, only two digits are the same!

Outline

Will concentrate on Error Detection and Correction.

- Detection: Check Digit
- Correction: Majority Rules and Generalization

Check Digit

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Think scanner at a supermarket....

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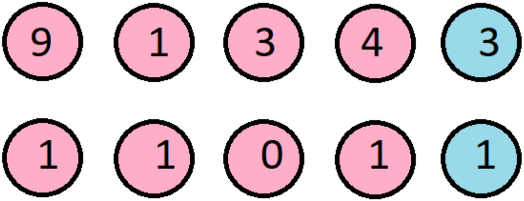

Last digit makes sum 0 mod 10 (or 0 mod 2).

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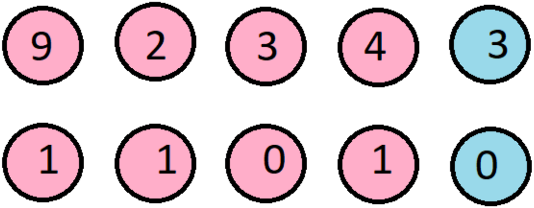
Think scanner at a supermarket....



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Next Steps

More involved methods detecting more: The Verhoeff algorithm catches single digit errors and flipping adjacent digits: https://en.wikipedia.org/wiki/Verhoeff_algorithm.

Want to detect where the error is:

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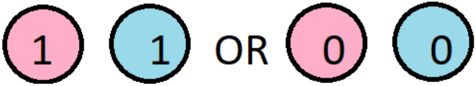
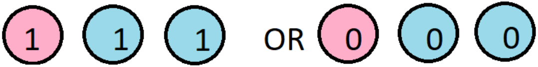
Want to detect where the error is: Tell me twice!



Majority Rules

Tell Me Three Times

Tell Me Three Times detects and *probably* corrects (need probability of an error small).



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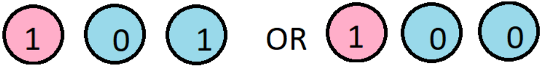

Tell Me Three Times

Crucially uses binary outcome: https://www.youtube.com/watch?v=RerJWv5vwxc and https://www.youtube.com/watch?v=vWCGs27_xPI.

What is the problem with this method?

Tell Me Three Times

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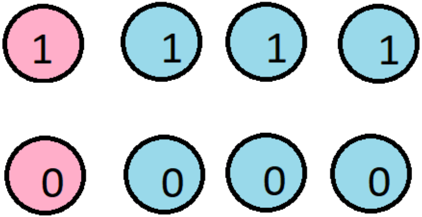
What is the problem with this method? Only one-third is information.

How can we do better?

Tell Me *n* Times

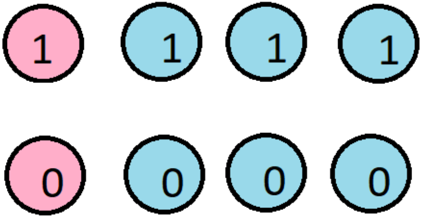


Tell Me Four Times: only 25% of message is data (general case just $1/n$).

Want to correct errors but still send a lot of information.

What's a success?

Tell Me n Times



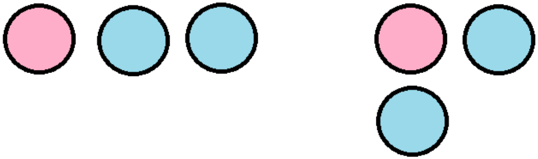
Tell Me Four Times: only 25% of message is data (general case just $1/n$).

Want to correct errors but still send a lot of information.

What's a success? Greater than 50% is data.

Tell Me Three Times (revisited)

Let's revisit Tell Me Three Times:



How should we do two data points?
How many check digits do you expect?

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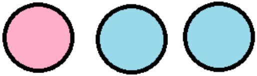

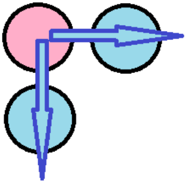

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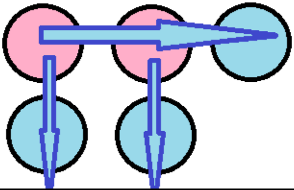
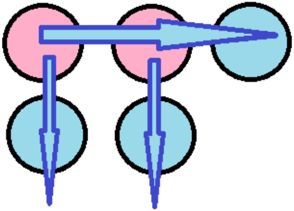


Two of Five

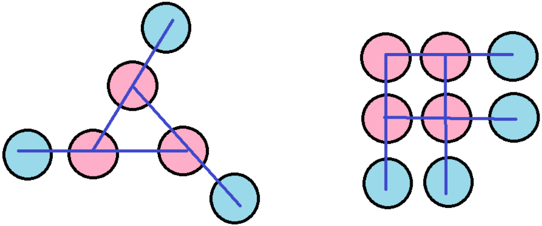
This is better: 2 of 5 or 40% of message is data!



Unfortunately still below 50%.

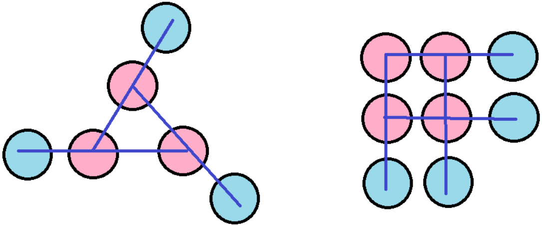
How many data points should we try next: 3, 4, 5, ...?

Three and Four Bits of Data



Which is better?

Three and Four Bits of Data

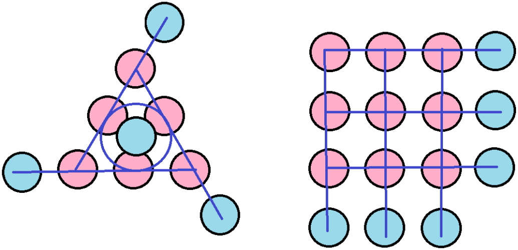


Which is better? Both 50% but fewer needed with triangle.

What should we do next: 5, 6, 7, 8, 9, ...?

Triangle and Square Numbers

$T_n = n(n+1)/2$ and $S_n = n^2$.



Both give 60% of the message is data. Can we continue?

Data on exactly two lines, check bits on one.

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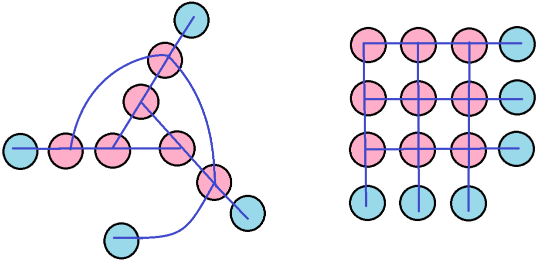

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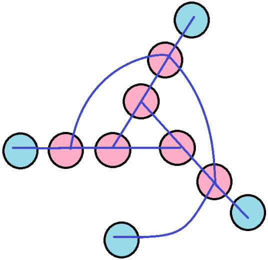
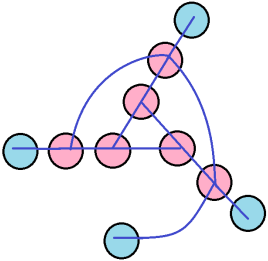
Triangle and Square Systems

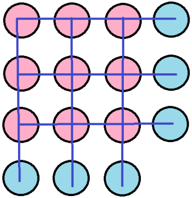


Triangle: $T_n = n(n+1)/2$ data, $n+1$ check, so $(n+2)(n+1)/2$ bits total and $n/(n+2)$ information.

Square: $S_n = n^2$ data, $2n$ check, so $n^2 + 2n$ bits total and $n/(n+2)$ information.

Triangle and Square Systems



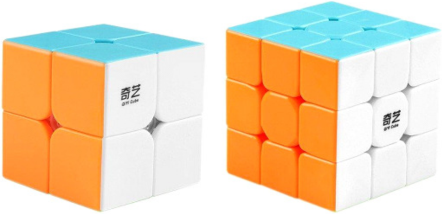


Can get as high a percentage information as desire, at a cost of longer string (and thus more likely to have two errors).

Generalizations

What is a better geometry to use?

Generalizations

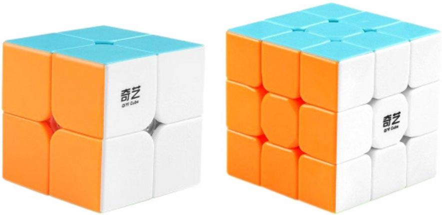


$2 \times 2 \times 2$: 8 data points, 6 check bits (for planes): info is $8/14 \approx 57\%$.

$3 \times 3 \times 3$: 27 data points, 9 check bits (for planes): info is $27/36 = 75\%$.

For 6×6 data square info is $36/48 = 75\%$, for T_7 is $28/36 \approx 77.78\%$.

Generalizations

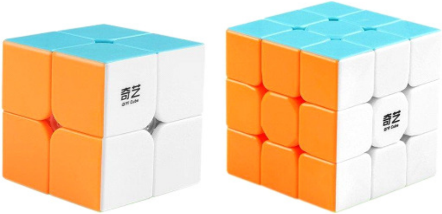


$4 \times 4 \times 4$: 64 data points, 12 check bits: info is $64/76 \approx 84.21\%$.

For 9×9 data square info is $81/99 \approx 81.82\%$.

For T_{11} triangle: 66 data points, info is $66/79 \approx 83.54\%$.

Generalizations



$n \times n \times n$: n^3 data points, $3n$ check bits: info is $n^2/(n^2+3)$.

Better percentage is information for large n; how should we generalize?

Generalizations

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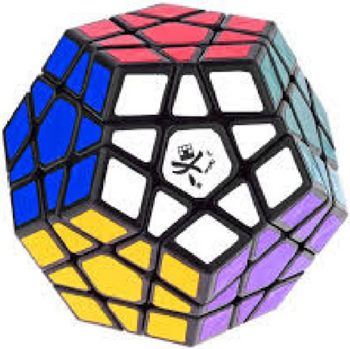
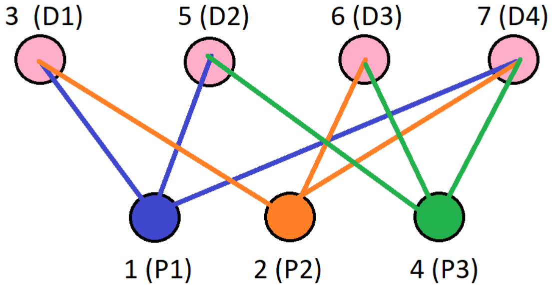


Other Approaches

Hamming Codes: Can send a message with 7 bits, 4 are data, and can correct one error: https://en.wikipedia.org/wiki/Hamming_code.

Extended binary Golay code: Can send a message with 24 bits, 12 are data, can correct any 3-bit errors and can detect some other errors: https://en.wikipedia.org/wiki/Binary_Golay_code.

Manhamming



- If no errors, all correct.
- If only one color error, is P1, P2 or P3.
- If just blue and orange is D1.
- If just blue and green is D2.
- If just orange and green is D3
- If all wrong is D4.

Comparison

Say want to transmit around $2^{12} = 4096$ bits of data.

Can do a square and cube; the Hamming code will do $2^{12} - 1 - 12$.

- Square: 4096 out of 4224 data: 96.9697%.
- Cube: 4096 out of 4144 data: 98.8417%.
- Hamming: 4083 out of 4095 data: 99.707%.

All converge to 100%, difference narrows as size increases.

Interleaving

Say transmit

01111010010101010101010101010101010101011110...

but a localized burst of noise, receive

01110111010101010101010101010101010101011110...

Interleaving

Transmit every fourth:

- 01000000001 $\mapsto$ 00000000001
- 10111111111 $\mapsto$ 11111111111
- 11000000001 $\mapsto$ 11000000001
- 10111111110 $\mapsto$ 11111111110

Steganography

Can you see the cat in the tree?

Transmitting Images

How to transmit an image?

- Have an $L \times W$ grid with LW pixels.
- Each pixel a triple, maybe (Red, Green, Blue).
- Often each value in $\{0, 1, 2, 3, \ldots, 2^n - 1\}$.
- $n = 8$ gives 256 choices for each, or 16,777,216 possibilities.

Steganography

Steganography: Concealing a message in another message: https://en.wikipedia.org/wiki/Steganography.

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Take one of the colors, say red, a number from 0 to 255.

Write in binary: $r_7 2^7 + r_6 2^6 + \cdots + r_1 2 + r_0$.

If change just the last or last two digits, very minor change to image.

Can hide an image in another.

If just do last, can hide a black and white image easily....

Can you see the cat in the tree?

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