Inrtoduwtion to Erorr Dwtetcion and Erorr Czrrectmon

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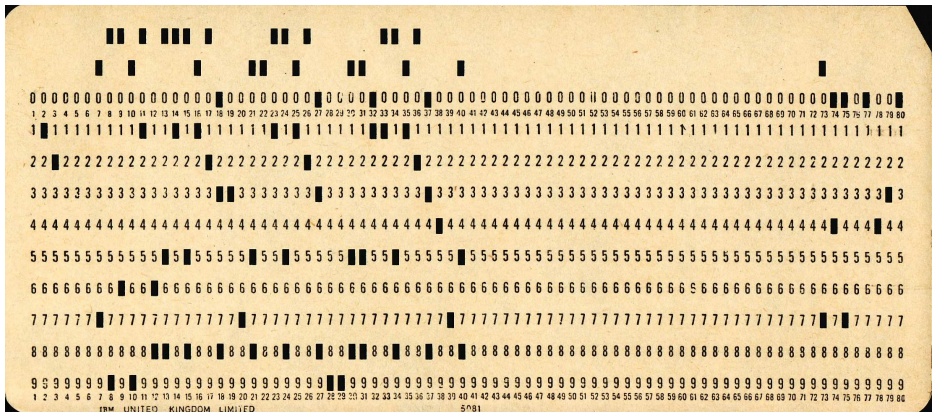

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Introduction to Error Detection and Error Correction

Steven J. Miller
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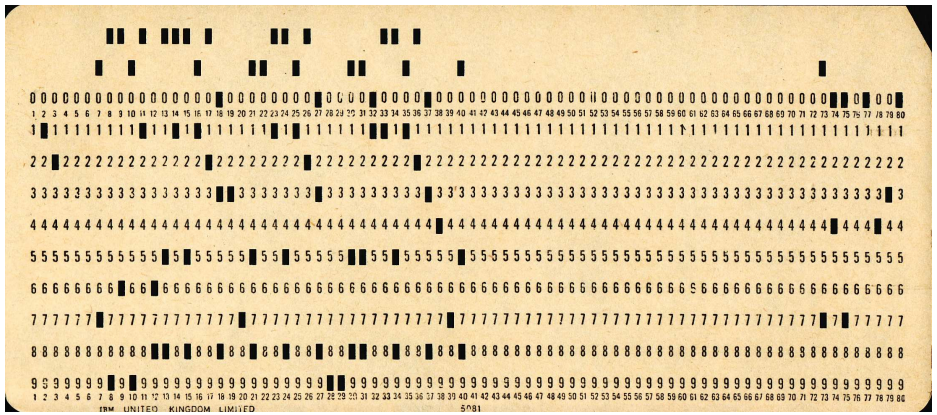

Burlington, June 19, 2019

Introduction

Cryptography Basics

Enough to send 0's and 1's:

- A = 00000, B = 00001, C = 00010, . . .
  Z = 11010, 0 = 11011, 1 = 11100, ...

Two major issues:

- Transmit message so only desired recipient can read.
- Ensure correct message received.

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Bit Error Dangers: RSA

If receive wrong bit in RSA, message completely different.

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Secret: $p = 15217$, $q = 17569$, $d = 80998505$.
Public: $N = pq = 267347473$, $e = 3141593$.
Note: $ed = 1 \bmod (p-1)(q-1)$.
Message: $M = 195632041$, send $M^e \bmod N$ or
$X = 121209473$.
Decrypt: $X^d \bmod N$ or 195632041.

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Decrypt: $X^d \bmod N$ or 195632041.

Imagine receive $\widetilde{X} = 121209483$.
Message 195632041
Decrypts 121141028, only two digits are the same!

Outline

Will concentrate on Error Detection and Correction.

- Detection: Check Digit
- Correction: Majority Rules and Generalization

Check Digit

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Think scanner at a supermarket....

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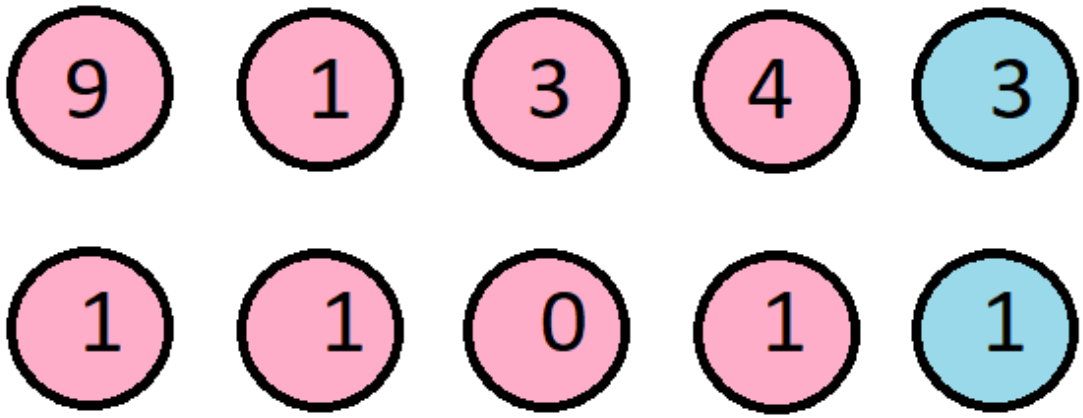

Last digit makes sum 0 mod 10 (or 0 mod 2).

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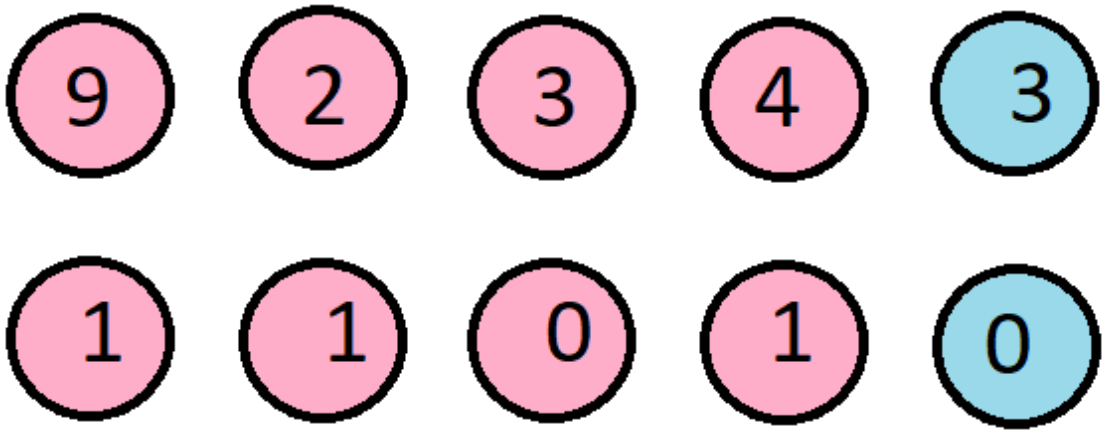
Think scanner at a supermarket....



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Next Steps

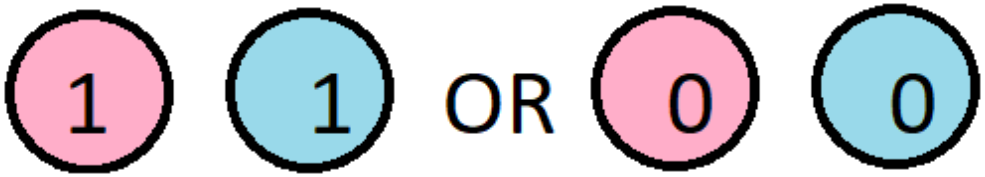
More involved methods detecting more: The Verhoeff algorithm catches single digit errors and flipping adjacent digits: https://en.wikipedia.org/wiki/Verhoeff_algorithm.

Want to detect where the error is:

Next Steps

More involved methods detecting more: The Verhoeff algorithm catches single digit errors and flipping adjacent digits: https://en.wikipedia.org/wiki/Verhoeff_algorithm.

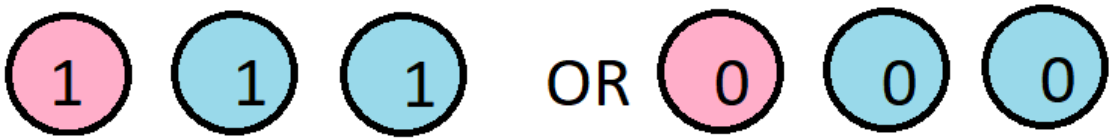
Want to detect where the error is: Tell me twice!



Majority Rules

Tell Me Three Times

Tell Me Three Times detects and *probably* corrects (need probability of an error small).



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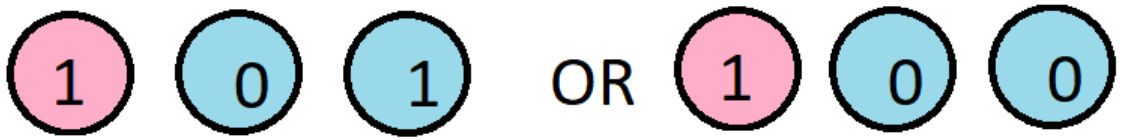

Tell Me Three Times

Crucially uses binary outcome: https://www.youtube.com/watch?v=RerJWv5vwxc and https://www.youtube.com/watch?v=vWCGs27_xPI.

What is the problem with this method?

Tell Me Three Times

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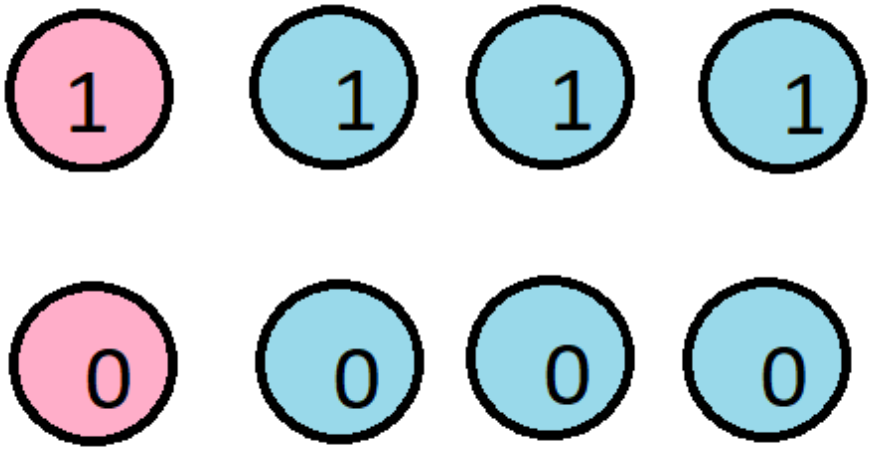
What is the problem with this method? Only one-third is information.

How can we do better?

Tell Me *n* Times

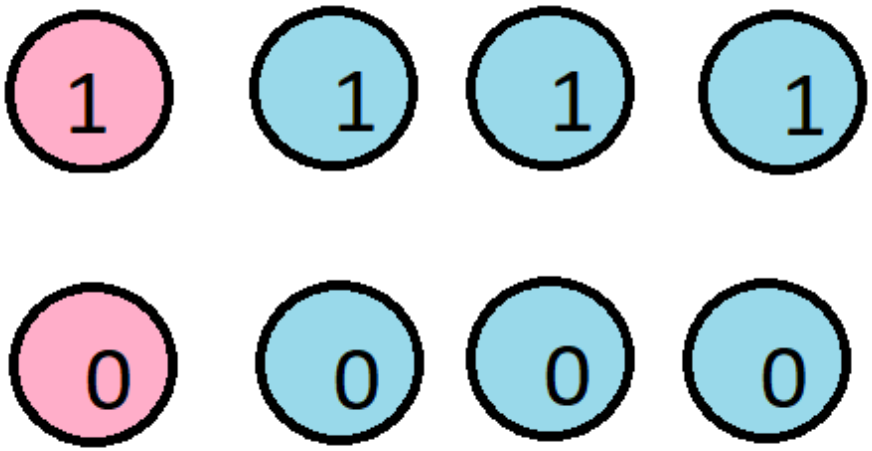


Tell Me Four Times: only 25% of message is data (general case just $1/n$).

Want to correct errors but still send a lot of information.

What's a success?

Tell Me *n* Times



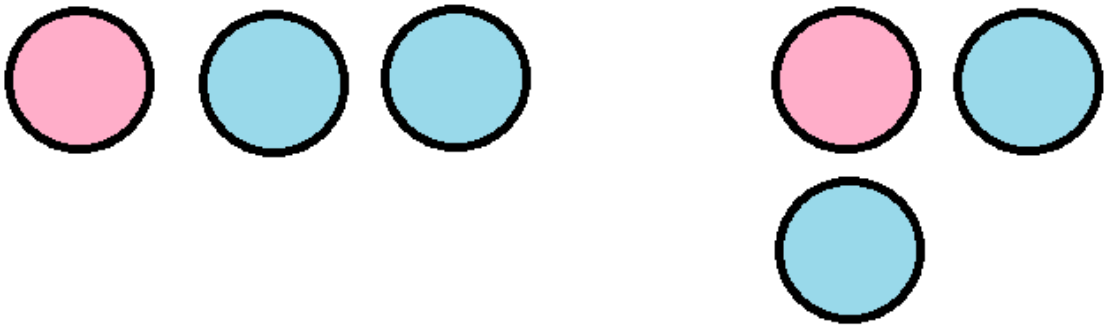
Tell Me Four Times: only 25% of message is data (general case just $1/n$).

Want to correct errors but still send a lot of information.

What's a success? Greater than 50% is data.

Tell Me Three Times (revisited)

Let's revisit Tell Me Three Times:



How should we do two data points?
How many check digits do you expect?

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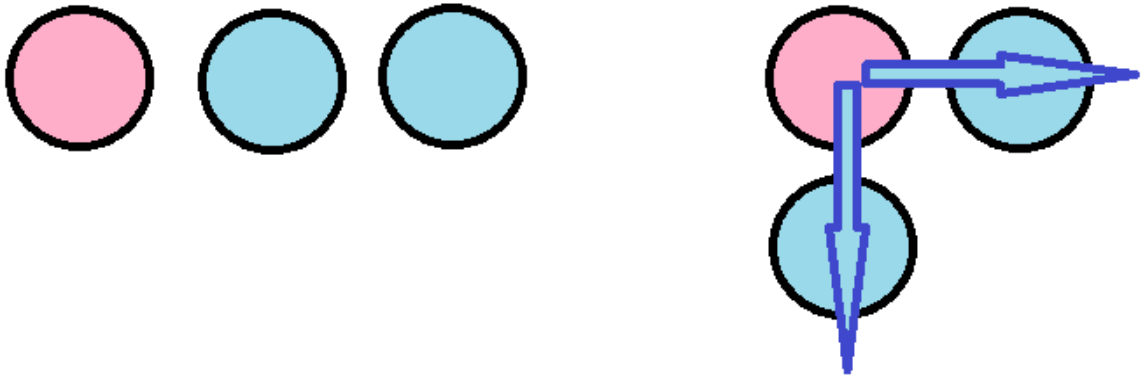

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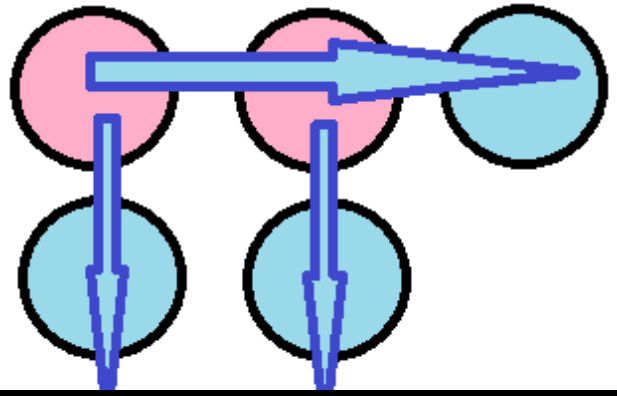
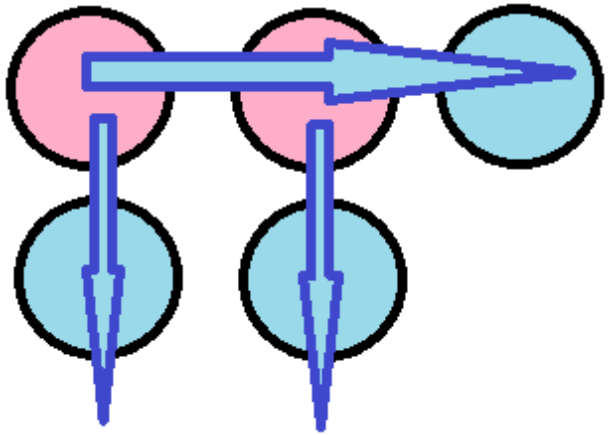


Two of Five

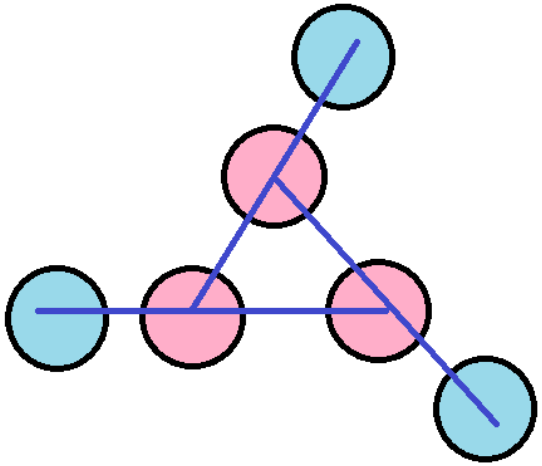
This is better: 2 of 5 or 40% of message is data!

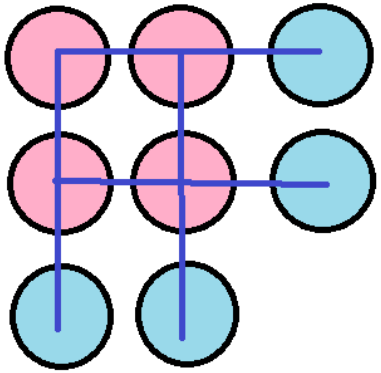


Unfortunately still below 50%.

How many data points should we try next: 3, 4, 5, ...?

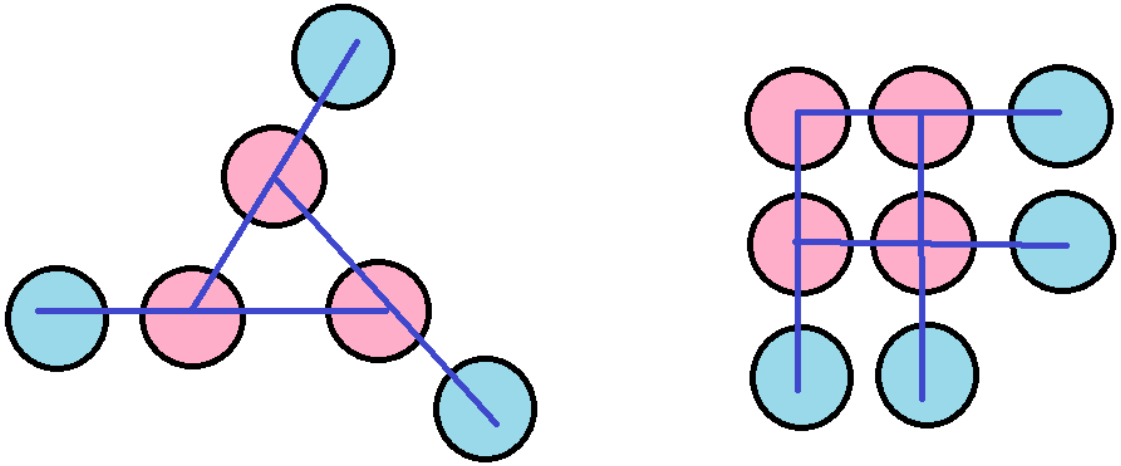
Three and Four Bits of Data





Which is better?

Three and Four Bits of Data

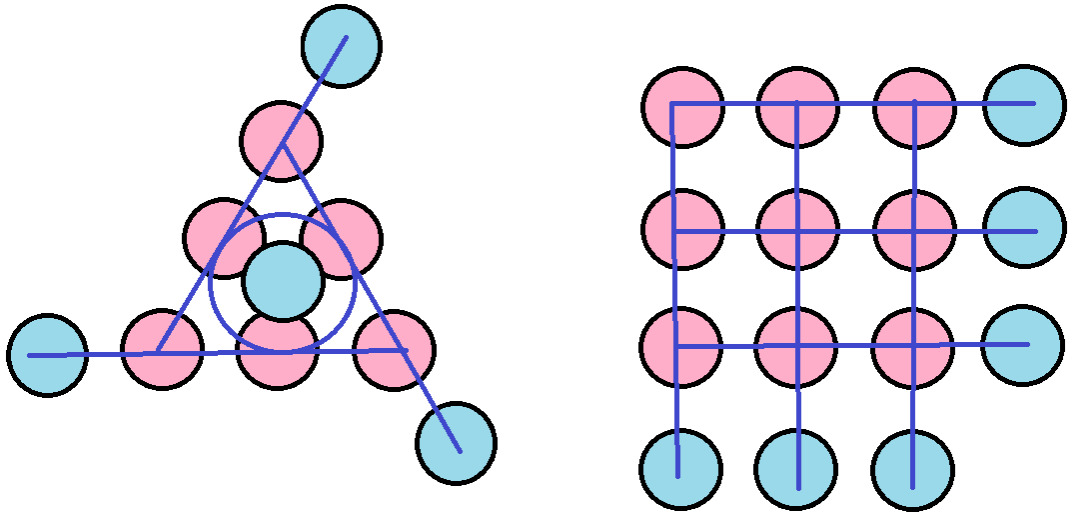


Which is better? Both 50% but fewer needed with triangle.

What should we do next: 5, 6, 7, 8, 9, ...?

Triangle and Square Numbers

$T_n = n(n+1)/2$ and $S_n = n^2$.



Both give 60% of the message is data. Can we continue?

Data on exactly two lines, check bits on one.

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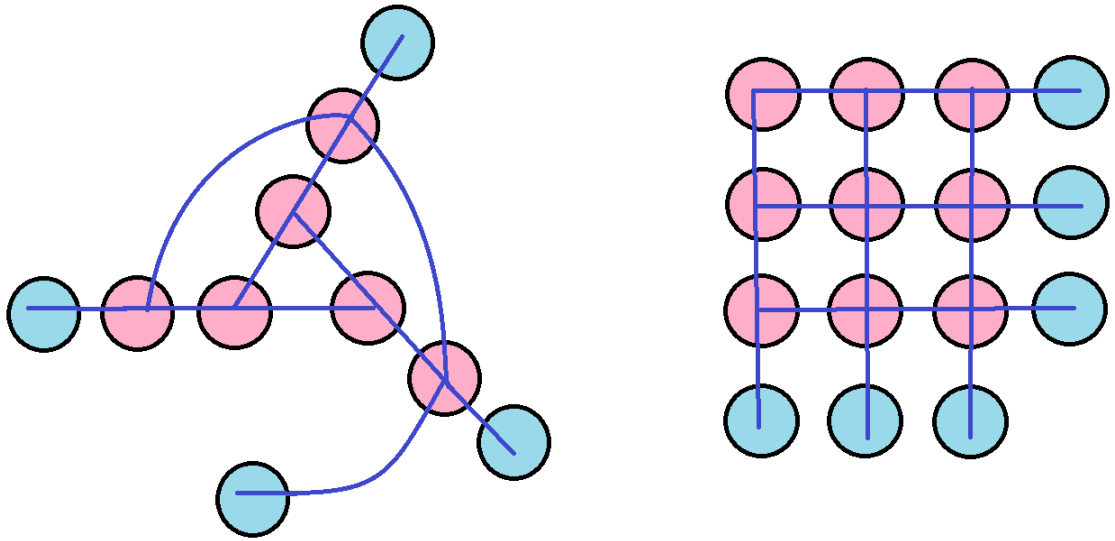

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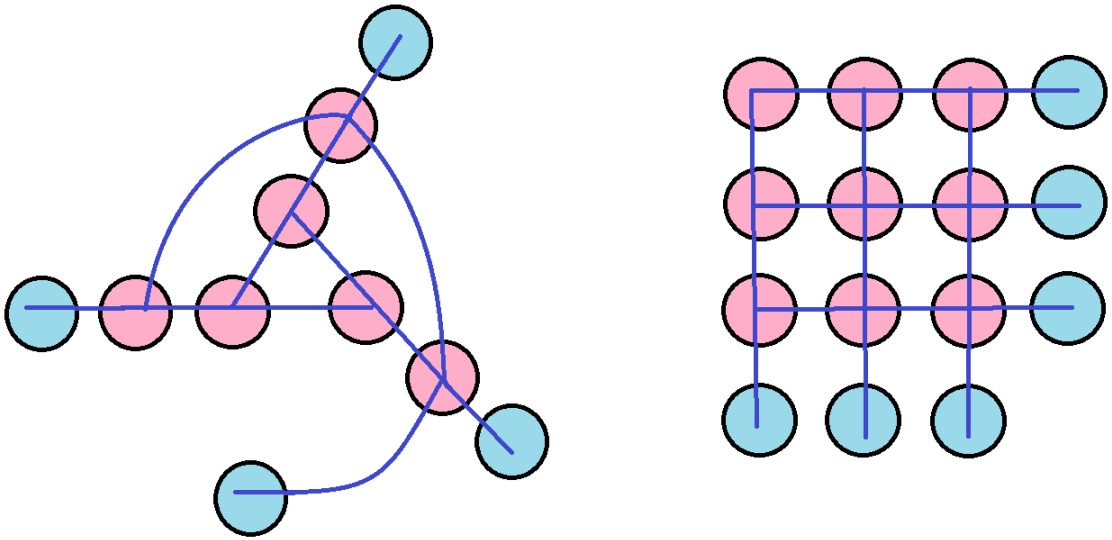
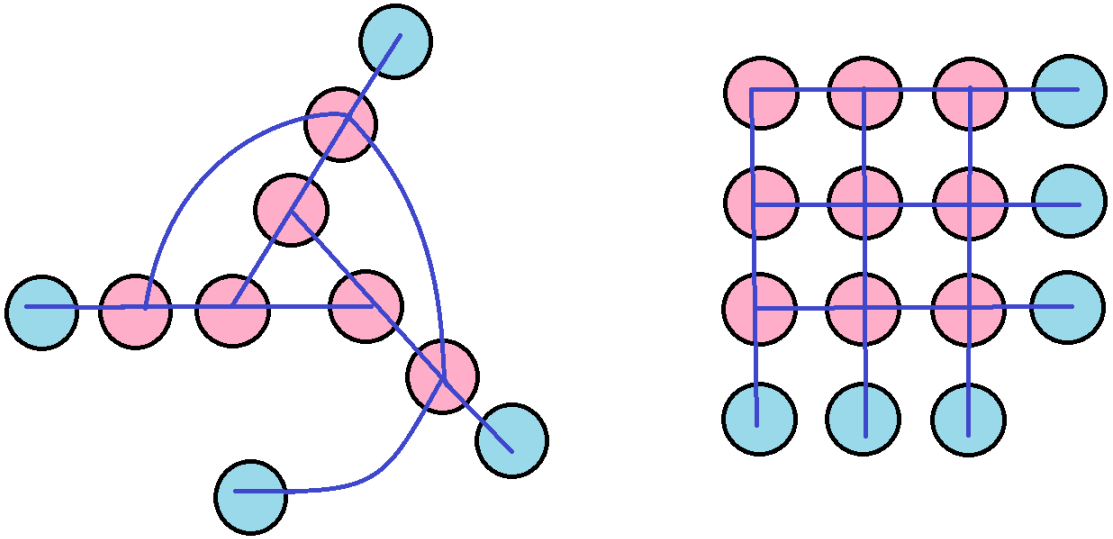
Triangle and Square Systems



Triangle: $T_n = n(n+1)/2$ data, $n+1$ check, so $(n+2)(n+1)/2$ bits total and $n/(n+2)$ information.

Square: $S_n = n^2$ data, $2n$ check, so $n^2 + 2n$ bits total and $n/(n+2)$ information.

Triangle and Square Systems



Can get as high a percentage information as desire, at a cost of longer string (and thus more likely to have two errors).

Generalizations

What is a better geometry to use?

Generalizations

$2 \times 2 \times 2$: 8 data points, 6 check bits (for planes): info is $8/14 \approx 57\%$.

$3 \times 3 \times 3$: 27 data points, 9 check bits (for planes): info is $27/36 = 75\%$.

For 6×6 data square info is $36/48 = 75\%$, for T_7 is $28/36 \approx 77.78\%$.

Generalizations

$4 \times 4 \times 4$: 64 data points, 12 check bits: info is $64/76 \approx 84.21\%$.

For 9×9 data square info is $81/99 \approx 81.82\%$.

For T_{11} triangle: 66 data points, info is $66/79 \approx 83.54\%$.

Generalizations

$n \times n \times n$: n^3 data points, $3n$ check bits: info is $n^2/(n^2+3)$.

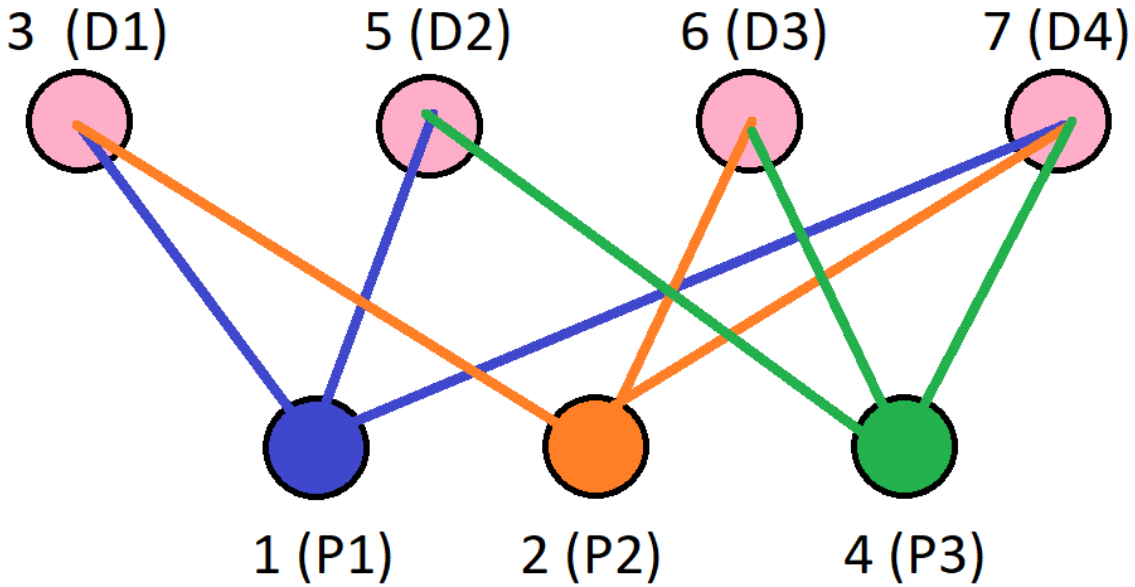
Better percentage is information for large n; how should we generalize?

Other Approaches

Hamming Codes: Can send a message with 7 bits, 4 are data, and can correct one error: https://en.wikipedia.org/wiki/Hamming_code.

Extended binary Golay code: Can send a message with 24 bits, 12 are data, can correct any 3-bit errors and can detect some other errors: https://en.wikipedia.org/wiki/Binary_Golay_code.

Manhamming



- If no errors, all correct.
- If only one color error, is P1, P2 or P3.
- If just blue and orange is D1.
- If just blue and green is D2.
- If just orange and green is D3
- If all wrong is D4.

Comparison

Say want to transmit around $2^{12} = 4096$ bits of data.

Can do a square and cube; the Hamming code will do $2^{12} - 1 - 12$.

- Square: 4096 out of 4224 data: 96.9697%.
- Cube: 4096 out of 4144 data: 98.8417%.
- Hamming: 4083 out of 4095 data: 99.707%.

All converge to 100%, difference narrows as size increases.

Interleaving

Say transmit

01111010010101010101010101010101010101011110...

but a localized burst of noise, receive

01110111010101010101010101010101010101011110...

Interleaving

Transmit every fourth:

- 01000000001 $\mapsto$ 00000000001
- 10111111111 $\mapsto$ 11111111111
- 11000000001 $\mapsto$ 11000000001
- 10111111110 $\mapsto$ 11111111110

Steganography

Can you see the cat in the tree?

Transmitting Images

How to transmit an image?

- Have an $L \times W$ grid with LW pixels.
- Each pixel a triple, maybe (Red, Green, Blue).
- Often each value in $\{0, 1, 2, 3, \ldots, 2^n - 1\}$.
- $n = 8$ gives 256 choices for each, or 16,777,216 possibilities.

Steganography

Steganography: Concealing a message in another message: https://en.wikipedia.org/wiki/Steganography.

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Take one of the colors, say red, a number from 0 to 255.

Write in binary: $r_7 2^7 + r_6 2^6 + \cdots + r_1 2 + r_0$.

If change just the last or last two digits, very minor change to image.

Can hide an image in another.

If just do last, can hide a black and white image easily....

Can you see the cat in the tree?

Can you see the cat in the tree?

Introduction to Cryptography: Alphabet Codes

Introduction to Cryptography: Alphabet Codes:

Steven J. Miller

http://www.williams.edu/Mathematics/sjmiller/public_html

VCTAL: Burlington, June 19, 2019

Caesar and Affine Ciphers

Encryption Functions

Function from alphabet to alphabet such that:

- Different letters go to different letters.
- All letters hit.

Do we need both conditions?

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Do we need both conditions?
Setup: x input letter, $f(x)$ is the encrypted letter.

Caesar Cipher

Cannot use $f(x) = c$ for a constant c: Why?
Next simplest function is $f(x) = x + c$ for some constant.

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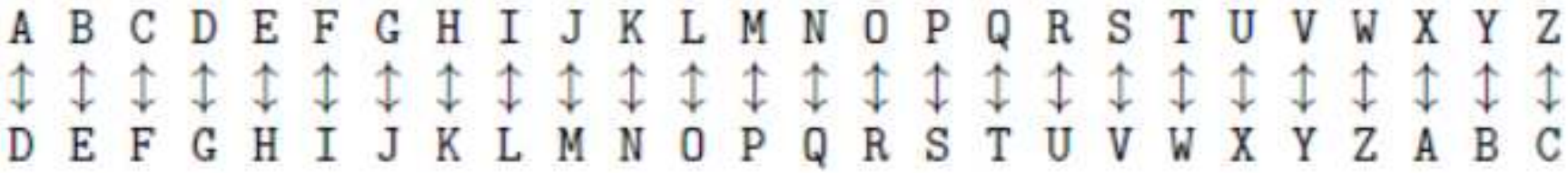
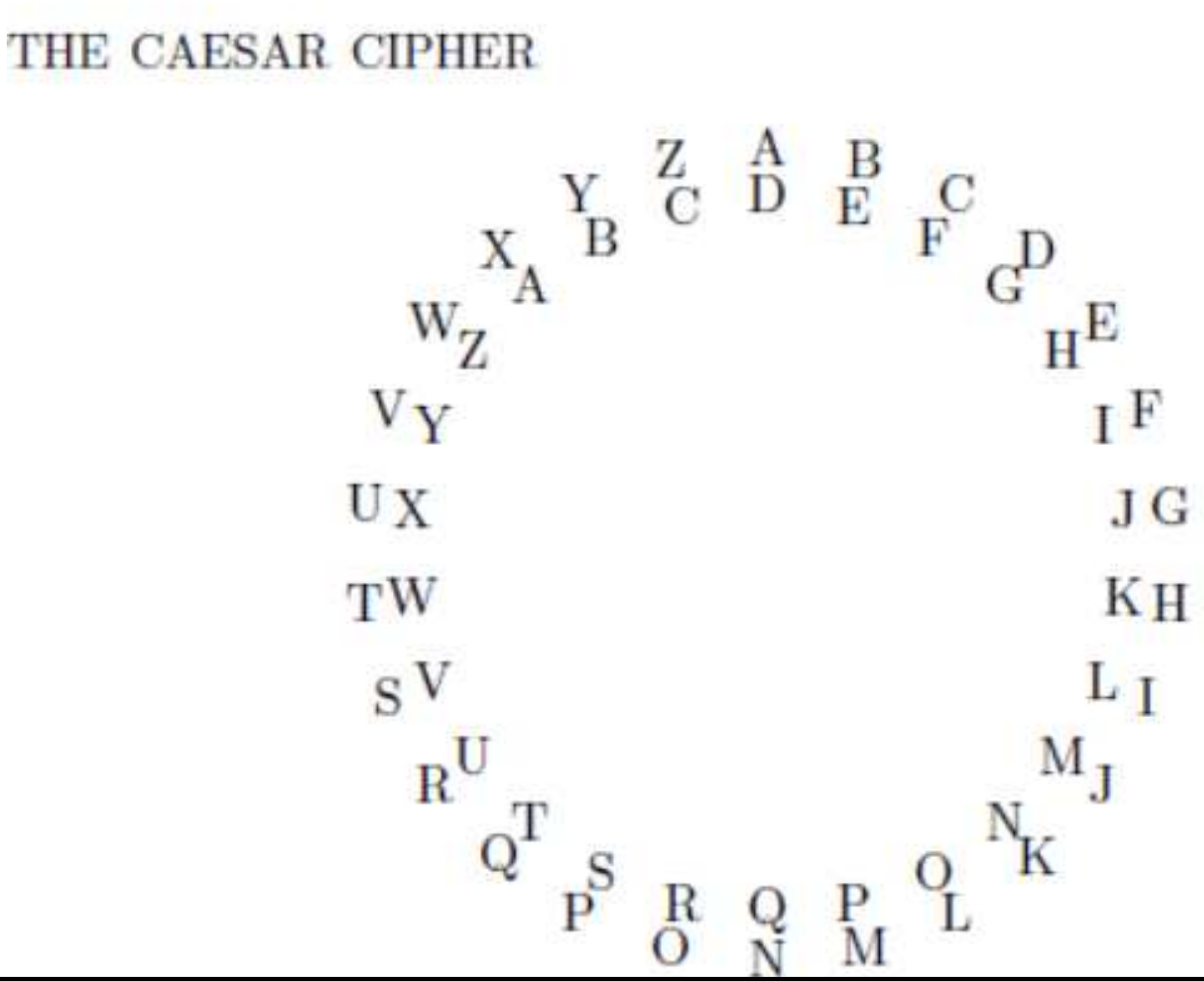
Next simplest function is $f(x) = x + c$ for some constant.

- $c = 3$ is the standard Caesar cipher: how many choices?
- $A \to D, B \to E, \ldots, W \to Z, X \to A, Y \to B, Z \to C$ (clock arithmetic).
- How do we decrypt?

Caesar Cipher: Illustration



Caesar Cipher: Illustration



Caesar Cipher: Example

	M	E	E	T	A	T	T	E	N
	12	4	4	19	0	19	19	4	13
add 3:	15	7	7	22	3	22	22	7	16
	P	H	H	W	D	W	W	H	Q

	P	H	H	W	D	W	W	H	Q
	15	7	7	22	3	22	22	7	16
subtract 3:	12	4	4	19	0	19	19	4	13
	M	E	E	T	A	T	T	E	N

Affine Cipher: $f(x) = ax + b$

Try more complicated function: $f(x) = ax + b$.

What b are possible?

Often best to start simple: take 6 letter alphabet: $\{A, B, C, D, E, F\}$.

To do calculations let $A = 1$, $B = 2$, $\ldots$, $F = 6$.

Do all b work? Do all a work?

Affine Cipher: II: $f(x) = ax + b$

Have $A = 1, B = 2, \ldots, F = 6$, look modulo 6.

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$$\begin{array}{ccccccccccc} A & +1 & \Rightarrow & B & +1 & \Rightarrow & C & +1 & \Rightarrow & D & +1 & \Rightarrow & E & +1 & \Rightarrow & F \\ \updownarrow & & & \updownarrow & & & \updownarrow & & & \updownarrow & & & \updownarrow & & & \updownarrow \\ C & +1 & \Rightarrow & D & +1 & \Rightarrow & E & +1 & \Rightarrow & F & +1 & \Rightarrow & A & +1 & \Rightarrow & B \end{array}$$

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Affine Cipher: II: $f(x) = ax + b$

Have $A = 1, B = 2, \ldots, F = 6$, look modulo 6.

Enough to study a and take $b = 0$: $f(x) = ax$.

- $a = 1$: works: get $\{1, 2, 3, 4, 5, 6\}$.
- $a = 2$: fails: get $\{2, 4, 6, 2, 4, 6\}$ (thus 4 and 6 will also fail).
- $a = 3$: fails: get $\{3, 6, 3, 6, 3, 6\}$ (and also get again 6 fails).
- $a = 5$: works: get $\{5, 4, 3, 2, 1, 6\}$.

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- $a = 3$: fails: get $\{3, 6, 3, 6, 3, 6\}$ (and also get again 6 fails).
- $a = 5$: works: get $\{5, 4, 3, 2, 1, 6\}$.

1, 5 work and 2, 3, 4, 6 fail: What's the pattern?

Affine Cipher: III: $f(x) = ax + b$

Alphabet is $A = 1, \ldots, F = 6$.

The a that work are relatively prime to 6....

What do you think works for 26 letter alphabet?

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Alphabet is $A = 1, \ldots, F = 6$.

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What do you think works for 26 letter alphabet?

Answer: 1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25: 12 choices.

Caesar cipher: 26 choices; Affine cipher: $12 \cdot 26 = 312$.

Can we go to higher degree polynomials?

Totient Function

Will use the Euler totient function $\phi(n)$ later.
$\phi(n)$ is the number of numbers from 1 to n that are relatively prime to n.

If p prime, $\phi(p) =$

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If $n = pq$ are distinct primes then $\phi(pq) = (p-1)(q-1)$:

- Lose $p, 2p, 3p, \ldots, qp$.
- Lose $q, 2q, 3q, \ldots, pq$.
- Double counted pq so add back:
  $pq - q - p + 1 = (p-1)(q-1)$.

Vigenére, and Permutation Ciphers

Vigenére

Issue is always send a letter to same new letter....

Take a keyphrase, write under and use for shifts:

D A D C A N A D D A B E D

A B C A B C A B C A B C A

E C G D C Q B F G B D H E

So D shifted to E, F, G; A shifted to B, C; ....
How secure is this?

Vigenére

How secure is the Vigenére cipher?

- https://www.telegraph.co.uk/news/worldnews/northamerica/usa/8225871/CIA-decodes-Civil-War-message-in-a-bottle-after-147-years.html
- https://owlcation.com/humanities/1863-Siege-of-Vicksburg-Secret-Message-Decoded
- http://archive.boston.com/news/nation/articles/2010/12/26/civil_war_note_finally_deciphered/
- http://cryptiana.web.fc2.com/code/civilwar4.htm
- https://en.wikipedia.org/wiki/Kasiski_examination

Permutation Ciphers

26 choices for A, 25 for B, ...

$26! \approx$

Permutation Ciphers

26 choices for A, 25 for B, ...

$26! \approx 4 \cdot 10^{26}$.

A lot more than affine case!

How secure are these?

Frequency Attacks

1	e	12.58%	14	m	2.56%
2	t	9.09%	15	f	2.35%
3	a	8.00%	16	w	2.22%
4	o	7.59%	17	g	1.98%
5	i	6.92%	18	y	1.90%
6	n	6.90%	19	p	1.80%
7	s	6.34%	20	b	1.54%
8	h	6.24%	21	v	0.98%
9	r	5.96%	22	k	0.74%
10	d	4.32%	23	x	0.18%
11	l	4.06%	24	j	0.15%
12	u	2.84%	25	q	0.12%
13	c	2.58%	26	z	0.08%

TABLE 1. Frequencies of letters in English text, from 9,481 English works from Project Gutenberg; see http://www.cryptograms.org/letter-frequencies.php.

Frequency Attacks

most common bigrams	
1	th
2	he
3	in
4	en
5	nt
6	re
7	er
8	an
9	ti
10	es
11	on
12	at
13	se
14	nd
15	or
16	ar
17	al

common trigrams	
1	the
2	and
3	tha
4	ent
5	ing
6	ion
7	tio
8	for
9	nde
10	has
11	nce
12	edt
13	tis
14	oft
15	sth

Why Primes?

Two Systems

p, q are 200 digit primes, $N = pq$ public, password p **or** q.

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Say every atom in the universe (about 10^{80} such) is a universe, and each atom there is a supercomputer checking 10^{15} items a second. About $3.2 \cdot 10^7$ seconds in a year, check about $3.2 \cdot 10^{182}$ per year.

Universe about 15 billion years old, so in the life of the universe would check about $5 \cdot 10^{192}$. Less than the number of primes to check!

RSA Description
(Rivest, Shamir, and Adleman)

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Imagine receive $\widetilde{X} = 12120948\,3$.
Message 195632041
Decrypts 12114102 8, only two digits are the same!

Implementation Questions

A lot of implementation issues.

- How do we find large primes? How large is large?
- How do we find e and d so that $ed = 1 \bmod (p-1)(q-1)$?
- How do we compute $M^e \bmod N$ efficiently?
- Can Eve determine d from e and N?

Introduction to Cryptography: RSA

Introduction to Cryptography: RSA: Steven J. Miller

http://www.williams.edu/Mathematics/sjmiller/public_html

VCTAL, Burlington, June 20

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Fermat's little Theorem

Euler totient function

$\phi(n)$ is the number of integers from 1 to n relatively prime to n.

$\phi(p) = p - 1$ and $\phi(pq) = (p-1)(q-1)$ if p, q distinct primes.

Do not need, but $\phi(mn) = \phi(m)\phi(n)$ if $\gcd(m, n) = 1$, and $\phi(p^k) = p^k - p^{k-1}$.

A lot of group theory lurking in the background, only doing what absolutely need.

Fermat's little Theorem

Fermat's little Theorem (FlT)

Let a be relatively prime to n. Then $a^{\phi(n)} = 1 \bmod n$.

Special cases: $a^{p-1} = 1 \bmod p$, $a^{(p-1)(q-1)} = 1 \bmod pq$.

Will only prove these two cases....

Proof of Fermat's little Theorem: $n = p$

Proof: Let $n = p$, let $\gcd(a, p) = 1$.

Consider $1, 2, \ldots, p-1$ and $a, 2a, \ldots, (p-1)a$.

Claim both sets are all residues modulo p.

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Note: General case: $x_1, \ldots, x_{\phi(n)}$ and $ax_1, \ldots, ax_{\phi(n)}$.

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Thus $a^{(p-1)(q-1)}$ is $1 \bmod p$ and is $1 \bmod q$.

$a^{(p-1)(q-1)} = 1 + \alpha p = 1 + \beta q$.

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Thus $\alpha p = \beta q$ so $q|\alpha$ and $p|\beta$, so $a^{(p-1)(q-1)} = 1 \bmod pq$. $\square$

Primality Tests from FlT

If $\gcd(a, n) = 1$ and $a^{n-1} \neq 1 \bmod n$ then n cannot be prime.

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If equalled 1 then n ***might be*** prime.

- If can take high powers, very fast!
- Can suggest candidate primes, and then use better, slower test for certainty.
- Carmichael numbers: Composites that are never rejected: 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, ... (OEIS A002997).

Fast Multiplication

Cost of Standard Polynomial Evaluation

Multiplication far more expensive than addition....

$f(x) = 3x^5 - 8x^4 + 7x^3 + 6x^2 - 9x + 2$: Cost is
$5 + 4 + 3 + 2 + 1 + 0 = 15$ multiplications.

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Thus $2S(d) = d \cdot (d+1)$ and claim follows.

Horner's Algorithm

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Horner's algorithm:

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Cost is degree d multiplications!

Useful also in fractal plotting.... Shows can often do common tasks faster.

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Horner is best in general, but maybe for special polynomials can do better?

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Recap

Horner takes us from order d^2 to order d.

Fast multiplication takes us to order $\log_2 d$, but only for special polynomials; these though are the ones used in RSA!

Euclidean Algorithm

Preliminaries

Input x, y with $y > x$.

Goals: find $\gcd(x, y)$, find a, b so that $ax + by = \gcd(x, y)$.

Lot of ways to go: non-constructive proofs of a, b but need values; Euclidean algorithm is *very* fast.

Euclidean Algorithm

Let $r_0 = y, r_1 = x$.

$$r_0 = q_1 r_1 + r_2, \quad 0 \leq r_2 < r_1.$$

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Continue until....

$$r_n = q_{n+1} r_{n+1} + r_{n+2}, \quad r_{n+2} \in \{0, 1\}.$$

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Note $\gcd(r_0, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3), \ldots$.

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Note $\gcd(r_0, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3), \ldots$.

Can 'climb upwards' to get a, b such that $ax + by = \gcd(x, y)$.

Implementing RSA

Implementing RSA

- Choose large primes p, q: Use FlT to get candidates.... If random choice is composite implement by 2 and try again.
- Use Euclidean algorithm to find e, d such that $ed = 1 \bmod \phi(pq)$; choose a candidate e randomly and apply Euclidean algorithm to $x = e$ and $y = (p-1)(q-1)$. If gcd equals 1 win, else increase e by 2 and try again.
- Use fast multiplication to compute $M^e \bmod pq$ efficiently, and also for that to the d^{th} power.