Leslie meets Prey and Predators: From Modeling to Research

Predator-Prey Modelling Group (SMALL 2024)
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AMS-SIMODE Special Session on Modeling Matters in Teaching and Learning Differential Equations
Joint Math Meetings, Seattle
January 8, 2025.

Overview

Introduce students to higher mathematics and research.

SMALL 2024 SUMMER REU: Strong students, varying backgrounds.

Pre-reqs: Calculus, linear algebra.

Springboard to research: combine / extend simple models.

Classical Predator-Prey Interactions

(Illustration from Premium Vector)

- Lotka-Volterra model: Evolution of predator / prey system.

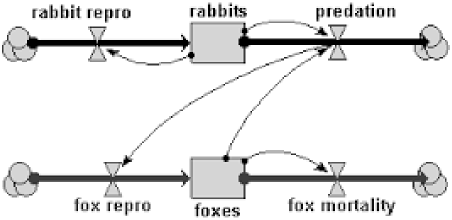
The Lotka-Volterra Equations

Definition

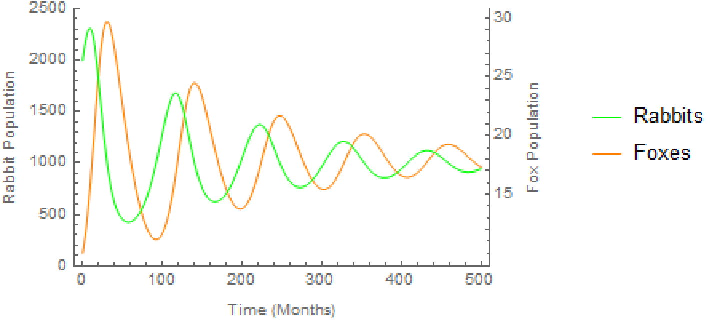
Prey population $x(t)$ and a predator $y(t)$, evolve by

$$\begin{cases} \frac{dx}{dt} & = & \alpha x - \beta xy \\ \frac{dy}{dt} & = & \delta xy - \gamma y. \end{cases}$$

Reasonable: look at signs, products.



Typical Dynamics of Lotka-Volterra



(Courtesy of Prof. Mats Vermeeren)

- Solutions often exhibit perpetual oscillations in both populations.
- Interplay inspired extensions and variations, including blending with Leslie matrices.

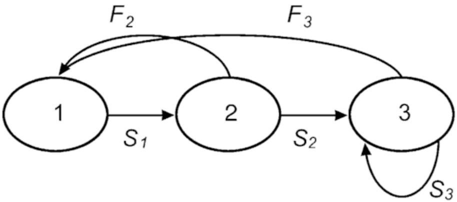
Leslie Matrices

Idea: split population into sub-populations by age.
Assume die when turn d; $a_n^{(1)}$ number newborns at time n, $a_n^{(2)}$ number of one-year-olds, ...

Population vector at time n: $\vec{p}_n := \left(a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(d-1)}\right)^T$.
Simple model: evolve by

$$\vec{p}_{n+1} \ := \ \mathbf{L}\vec{p}_n \ = \ \underbrace{\begin{bmatrix} f_1 & f_2 & \cdots & f_{d-1} \\ s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{d-2} \end{bmatrix}}_{\substack{\text{Leslie matrix:} \\ f_i\text{- fertility rates} \\ s_i \text{ - survival rates}}} \vec{p}_n,$$

Example (3 Age Groups)

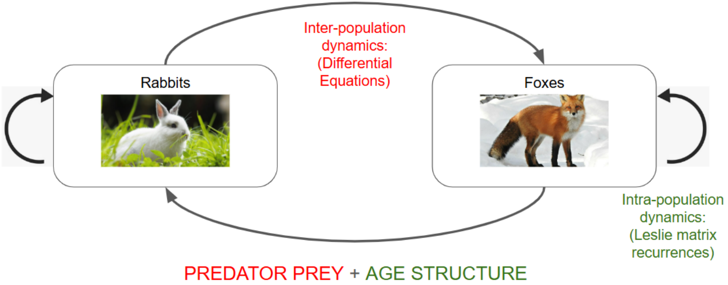


Leslie Matrix for this Scenario.

$$\mathbf{L} = \begin{bmatrix} 0 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & S_3 \end{bmatrix}$$

Research Problem

Research problem: Combine Predator-Prey and Leslie.



Gather data and conjecture, build simple models, ...

Student Background at the Start of SMALL

Saad: Senior, math and physics major

Daniel: Junior, math and physics

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Relevant Background Courses.

- Linear Algebra
- Programming in MATLAB and Python
- Quantum Mechanics

Whales and Plankton

Question

System of Whales and Plankton; Whales consume Plankton for food. Which parameters affect the population evolution? Is there a closed form solution?

A blue whale in the eastern North Pacific Ocean ingests an average of 16 tonnes of krill in a single day of feeding.

Factors

Factors of Population Evolution.

1. $f > 0$, fertility rate of Whales,
2. $F > 0$, fertility rate of Plankton,
3. $k > 0$, predation rate,
4. $m > 0$, growth multiplier of the whale induced by the Plankton.

Overconsumption

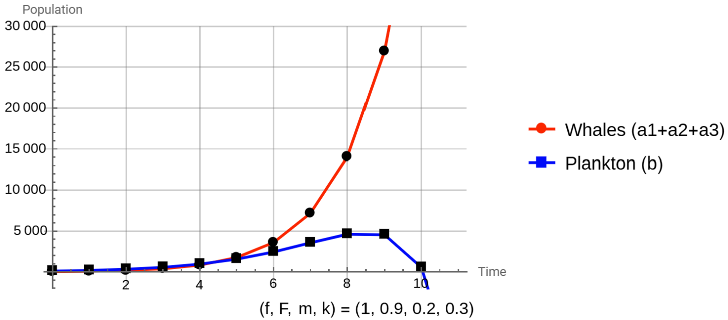


Figure: Plankton goes extinct.

Underconsumption

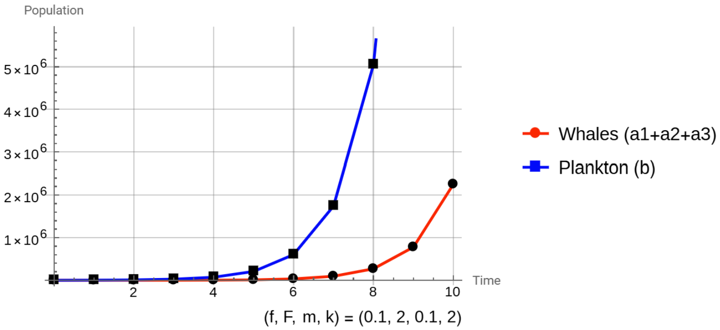


Figure: Both population grows exponentially.

Unrealistic Cases

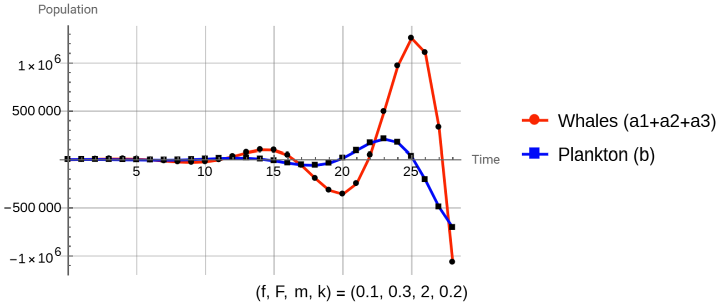


Figure: Unrealistic model.

Competitive Model

Definition (Leslie Competitive Predator-Prey Model)

Let $\vec{\alpha}_n$, $\vec{\beta}_n$ be the population vectors of the predator and prey species at time n. Define the competitive model as

$$\begin{aligned}\vec{\alpha}_{n+1} &= \max(L\vec{\alpha}_n - km\vec{\beta}_n, \vec{0})\\ \vec{\beta}_{n+1} &= \max(L\vec{\beta}_n - k\vec{\alpha}_n, \vec{0}),\end{aligned}$$

where $k, m \in (0,1)$ are interaction and competitive advantage ratios, both between 0 and 1.

The Perron-Frobenius Theorem in Action

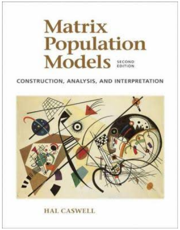
Definition (The Perron-Frobenius Theorem (Simplified))

For a nonnegative (and irreducible) square matrix M, there is a *dominant eigenvalue* (the Perron root) with a corresponding positive eigenvector. This eigenvalue governs the long-term behavior in Leslie recurrence models.

- **Why it Helps:**
  - **Population Growth Rate:** The largest eigenvalue of the Leslie (or migration) matrix determines if over or under-consumption happens.
  - **Stable Age Distribution:** The corresponding eigenvector tells us the proportion of individuals across age classes in the long run.

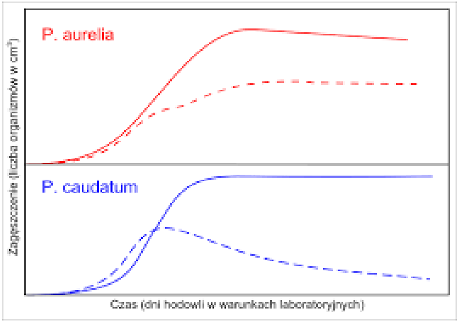
Foundational Reference

Hal Caswell - Senior Scientist in the Biology Department of the Woods Hole Oceanographic Institution



Matrix Population Models: Construction, Analysis, and Interpretation (Caswell 2001)

Applying the Competitive Model



Population evolution of Parnaceum Aurelia and Parnaceum Caudatum (Gause, 1930s).

Challenge.

Conventional methods fail due to the recursive structure of the model.

Supervised Machine Learning to the Rescue

Definition (Error of the fit)

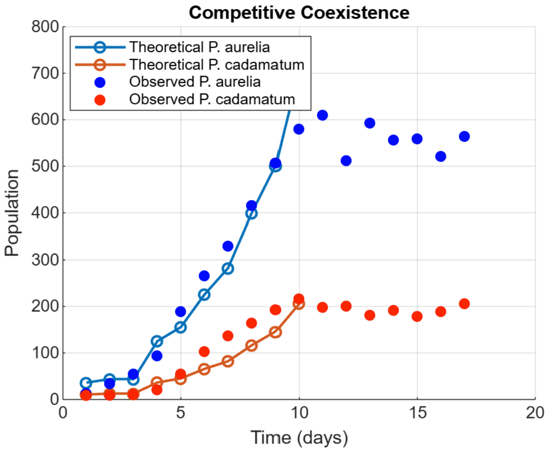
Let $\overline{\alpha}_t, \overline{\beta}_t$ be the total population of respective species predicted by the competitive model, while $p(t), q(t)$ represent the real time populations. Then, we define the residual as

$$\chi(f, k, \beta_0, m) \;:=\; \left(\sum_{t \,\leq\, T} \left(\frac{\overline{\alpha}_t - p(t)}{p(t)} \right)^2 + \sum_{t \,\leq\, T} \left(\frac{\overline{\beta}_t - q(t)}{q(t)} \right)^2 \right).$$

<u>Machine Learning Scheme</u>

1. Evaluate χ for random choice of the four parameters. Among the random tuples, choose the tuple that minimizes the value of χ.
2. Perform a general gradient descent involving all four parameters, starting from the point chosen in step 1. Find the admissible value of k, m first - then use these to optimize for the remaining.

Admissible Fit



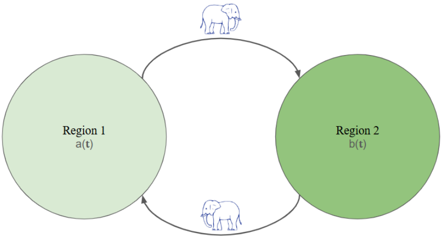
$$\chi = 3.11 < 10.351$$

Migratory Populations: The Big Picture

- Real populations often occupy multiple regions.
- Individuals move between areas due to resources, climate, or human influence.
- Understanding spatial structure is crucial: migration can alter stability, growth, and distribution.

Incorporating Migration into Leslie Models

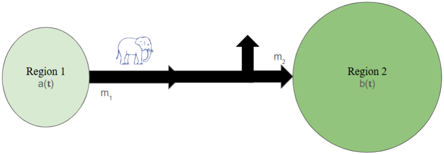
- Classic Leslie matrices model age-structured populations without spatial movement.
- We extend these models to include migration between regions.
- Two main scenarios:
  1. Constant (unidirectional) migration
  2. Constant proportion (fraction-based) migration.



Constant Unidirectional Migration: Setup

- Assume a fixed vector m_1 moves out of population a each step.
- Assume a fixed vector m_2 moves into population b each step.
- System for two populations $a(t), b(t)$:

$$a(t) = \mathbf{L_1} a(t-1) - m_1, \quad b(t) = \mathbf{L_2} b(t-1) + m_2.$$



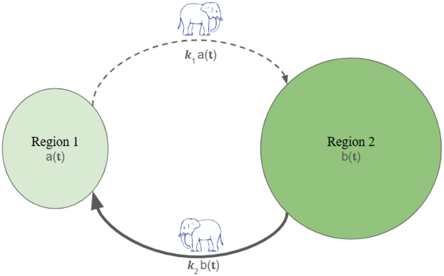
Closed Form Solutions.

$$\begin{aligned} a(t) &= \mathbf{L_1}^t a(0) - (\mathbf{I} - \mathbf{L_1})^{-1}(\mathbf{I} - \mathbf{L_1}^t) m_1. \\ b(t) &= \mathbf{L_2}^t b(0) + (\mathbf{I} - \mathbf{L_2})^{-1}(\mathbf{I} - \mathbf{L_2}^t) m_2, \end{aligned}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

Constant Proportion Migration: Motivation

- Constant numbers might not be realistic: migration often depends on current population size.
- Constant proportion migration: a fixed fraction of each population moves each step.
- Reflects density-dependent dispersal: more individuals → more migrants.



Constant Proportion Migration: Setup

Definition

With k_1, k_2 as fractions migrating, for two populations $a(t), b(t)$:

$$a(t) = \mathbf{L}a(t-1) - k_1 \cdot a(t-1) + k_2 \cdot b(t-1)$$

$$b(t) = \mathbf{L}b(t-1) - k_2 \cdot b(t-1) + k_1 \cdot a(t-1).$$

- Migration is now proportional to $a(t-1)$ and $b(t-1)$.
- System complexity increases (which is why we assume the same Leslie matrix $\mathbf{L}$ for both populations), but more realistic.

Constant Proportion Migration: Closed-Form Expressions

Definition (Closed-Form for Two-Population System)

Let $\Sigma(t) := a(t) + b(t)$ and $\Delta(t) := a(t) - b(t)$, with migration rates k_1, k_2 and $\kappa = k_1 + k_2$, $\Delta_k = k_2 - k_1$. Then:

$$\begin{aligned} \Sigma(t) &= \mathbf{L}^t \, \Sigma(0), \\ \Delta(t) &= \mathbf{L}^t \left(\frac{\Delta_k}{\kappa} \Sigma(0) \right) + (\mathbf{L} - \kappa \mathbf{I})^t \Big(\Delta(0) + \frac{\Delta_k}{\kappa} \Sigma(0) \Big). \end{aligned}$$

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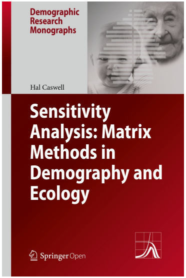
Sketch of Proof (Generating Functions):

1. *Define* $X(z) = \sum_{t \geq 0} \Sigma(t) z^t$ *and* $Y(z) = \sum_{t \geq 0} \Delta(t) z^t$.
2. *From the proportion-migration recurrences, derive functional equations for* $X(z)$ *and* $Y(z)$.
3. *Solve* $X(z) = (\mathbf{I} - \mathbf{L}z)^{-1} \Sigma(0)$ *and invert the generating function (via series expansion) to obtain closed form.*

The Inspiration Behind Our Proof

Challenges.

- Analytical solutions not feasible in arbitrary case → Simplified problem by fixing 2 populations and solving for arbitrary ages.
- Solution for the 1 age only is trivial - solve coupled recurrence using generating functions → Inspiration for our proof.



Sensitivity Analysis (Caswell, 2019)

Convergent Behavior

- If $\lambda_{\max}(\mathbf{L} - \kappa\mathbf{I}) < 1$, populations tend to die out or stabilize at a consistent level.
- This scenario can be interpreted as sub-critical growth rates dominated by mortality.

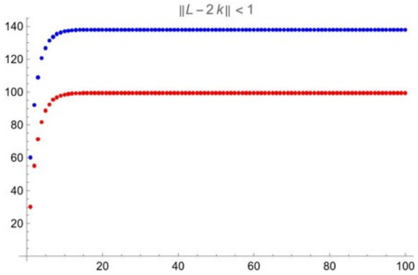


Illustration of convergent behavior(in the case when $k_1 = k_2 = k \implies \kappa = 2k$*).*

Divergent / Exponential Behavior

- If $\lambda_{\max}(\mathbf{L} - \kappa\mathbf{I}) > 1$, each population can grow exponentially in isolation.
- Populations may explode unless factors like limited resources or predator pressure are included.

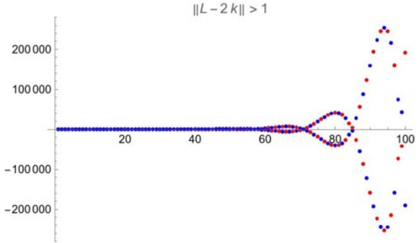


Illustration of divergent behavior(in the case when $k_1 = k_2 = k$).

Oscillatory Behavior

- Complex eigenvalues ($\lambda_{\max}(\mathbf{L} - \kappa\mathbf{I}) \approx 1$) can induce sustained oscillations or cycles.
- This manifests in a repeating boom-bust population cycle (similar to predator-prey) in each region.

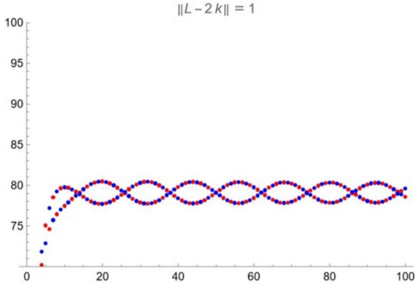


Illustration of harmonic behavior(in the case when $k_1 = k_2 = k$).

Conclusions

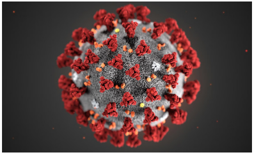
Key Takeaways.

- **Population Modeling with Predator Prey/Migration:**
  - Extensions of classic Leslie models capture spatial flow.
  - Analytical solutions help clarify how migration affects stability.
- **Applications and Practical Use:**
  - Empirical data + Machine Learning = parameter estimation.
  - Results inform conservation strategies and resource management.

Potential Applications

Modeling marine ecosystems

Conservation efforts for migratory birds

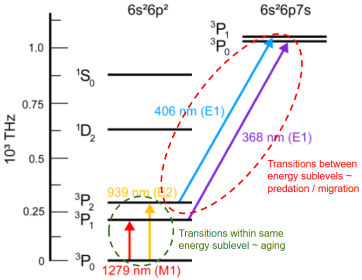


Epidemic control

Future Directions

Open Challenges.

- Multi-region and non-linear effects demand numerical approaches.
- Quantum-inspired methods offer potential new perspectives.



Energy level transitions in ^{208}Pb and its analogy to predator-prey and aging.

Acknowledgements

Thank you – Questions?

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