The Far Difference Game and Generalizations

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Background

We index Fibonacci numbers as follows:

$F_1 = 1,\ F_2 = 2,\ F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Theorem (Zeckendorf, 1972)

Every non-negative integer can be written uniquely as the sum of distinct, non-adjacent Fibonacci numbers.

Example:

$$2022 = F_{16} + F_{13} + F_8 + F_6 + F_1 = 1597 + 377 + 34 + 13 + 1$$

Far Difference Decomposition

We index Fibonacci numbers as follows:

$F_1 = 1,\ F_2 = 2,\ F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

The following theorem by Hannah Alpert [Alp09] defines far-difference decomposition:

Theorem (Alpert 2009)

Every non-negative integer can be uniquely decomposed under the following restrictions:

- *The terms are distinct Fibonacci numbers, each of which is either added or subtracted*
- *Every two terms of the same sign are at least 4 apart in index*
- *Every two terms of different sign are at least 3 apart in index*

Example:

$$2022 = F_{17} - F_{14} + F_9 - F_5 + F_1 = 2584 - 610 + 55 - 8 + 1$$

Summand Minimality

Theorem (Alpert 2009)

The Far Difference Decomposition is summand minimal among decompositions into signed Fibonacci numbers

The theorem was proved by considering the set of all summand minimal decompositions.

We present another proof using transformations of terms as moves within a game.

Far Difference Game

In 2018, Baird, Epstein, Flint, and Miller [Bai+18] introduced a game based on the Zeckendorf Decomposition. We now introduce an analogous game based instead on the Far-Difference Decomposition.

F_1	F_2	F_3	...	F_n
$N\oplus$	0	0	...	0

The game board is an infinite string of playing squares $1, 2, 3, 5, 8, \ldots$ and the game begins with N tokens on square 1. Two players alternate playing one of the following types of moves:

Far Difference Game Moves

Move Name	k-3	k-2	k-1	k+0	k+1		k-3	k-2	k-1	k+0	k+1
$I_{3,k}^+$	$\oplus$			$\oplus$		$\mapsto$		$\ominus$			$\oplus$
$I_{3,k}^-$	$\ominus$			$\ominus$		$\mapsto$		$\oplus$			$\ominus$
$I_{2,k}^+$		$\oplus$		$\oplus$		$\mapsto$	$\ominus$				$\oplus$
$I_{2,k}^-$		$\ominus$		$\ominus$		$\mapsto$	$\oplus$				$\ominus$
C_k^+			$\oplus$	$\oplus$		$\mapsto$					$\oplus$
C_k^-			$\ominus$	$\ominus$		$\mapsto$					$\ominus$
S_{k-1}^+			$2\oplus$			$\mapsto$		$\ominus$			$\oplus$
S_{k-1}^-			$2\ominus$			$\mapsto$		$\oplus$			$\ominus$
$D_{2,k}^+$		$\ominus$		$\oplus$		$\mapsto$			$\oplus$		
$D_{2,k}^-$		$\oplus$		$\ominus$		$\mapsto$			$\ominus$		
$D_{1,k}^+$			$\ominus$	$\oplus$		$\mapsto$		$\oplus$			
$D_{1,k}^-$			$\oplus$	$\ominus$		$\mapsto$		$\ominus$			

Sample Game ($N = 10$)

1	2	3	5	8	13
10$\oplus$	0	0	0	0	0
8$\oplus$	1$\oplus$	0	0	0	0
6$\oplus$	2$\oplus$	0	0	0	0
4$\oplus$	3$\oplus$	0	0	0	0
3$\oplus$	2$\oplus$	1$\oplus$	0	0	0
1$\oplus$	2$\oplus$	0	1$\oplus$	0	0
0	1$\oplus$	1$\oplus$	1$\oplus$	0	0
1$\ominus$	0	1$\oplus$	0	1$\oplus$	0
1$\ominus$	1$\ominus$	0	0	0	1$\oplus$
0	0	1$\ominus$	0	0	1$\oplus$

The first player to be unable to make a move loses (in this case, player 2).

Invariants and Monovariants

Invariants are quantities that stay constant throughout the game:

- The sum of all token values

Monovariants are quantities that can either only increase or only decrease throughout the game:

- The number of tokens

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Game Termination

Conjecture

Games must be finite in length.

Proposition (SMALL 2022)

If the game terminates, it must do so at the Far-Difference Decomposition.

This follows from the fact that the game finishes only when the resulting board satisfies the criterion for far-difference decomposition.

No Loops

Proposition (SMALL 2022)

The Far-Difference game contains no loop, and must terminate if the largest achievable summand is bounded.

Given a game where no tokens exceed square M, we define the following sum:

$$\sum_{i=1}^{M} a_i \left(1 - \frac{2^i}{2^M}\right)$$

where a_i represents the number of tokens of either sign within square i. This sum is always positive. Furthermore, it is a decreasing monovariant for all of the moves which consume no tokens.

The Bullet

We consider configurations of tokens which can—via some sequence of legal moves—travel infinitely to the right without leaving tokens behind.

Proposition (SMALL 2022)

If a configuration C can travel infinitely to the right, then it must be able to do so periodically. We call any such configuration a "bullet."

Such a bullet would be analogous to the "gliders" and "spaceships" in Conway's Game of Life.

Sketch of Periodicity

Theorem (SMALL 2022)

If a configuration C can travel infinitely to the right, then it must be able to do so periodically. We call any such configuration a "bullet."

It is sufficient to prove that any such configuration C can travel to the right with a bounded width.

Suppose that C cannot avoid a gap composed of zeroes of arbitrary length.

Either the configuration on the left, C_L, is capable of bridging the gap on its own, or C_R is able to send tokens to interact with C_L.

Corollary

The weighted sum of the tokens of an infinitely-traveling configuration is 0.

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Bullet Properties

Lemma (SMALL 2022)

Any bullet is a linear combination of the following blocks (which we call a "F-units") at position k.

F_{k-2}	F_{k-1}	F_k
$\ominus$	$\ominus$	$\oplus$

We prove this lemma by induction:

Base case: Show that if two adjacent squares sum to 0 and can move periodically, then both must be 0.

Inductive step: Remove enough F-units from the end of the bullet to decrease length by 1.

Example of F-Unit Decomposition

F_{k-2}	F_{k-1}	F_k	F_{k+1}	F_{k+2}
$\ominus$	$\ominus$		$\ominus$	$\oplus$
		$\ominus$	$\ominus$	$\oplus$
$\ominus$	$\ominus$	$\oplus$		

Bullet Moves

Lemma (SMALL 2022)

A bullet can only contain $I_3^{\pm}$, $I_2^{\pm}$, and $S^{\pm}$ moves, and must not contain token cancellation at any point.

Lemma (SMALL 2022)

The combined number of I_3^{+}, I_2^{+}, and S^{+} moves performed is equal to the combined number of I_3^{-}, I_2^{-}, and S^{-} moves performed within a period of a bullet. Thus, the number of moves within a period of a bullet is even.

Results from the Game

Proposition (SMALL 2022)

Any game state with sum equal to N can be transformed into a game state that includes the far-difference decomposition of N and some bullets.

Observe that after firing a finite number of bullets, we have a bounded configuration that sums to $\sum t_i F_i = N$.

t_1	t_2	$\dots$	t_n	0	$\dots$	0	B_1	0	$\dots$	0	B_2	$\dots$

We then perform moves on t_i to achieve the far-difference decomposition.

Corollary

The far-difference decomposition is summand-minimal.

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Tribonacci Far Difference Game

Similar to the Fibonacci Far Difference decomposition, we can define a decomposition for the Tribonacci sequence

$$T_1 = 1,\ T_2 = 2,\ T_3 = 4,\ T_n = T_{n-1} + T_{n-2} + T_{n-3} \text{ for } n \geq 4.$$

Theorem (SMALL 2022)

Every positive integer can be uniquely decomposed under the following restrictions:

- *The terms are distinct Tribonacci numbers, each of which is either added or subtracted.*
- *Every two terms of the same sign are at least 3 apart in index.*
- *Every two terms of different sign are at least 2 apart in index.*

The restrictions for Tetranacci sequence and higher order Fibonacci sequences are much more complicated.

Tribonacci Game Moves

Again, we define moves in order to transform illegal configurations into the legal one.

Move Name	k-3	k-2	k-1	k+0	k+1		k-3	k-2	k-1	k+0	k+1
$I_{2,k}^{+}$		$\oplus$		$\oplus$		$\mapsto$			$\ominus$		$\oplus$
$I_{2,k}^{-}$		$\ominus$		$\ominus$		$\mapsto$			$\oplus$		$\ominus$
$I_{1,k}^{+}$			$\oplus$	$\oplus$		$\mapsto$		$\ominus$			$\oplus$
$I_{1,k}^{-}$			$\ominus$	$\ominus$		$\mapsto$		$\oplus$			$\ominus$
S_{k}^{+}				$2\oplus$		$\mapsto$	$\oplus$				$\oplus$
S_{k}^{-}				$2\ominus$		$\mapsto$	$\ominus$				$\ominus$
$D_{1,k+1}^{+}$				$\ominus$	$\oplus$	$\mapsto$	$\ominus$			$\oplus$	
$D_{1,k+1}^{-}$				$\oplus$	$\ominus$	$\mapsto$	$\oplus$			$\ominus$	

Infinite Loop

However, unlike the Far Difference Game on Fibonacci numbers, this game on Tribonacci numbers has multiple infinite loops, such as the following example:

1	2	4	7	13	24	44
		$\ominus$		$\oplus$	$\ominus$	$\oplus$
		2$\ominus$		$\oplus$	$\oplus$	
		2$\ominus$	$\ominus$			$\oplus$
	$\oplus$	$\ominus$		$\ominus$		$\oplus$
	$\oplus$		$\oplus$		$\ominus$	$\oplus$
		$\ominus$		$\oplus$	$\ominus$	$\oplus$

Future Work

- Termination of the far-difference game
- Generalization to far-difference decomposition of PLRS to prove summand-minimality
- Winning strategies, random games, shortest and longest games

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References

[Alp09] Hannah Alpert. “Differences of Multiple Fibonacci Numbers”. In: *Integers* 9.6 (Dec. 1, 2009). Publisher: De Gruyter Section: Integers, pp. 745–749. ISSN: 1867-0652. DOI: 10.1515/INTEG.2009.061. URL: https://www.degruyter.com/document/doi/10.1515/INTEG.2009.061/html (visited on 07/24/2022).

[Bai+18] Paul Baird-Smith, Alyssa Epstein, Kristen Flint, and Steven J Miller. “The Zeckendorf Game”. In: *Combinatorial and Additive Number Theory, New York Number Theory Seminar*. Springer. 2018, pp. 25–38.