The Far Difference Game

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Far Difference Decomposition

We index Fibonacci numbers as follows:

$$F_1 = 1, F_2 = 2, \ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

The following theorem by Hannah Alpert [Alp09] defines far-difference decomposition:

Theorem (Alpert 2009)

Every positive integer can be uniquely decomposed under the following restrictions:

- *The terms are distinct Fibonacci numbers, each of which is either added or subtracted*
- *Every two terms of the same sign are at least 4 apart in index*
- *Every two terms of different sign are at least 3 apart in index*

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Example: $2358 = F_{17} - F_{12} + F_5 - F_1 = 2584 - 233 + 8 - 1$

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We present another proof using transformations of terms as moves within a game.

Far Difference Game

In 2018, Baird et al. [Bai+18] introduced a game based on the Zeckendorf Decomposition. We now introduce an analogous game based instead on the Far-Difference Decomposition.

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F_1	F_2	F_3	...	F_n
$N\oplus$	0	0	...	0

The game board is an infinite string of playing squares $1, 2, 3, 5, 8...$ and the game begins with N tokens on square 1. Two players alternate playing one of the following types of moves:

Far Difference Game Moves

Move Name	k-3	k-2	k-1	k+0	k+1		k-3	k-2	k-1	k+0	k+1
$I_{3,k}^{+}$	$\oplus$			$\oplus$		$\mapsto$		$\ominus$			$\oplus$
$I_{3,k}^{-}$	$\ominus$			$\ominus$		$\mapsto$		$\oplus$			$\ominus$
$I_{2,k}^{+}$		$\oplus$		$\oplus$		$\mapsto$	$\ominus$				$\oplus$
$I_{2,k}^{-}$		$\ominus$		$\ominus$		$\mapsto$	$\oplus$				$\ominus$
C_{k}^{+}			$\oplus$	$\oplus$		$\mapsto$					$\oplus$
C_{k}^{-}			$\ominus$	$\ominus$		$\mapsto$					$\ominus$
S_{k-1}^{+}			$2\oplus$			$\mapsto$		$\ominus$			$\oplus$
S_{k-1}^{-}			$2\ominus$			$\mapsto$		$\oplus$			$\ominus$
$D_{2,k}^{+}$		$\ominus$		$\oplus$		$\mapsto$			$\oplus$		
$D_{2,k}^{-}$		$\oplus$		$\ominus$		$\mapsto$			$\ominus$		
$D_{1,k}^{+}$			$\ominus$	$\oplus$		$\mapsto$		$\oplus$			
$D_{1,k}^{-}$			$\oplus$	$\ominus$		$\mapsto$		$\ominus$			

Sample Game (N=10)

1	**2**	**3**	**5**	**8**	**13**
10$\oplus$	0	0	0	0	0
8$\oplus$	1$\oplus$	0	0	0	0
6$\oplus$	2$\oplus$	0	0	0	0
4$\oplus$	3$\oplus$	0	0	0	0
3$\oplus$	2$\oplus$	1$\oplus$	0	0	0
1$\oplus$	2$\oplus$	0	1$\oplus$	0	0
0	1$\oplus$	1$\oplus$	1$\oplus$	0	0
1$\ominus$	0	1$\oplus$	0	1$\oplus$	0
1$\ominus$	1$\ominus$	0	0	0	1$\oplus$
0	0	1$\ominus$	0	0	1$\oplus$

The first player to be unable to make a move loses (in this case, player two).

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Game Termination

Conjecture

Games must be finite in length.

Proposition (SMALL 2022)

If the game terminates, it must do so at the Far-Difference Decomposition.

This follows from the fact that the game finishes only when the resulting board satisfies the criterion for far-difference decomposition.

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Given a game where no tokens exceed square M, we define the following sum:

$$\sum_{i=1}^{M} a_i \left(1 - \frac{2^i}{2^M}\right)$$

where a_i represents the number of tokens of either sign on square i.

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$$\sum_{i=1}^{M} a_i \left(1 - \frac{2^i}{2^M}\right)$$

where a_i represents the number of tokens of either sign on square i. This sum is always positive. Furthermore, it is a decreasing monovariant for all of the moves which consume no tokens.

The Bullet

We consider configurations of tokens which can—via some sequence of legal moves—travel infinitely to the right without leaving tokens behind.

Theorem (SMALL 2022)

If a configuration C can travel infinitely to the right, then it must be able to do so periodically. We call any such configuration a "bullet."

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Suppose that C cannot avoid a gap composed of zeroes of arbitrary length.

Either the configuration on the left, C_L, is capable of bridging the gap on its own, or C_R is able to send tokens to interact with C_L.

Corollary

The weighted sum of the tokens of an infinitely-traveling configuration is 0.

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Bullet Properties

Lemma (SMALL 2022)

Any bullet is a linear combination of the following block called a "F-unit" at position k. We denote this U_k.

F_{k-2}	F_{k-1}	F_k
$\ominus$	$\ominus$	$\oplus$

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We prove this lemma by induction:

Base case: If we have tokens on only two squares, we know they must sum to zero and can move periodically. We prove that both squares are empty.

Proof

More specifically: write $XF_m + YF_{m+1} = 0$, and $XF_{m+p} + YF_{m+p+1} = 0$, where p is the bullet period. Then by subtraction and substitution we get:

$$X(F_{m+1}F_{m+p} - F_mF_{m+p+1}) = 0$$

Since $gcd(F_{m+p}, F_{m+p+1}) = gcd(m+p, m+p+1) = 1$ and $F_{m+p} > F_m$, we have $(F_{m+1}F_{m+p} - F_mF_{m+p+1}) \neq 0$
Thus $X = 0$, meaning $Y = 0$ as well.

Working inductively, any longer sequence with a_k tokens in the rightmost square can be shortened by subtracting $a_k \cdot U_k$. We then use the inductive hypothesis for the remaining configuration.

Example of F-Unit Decomposition

F_{k-2}	F_{k-1}	F_k	F_{k+1}	F_{k+2}
$\ominus$	$\ominus$		$\ominus$	$\oplus$
		$\ominus$	$\ominus$	$\oplus$
$\ominus$	$\ominus$	$\oplus$		

Bullet Period

Lemma

The combined number of I_3^+, I_2^+, and S^+ moves performed is equal to the combined number of I_3^-, I_2^-, and S^- moves performed within a period of a bullet. Thus, the number of moves within a period of a bullet is even.

The number of negative and positive tokens must be unchanged after one period of the bullet. Any odd number of moves within a bullet will either increase or decrease these values, so the bullet period is even.

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Results from the Game

Proposition

Any game state with sum equal to N can be transformed into a game state that includes the far-difference decomposition of N and some bullets.

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Observe that after firing a finite number of bullets, we have a bounded configuration that sums to $\sum t_i * F_i = N$.

t_1	t_2	...	t_n	0	...	0	‖	B_1	‖ 0	...	0 ‖	B_2	‖ ...

We then perform moves on t_i to achieve the far-difference decomposition.

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We then perform moves on t_i to achieve the far-difference decomposition.

As the number of tokens is non-increasing, it suffices to reach the far difference decomposition with potential extra tokens. This then implies summand minimality of the far-difference decomposition.

Future Work

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- Winning strategies, random games, shortest and longest games

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