Continuing Analysis of the Zeckendorf Game

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Table of Contents

Table of Contents

Zeckendorf Decomposition

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Theorem (Zeckendorf, 1972 [3])

Every positive integer can be uniquely written as the sum of non-adjacent Fibonacci numbers.

For example,

$$2022 = 1597 + 377 + 34 + 13 + 1 = F_{16} + F_{13} + F_8 + F_6 + F_1.$$

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- Each turn one of two kinds of moves can be chosen: Split or Combine.
- Players alternate turns and the last player to move wins.

Game Rules

Split Move:

$$S_k : F_k \wedge F_k \mapsto F_{k-2} \wedge F_{k+1} \quad (k > 2)$$

$$S_2 : F_2 \wedge F_2 \mapsto F_1 \wedge F_3$$

F_1	F_2	F_3	F_4	...	F_n
0	0	2	0	...	0

$\mapsto$

F_1	F_2	F_3	F_4	...	F_n
1	0	0	1	...	0

Game Rules Cont.

Combine Move:

$$C_k : F_{k-1} \wedge F_k \mapsto F_{k+1} \quad (k > 1)$$

$$C_1 : F_1 \wedge F_1 \mapsto F_2$$

F_1	F_2	F_3	F_4	...	F_n
0	1	1	0	...	0

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F_1	F_2	F_3	F_4	…	F_n
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- Note the total weighted sum remains constant at N.
- Furthermore, the game is guaranteed to terminate at the Zeckendorf decomposition.

Who Wins?

Theorem (Baird et al., 2018)

For all $N > 2$, Player 2 can always win, though no explicit winning strategy is known.

Table of Contents

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Theorem (Cusenza et al., 2020)

The upper bound for game length is $\phi^2 N - IZ(N) - \phi Z(N)$, where $IZ(N)$ is the sum of the indices in the Zeckendorf Decomposition.

For example, $IZ(2022) = 16 + 13 + 8 + 6 + 1 = 44$ and $Z(2022) = 5$.

Game Moves

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Theorem (SMALL 2022)

Let n be the largest summand in the Zeckendorf Decomposition of N, we get that for any $2 \leq k \leq n-1$, the following sum is constant:

$$MS_k + MC_k + MC_{k+1} + \ldots + MC_{n-1}$$

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$$MS_k + MC_k + MC_{k+1} + \ldots + MC_{n-1}$$

We prove this by relabeling the board as follow

1	2	3	...	F_k	$F_{k+1} - 1$	$F_{k+2} - 2$	$F_{k+3} - 4$	...

where after the k^{th} bin, the value of a bin is equal to the sum of the values of the two bins which precede it, subtract 1

Game Moves

Proposition (SMALL 2022)

For any game with starting tokens N, we have that $MC_1 - MS_2 \approx (2 - \phi)N$

We prove this with a relabeling of the board

2	3	5	...	F_{k+1}	...

and observing that the sum of token values goes from $2N$ to $\sim \phi N$ with the change coming only from C_1 or S_2 moves.

Table of Contents

Existence of All Game Lengths

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Theorem (SMALL 2022)

For any beginning number of tokens N, let the length of the longest game be M. For any k such that $N - Z(N) \leq k \leq M$, there exists a Zeckendorf game of length k starting with input N.

Proof Sketch for All Game Lengths

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- Play on input $N - 1$, then finish the game, to get interval $I_1 = [L_1, R_1]$ of achievable game lengths by the induction hypothesis. Similarly, play on input $F_n - 1$, then finish the game, to get interval $I_2 = [L_2, R_2]$.

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- We denote the games with length R_1 and L_2 by $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

$$\mathcal{G}_1 : [(\text{Longest on } F_n - 1) \to (\text{longest remaining on } N - 1)] \to (\text{combine moves on } N)$$

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- The rest of the proof proceeds by showing that $L_2 \leq R_1$, so that $I_1 \cup I_2$ is also an interval of achievable game lengths.

Table of Contents

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Conjecture (Baird et al., 2018)

As $N \to \infty$, the number of moves in a random game where all legal moves are equally likely converges in distribution to a Gaussian with mean and variance approximately $0.215N$.

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Baird et al. verify this result empirically, executing 9999 random Zeckendorf games for $N = 200$, and plotting the empirical distribution against a Gaussian.

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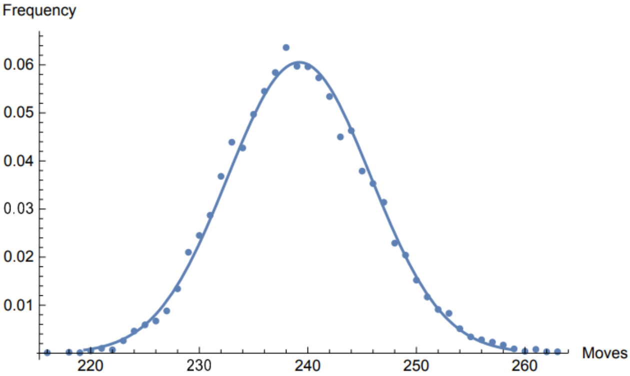

Figure: Result of running 9,999 simulations of random Zeckendorf games on input $N = 200$, plotted against a Gaussian.

Winning Odds

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Theorem (SMALL 2022)

As $N \to \infty$, the probability that each player wins in a random game approaches $1/2$. Explicitly,

$$\lim_{N\to\infty} \mathbb{P}_N(\textit{Player 1 wins}) = \lim_{N\to\infty} \mathbb{P}_N(\textit{Player 2 wins}) = \frac{1}{2}$$

where $\mathbb{P}_N$ denotes the probability measure induced by playing random Zeckendorf games with input N.

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- For $k \geq 2$, if bin $k - 1$ has height at least 1 and bin k has height at least 2, the move C_k and sequence (S_k, C_{k-1}) are both legal and have the same effect on the game.

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- Partition the collection of all Zeckendorf games on input N into those with the same **base sequence**, obtained from replacing certain instances of (S_k, C_{k-1}) by C_k. Let us index these sets by $\{\mathcal{A}_i : i \in \mathcal{I}\}$.

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- By the law of total probability,

$$\mathbb{P}_N(\text{P1 wins}) = \sum_{i \in \mathcal{I}} \mathbb{P}_N\left(\text{P1 wins} \mid \mathcal{A}_i\right) \cdot \mathbb{P}_N\left(\mathcal{A}_i\right).$$

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- The event P1 wins, conditioned on the event $\mathcal{A}_i$, can be understood as a fixed constant added to a binomial random variable with exploding variance, which converges to being odd roughly half the time.

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