The Fibonacci Quilt Sequence

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Outline

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Fibonacci sequence

Theorem (Zeckendorf's Theorem)

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = 1,\ F_2 = 2.$

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Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

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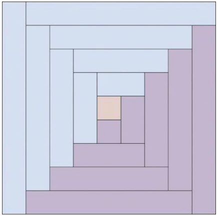
and $F_1 = 1$, $F_2 = 2$.

Thus, if we create an increasing sequence of positive integers such that any positive number can be written uniquely as a sum of non-consecutive terms, we construct the sequence

$$1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Expanding to a 2-Dimensional Construction



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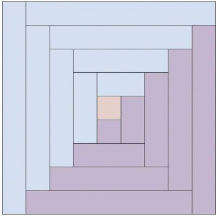
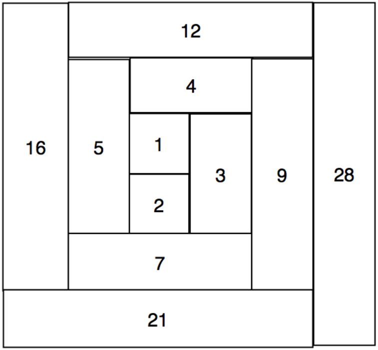


Beginning in the center with the 1 by 1 square and spiraling out counter clockwise, we construct an increasing sequence of positive integers where every positive integer can be expressed as a sum of terms that do not share an edge in the Fibonacci Quilt.

The Fibonacci Quilt Sequence



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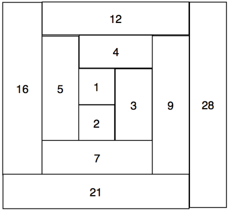


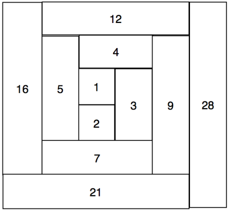
Theorem (Fibonacci Quilt Numbers)

The terms of the Fibonacci Quilt Sequence $\{a_n\}$ *satisfy the recurrence relation*

$$a_{n+1} = a_{n-1} + a_{n-2}$$

for $n \geq 4$ *and* $a_i = i$ *otherwise.*

The Fibonacci Quilt Sequence



Lemma (Recurrence Relation)

The terms of the Fibonacci Quilt Sequence $\{a_n\}$ *satisfy the recurrence relation*

$$a_{n+1} = a_n + a_{n-4}$$

for $n \geq 6$ *and* $a_i = i$ *for* $i \leq 5$ *and* $a_6 = 7$.

Proof of Lemma: $a_{n+1} = a_n + a_{n-4}$ for $n \geq 6$

Proof.

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If a_{n+1} is an element of the FQ Sequence where $n \geq 10$, then a_{n+1} cannot be legally decomposed from any previous elements of the FQ Sequence. Thus, $a_{n+1} - 1$ is the largest positive integer that can be legally decomposed using the elements from the set $\{a_1 < a_2 < \cdots < a_n\}$.

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Then $(a_{n+1} - 1) - (a_n)$ must be the largest number that can be legally decomposed using elements from the set $\{a_1 < a_2 < \cdots < a_{n-5}\}$, which equals $a_{n-4} - 1$ for $n \geq 10$ by construction. Thus $a_{n+1} = a_n + a_{n-4}$ for $n \geq 6$. □

Proof of Theorem: $a_{n+1} = a_{n-1} + a_{n-2}$ for $n \geq 4$

Proof.

Using the lemma and an inductive strategy, we first confirm that the recursive rule is satisfied for $n = 4$ and $n = 5$.

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Then

$$\begin{aligned} a_{n+1} &= a_n + a_{n-4} \\ &= (a_{n-1} + a_{n-5}) + a_{n-4} \\ &= a_{n-1} + (a_{n-4} + a_{n-5}) \\ &= a_{n-1} + a_{n-2}. \end{aligned}$$

□

Not a Positive Linear Recurrence

$a_{n+1} = a_{n-1} + a_{n-2}$ for $n \geq 4$ is not a Positive Linear Recurrence as the leading coefficient in the recurrence relation is zero.

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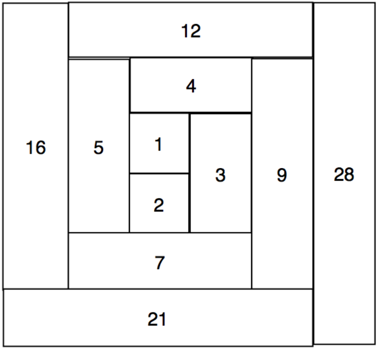
This leads to new complications and questions we can ask.

Non-Unique Decompositions



We find that we no-longer have unique decompositions. In particular, $11 = 9+2$ and $11=7+4$ are both legal decompositions.

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Theorem (Growth Rate of Number of Decompositions)

There is computable $\lambda > 1$ and a $C > 0$ such that $d_{ave}(n) \sim C\lambda^n$. Thus the average number of decompositions of integers in $[0, a_{n+1})$ tends to infinity exponentially fast.

Number of Legal Decompositions

We define the following:

- d_n: the number of FQ-legal decompositions using only elements of $\{a_1, a_2, \ldots, a_n\}$.

$$d_n = \sum_{m=0}^{a_{n+1}-1} d(m)$$

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- Similarly we define c_n and b_n where
  - c_n counts decompositions where a_n is a summand, and
  - b_n counts decompositions where a_n *and* a_{n-2} are both summands.

Number of Legal Decompositions

Using brute force we can compute the first few values of the sequences.

n	a_n	d_n	c_n	b_n
0		1	1	0
1	1	2	1	0
2	2	3	1	0
3	3	4	1	0
4	4	6	2	1
5	5	8	2	1
6	7	11	3	1
7	9	15	4	1
8	12	21	6	2
9	16	30	9	3
10	21	42	12	4
11	28	59	17	6

Table: First few terms.

Number of Legal Decompositions

Lemma

For $n \geq 7$ we have

$$d_n = c_n + d_{n-1}, \tag{1}$$

$$c_n = d_{n-5} + c_{n-2} - b_{n-2}, and \tag{2}$$

$$b_n = d_{n-7}. \tag{3}$$

Thus $d_n = d_{n-1} + d_{n-2} - d_{n-3} + d_{n-5} - d_{n-9}$, *implying* $d_{ave}(n) \approx C(1.05459)^n$.

Greedy Algorithm

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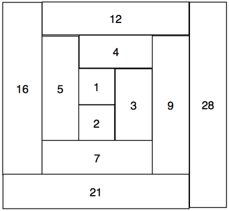

For the FQ Sequence we do not have unique decompositions and the greedy algorithm does not always successfully terminate with a legal decomposition.

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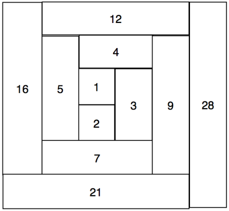

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We have $6 > a_5$, but $6 = a_4 + a_2$ is the only legal decomposition.

If $N = a_n + 6 < a_{n+1}$, then N does not have a legal decomposition by applying the Greedy Algorithm.

Greedy Algorithm

In fact, if $N = a_n + x$ and $N < a_{n+1}$, then N and x either both have a successful legal decomposition by applying the Greedy Algorithm, or neither has one.

The "Greedy Fails" less than $a_{18} = 200$ are 6, 27, 34, 43, 55, 71, 92, 113, 120, 141, 148, 157, 178, 185, 194.

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The "Greedy Fails" less than $a_{18} = 200$ are 6, 27, 34, 43, 55, 71, 92, 113, 120, 141, 148, 157, 178, 185, 194.

h_n: number of integers in the interval $[1, a_{n+1})$ where the greedy algorithm successfully terminates in a legal decomposition.

Greedy Algorithm for Decompositions

n	a_n	h_n	ρ_n
1	1	1	100.0000
2	2	2	100.0000
3	3	3	100.0000
4	4	4	100.0000
5	5	5	83.3333
6	7	7	87.5000
7	9	10	90.9091
8	12	14	93.3333
9	16	19	95.0000
10	21	25	92.5926
11	28	33	91.6667

Table: First few terms, yields $h_n = h_{n-1} + h_{n-5} + 1$.

Now we have $h_n = h_{n-1} + h_{n-5} + 1$ and $\rho_n \sim \frac{h_n}{a_{n+1}}$.

Using the Generalized Binet Formula, we find the Greedy Algorithm for Decompositions yields a legal decomposition for about 92.627% of the integers.

Thank You

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