Random Fibonacci Tilings

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Affiliated with: University of Rhode Island, Williams College

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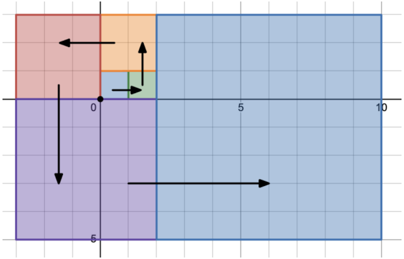
Williams SMALL
Probability and Number Theory

Introduction

The construction of the standard Fibonacci spiral.



Introduction

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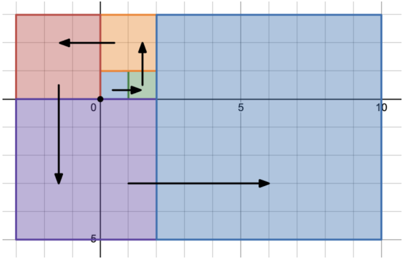

Idea

Odd indexed turns: randomly place a square left/right.
Even indexed turns: randomly place the square up/down.

Example

Consider the sequence:

$$\begin{bmatrix} \textbf{Left} & \textbf{Up} & \text{Left} & \textbf{Up} & \textbf{Left} \\ \text{Right} & \text{Down} & \textbf{Right} & \text{Down} & \text{Right} \end{bmatrix}$$

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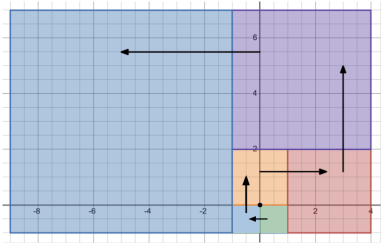
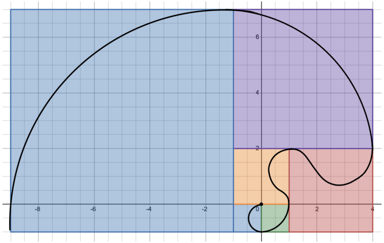
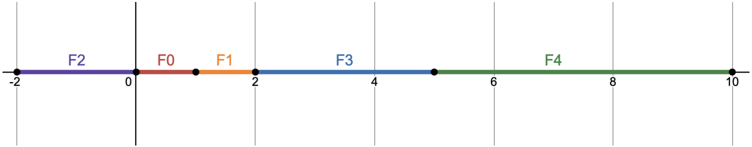
This yields the following tiling:





One-Dimensional General Case

Set $F_0 = F_1 = 1...$
We consider a 1-D version of our problem.



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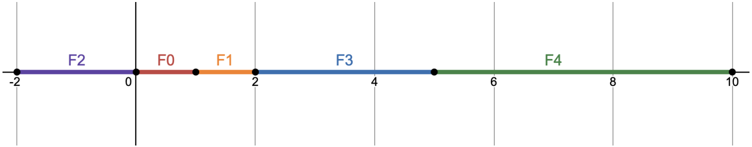

Let $b \in \mathbb{Z}^+$, and let X_i be a random variable

$$X_i := \begin{cases} F_i & \text{probability } p \\ 0 & \text{probability } 1-p. \end{cases}$$

Then,

$$\mathbb{E}(X_i) = pF_i$$

One-Dimensional Expectation

Let T_x be a variable representing the index of the first flip to cover b.

Theorem

$$\mathbb{E}(T_x) = \log_{\varphi}\left(\frac{\sqrt{5}\, b}{p}\right) - 3 + O(1) = \log_{\varphi}(b/p) + O(1)$$

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By linearity of expectation and the identity $\sum_{i=0}^{n} F_i = F_{n+2} - 1$:

$$\mathbb{E}(X_0 + \cdots + X_n) \;=\; p\sum_{i=0}^{n} F_i \;=\; p(F_{n+2} - 1).$$

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To cover b, we set $\mathbb{E}(X_0 + \cdots + X_n) \;\geq\; b$:

$$\begin{aligned} p(F_{n+2} - 1) \;\geq\; b \quad &\iff \quad F_{n+2} \geq \frac{b}{p} + 1 \\ &\iff \quad \frac{\varphi^{n+3}}{\sqrt{5}} \gtrsim \frac{b}{p} + 1 \\ &\iff \quad n = \log_{\varphi}\left(\frac{\sqrt{5}\,b}{p}\right) - 3 + O(1). \end{aligned}$$

Two-Dimensional Case: Parity Decomposition

To cover (a, b), we decompose movement along each axis by step parity $\pmod 2$:

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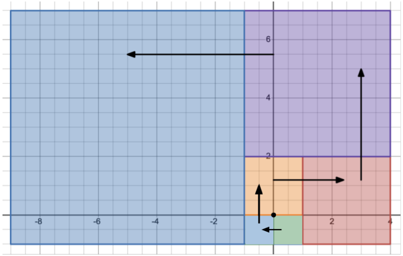

- **Odd-indexed steps ($i = 1, 3, 5, \ldots$):** Contribute only to the width (x-axis)
- **Even-indexed steps ($i = 0, 2, 4, \ldots$):** Contribute only to the height (y-axis)

Two-Dimensional Expectation

Theorem

$$\mathbb{E}\max\{T_x, T_y\} \;=\; \frac{1}{2}\cdot\left(\mathbb{E}T_x + \mathbb{E}T_y + \mathbb{E}|T_x - T_y|\right)$$

Two-Dimensional Counting & Expectations

Using Fibonacci sum identities modulo 2:

- **Horizontal (x-axis, k odd steps):**

$$\sum_{j=1}^{k} F_{2j-1} = F_{2k} - 1 \implies \mathbb{E}(S_x) = p(F_{2k} - 1) \geq a \implies F_{2k} \geq \frac{a}{p} + 1$$

$$\implies T_x = 2k - 1 = \log_\varphi\left(\frac{\sqrt{5}\,a}{p}\right) - 2 + O(1)$$

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- **Vertical (y-axis, $k+1$ even steps):**

$$\sum_{j=0}^{k} F_{2j} = F_{2k+1} \implies \mathbb{E}(S_y) = pF_{2k+1} \geq b \implies F_{2k+1} \geq \frac{b}{p}$$

$$\implies T_y = 2k = \log_\varphi\left(\frac{\sqrt{5}\,b}{p}\right) - 2 + O(1)$$

Two-Dimensional Expectation

Theorem

$$\mathbb{E}\max\{T_x, T_y\} = \frac{1}{2}\cdot\left(\mathbb{E}T_x + \mathbb{E}T_y + \mathbb{E}|T_x - T_y|\right)$$

Where $\mathbb{E}T_x$ and $\mathbb{E}T_y$ are calculated from the modulo 2 sub-sequences:

$$\mathbb{E}T_x = \log_\varphi\left(\frac{\sqrt{5}\,a}{p}\right) - 2 + O(1), \qquad \mathbb{E}T_y = \log_\varphi\left(\frac{\sqrt{5}\,b}{p}\right) - 2 + O(1).$$

D-Dimensional Case

To cover $(a_1, a_2, \ldots, a_d)$, we must cover it on each the of d axes. This breaks into d 1-D cases.

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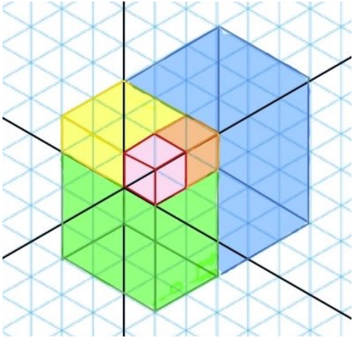


- $k \pmod{d}$-indexed flips contributes only to the k^{th} axis.

D-Dimension Expectation

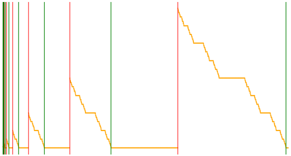
Theorem

$$\mathbb{E}\max\{T_{x_1},\dots,T_{x_n}\} = \frac{1}{2^{n-1}}\left(\mathbb{E}[T_{x_1}] + \sum_{k=2}^{n} 2^{k-2}\left(\mathbb{E}[T_{x_k}] + \mathbb{E}\left|\max\{T_{x_1},\dots,T_{x_{k-1}}\} - T_{x_k}\right|\right)\right).$$

Open Questions and Possible Directions

Open Question

Can we find a closed form for $E_{n_b}(b)$ in 2 dimensions and higher?



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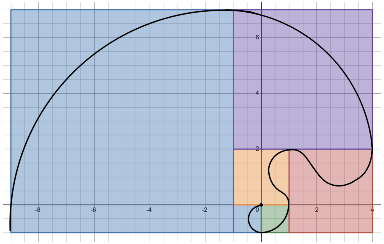
Is there an algorithm that allows us to generalize the Fibonacci Spiral to the fair coin randomized tiling problem?



Thank You

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